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A 2-DIMENSIONAL ALGEBRAIC VARIETY WITH 27 RECTILINEAR GENERATORS AND 108 TRISECANTS AND ITS CONNECTION WITH THE MAXIMAL EXCEPTIONAL SIMPLE LIE GROUP

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Abstract. A 2-dimensional algebraic variety in 4-dimensional projective space determining a regular configuration is considered and its connection with simple exceptional Lie group E_8 is found.

1. Galois and Weyl groups connected with simple Lie groups

With every complex and real compact simple Lie group two finite groups called "Galois and Weyl groups" are connected. Galois group Γ of a simple Lie group is the Galois group of characteristic equation of this group, Weyl group W of this Lie group is the group generated by reflections determined by root vectors of this Lie group (see [1, pp. 63 and 69]).

É. Cartan [2] has proved that for simple Lie groups B_n, C_n, G_2, F_4, E_7 and E_8 groups V and W are isomorphic, and for simple Lie groups A_n, D_n, and E_6 group W is an invariant subgroup of group Γ, and the quotient group Γ/W for group D_4 consists of 3! elements, for other groups A_n, D_n, and E_6 consists of 2 elements. Note that the Dynkin diagrams for simple Lie groups with isomorphic Γ and W have no symmetry, for group D_4 has 3-lateral symmetry, for other groups A_n, D_n, and E_6 have 2-lateral symmetries (see [1, pp. 70 and 72]).

Weyl groups of simple Lie groups A_n, B_n, C_n, D_n, G_2, and F_4 are isomorphic to the groups of symmetries of n-dimensional regular simplex, and the n-dimensional cube and cocube, and n-dimensional semicube, regular plane hexagon, and 4-dimensional bicube, respectively (see [1, p. 71]).

É. Cartan [3, pp. 35–43 and 43–50] has established that the Galois group of simple Lie group E_6 is isomorphic to the group of symmetries of a smooth cubic

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surface S_3 in projective space P^3 with 27 rectilinear generators and that the Galois group of simple Lie group E_7 is isomorphic to the group of symmetries of quartic curve C_4 in plane P^2 with 28 double tangents (see [1, p. 72]).

The equation of surface S_3 can be reduced to the form $ace = bdf$, where a, b, c, d, e, f are linear polynomials. If we denote the intersection of two planes $a = 0$ and $b = 0$ by ab, then 9 of these 27 rectilinear generators are lines: ab, ad, af, cb, cd, cf, eb, ed, ef. If we denote the 3 rectilinear generators which meet the lines ab, cd, ef by $(ab, cd, ef)_i$, $i = 1, 2, 3$, etc., then the other 18 rectilinear generators are $(ab, cd, ef)_i$, $(ad, cf, eb)_i$, $(af, cb, ed)_i$, $(ab, cf, ed)_i$, $(ad, cb, ef)_i$, and $(af, cd, eb)_i$.

The Galois group of the simple Lie group E_7 also admits the 3-dimensional geometric interpretation as the group of symmetries of a quartic surface S_4 in space P^3 with 16 rectilinear generators. The equation of this surface can be reduced to the form $aceg = bdfh$. The 16 rectilinear generators of this surface are straight lines ab, ad, af, ah, bc, be, bg, cd, cf, ch, de, dg, ef, eh, fg, gh which are lines of intersection of two tetrahedra bounded by planes $a = 0$, $c = 0$, $e = 0$, $g = 0$ and $b = 0$, $d = 0$, $f = 0$, $h = 0$. The configuration of 28 double tangents of curve C_4 can be obtained from the configuration of 16 rectilinear generators of surface S_4 and of 12 lines ac, ae, ag, bd, bf, bh, ce, cg, df, dh, eg, fh being edges of mentioned tetrahedra by the projection from a point onto a plane.

É. Cartan [4, pp. 45 and 43) has found that the numbers of elements of these groups are $9!/7$ and $12!/11.5.3.2$, respectively. These groups are subgroups of groups of permutations of 9 rectilinear generators of surface S_3 and of 12 edges of mentioned tetrahedra.

2. Algebraic variety V^2 and Weyl group of simple Lie group E_8

Let us consider 2-dimensional algebraic variety V^2 in projective space $P4$ being the intersection of two smooth cubic hypersurfaces whose equations can be reduced to the form $adg = beh = cfi$, where $a, b, c, d, e, f, g, h, i$ are linear polynomials. If we denote the intersection of three hyperplanes $a = 0$, $b = 0$, and $c = 0$ by abc, then 27 rectilinear generators of this variety are lines: abc, abf, abi, aec, aef, aei, ahc, ahf, ahi, dbc, dbf, dbi, dec, def, dei, dhc, dhf, dhi, gbc, gbf, gbi, gec, gef, gei, ghc, ghf, ghi.

Since the dimension d of the intersection of i-dimensional and j-dimensional planes in a projective space and the dimension s of the plane generated by these two planes are connected by the equality

$$i + j = d + s \tag{1}$$

(see [1, p. 8]), for every given 3 skew straight lines in 4-dimensional space P^4 there are 3 trisecants, that is 3 straight lines meeting all given 3 lines, because two from given 3 lines generate a 3-dimensional plane which according to equality (1) meets the third given line at one point, and through this point one of 3 trisecants passes. If we denote the 3 rectilinear trisecants which meet the lines abc, def, and ghi by

$(abc, def, ghi)_i$, $i = 1, 2, 3$, etc., then there are following 108 trisecants:

$$
\begin{array}{llll}
(abc, def, ghi)_i, & (abc, gef, dhi)_i, & (abc, dhf, gei)_i, & (abc, dei, ghf)_i, \\
(abf, dec, ghi)_i, & (abf, gec, dhi)_i, & (abf, dhc, gei)_i, & (abf, dei, ghc)_i, \\
(abi, def, ghc)_i, & (abi, gef, dhc)_i, & (abi, dhf, gec)_i, & (abi, dec, ghf)_i, \\
(aec, dbf, ghi)_i, & (aec, gbf, dhi)_i, & (aec, dhf, gbi)_i, & (aec, dbi, ghf)_i, \\
(aef, dbc, ghi)_i, & (aef, gbc, dhi)_i, & (aef, dhc, gbi)_i, & (aef, dbi, ghc)_i, \\
(aei, dbf, ghc)_i, & (aei, gbf, dhc)_i, & (aei, dhf, gbc)_i, & (aei, dbc, ghf)_i, \\
(ahc, def, gbi)_i, & (ahc, gef, dbi)_i, & (ahc, dbf, gei)_i, & (ahc, dei, gbf)_i, \\
(ahf, dec, gbi)_i, & (ahf, gec, dbi)_i, & (ahf, dbc, gei)_i, & (ahf, dei, gbc)_i, \\
(ahi, def, gbc)_i, & (ahi, gef, dbc)_i, & (ahi, dbf, gec)_i, & (ahi, dec, gbf)_i.
\end{array}
$$

Thus on variety V^2 there are 27 rectilinear generators and 108 rectilinear trisecants.

The surface S_3 can be obtained from the variety V^2 by the projection from a point onto a hyperplane, this projection maps rectilinear generators and trisecants on variety V^2 to rectilinear generators on surface S_3.

The method used by É. Cartan in [3] allows to establish that Weyl group of simple Lie group E_8 is isomorphic to the group of symmetries of variety V^2. This group is isomorphic also to the group of symmetries of polyhedron defined by H. S. M. Coxeter [5]. This polyhedron has 240 vertices, 16 of these vertices are represented by 8 octonion basis elements and their products by -1, $224 = 14 \cdot 16$ other vertices are represented by linear combinations of these elements (see [1, pp. 71–71]).

Coxeter [5] has established that the number of elements of this group is equal to $16!/13 \cdot 11 \cdot 7 \cdot 5 \cdot 3 \cdot 2$. This group is a subgroup of group of permutations of 16 vertices of mentioned polyhedron.

3. Geometric interpretations of compact simple Lie groups E_6, E_7, and E_8

In book [1, pp. 340–350] following geometric interpretations of compact simple Lie groups E_6, E_7, and E_8 are presented: they are groups of motions in Hermitian elliptic planes over tensor products of alternative division algebra O of octonions by division algebras C of complex numbers, H of quaternions, and O, respectively. The coordinates x^0, x^1, x^2 of points in these planes are triples of elements in mentioned tensor products belonging to one associative subalgebra of these tensor products determined by two arbitrary elements of these algebras and defined up to right multiplier which is not a zero divisor from the same subalgebra. Straight lines in these planes are determined by equations

$$u_0x^0 + u_1x^1 + u_2x^2 = 0, \tag{2}$$

where coefficients u_i are triples of elements in mentioned tensor products belonging to one associative subalgebra of these tensor products determined by two arbitrary elements of these algebras and defined up to left multiplier whic is not a zero divisor from the same subalgebra.

The transformations of coordinates x^1 under motions in mentioned planes are not linear representations, the infinitesimal transformations in Lie algebras of these groups were found by Vinberg [6] in 1964 (see [1, pp. 79–80]). In book [1, pp. 342 and 346] was proved that straight lines in mentioned planes admit interpretations as the manifolds of straight lines in real elliptic space S^9, of 3-dimensional planes in S^{11}, and of 7-dimensional planes in S^{15}, respectively.

4. Connection between variety V^2 and Hermitian elliptic plane over the tensor product of two algebras O

Vinberg [7] has proved in 2003 that two straight lines in the plane over the tensor product of two algebras O meet at 135 points. This result can be also obtained as follows: if equations of two straight lines are $x^0 = 0$ and $x^1 = 0$, the coordinates of common points of these lines are 0, 0, x^2. If x^2 is not a zero divisor, it can be divided by itself and reduced to 1. This is impossible, if x^2 is a zero divisor, and in this case there are many common points of the lines with different coordinates x^2.

From paper [7] following results follow: in a straight line over tensor product of two algebras O the finite group of transformations consisting of $256 = 2^8$ elements and being a subgroup of Weyl group of simple Lie group E_8 corresponding to transformations of 16 vertices of Coxeter's polyhedron represented by octonion basis elements and their products by -1 can be determined. This group contains identity E, 135 involutive elements which are reflections in common points of considered line with another line, and 120 other involutive elements. If points of considered straight line are represented by 7-dimensional planes in space S^{15}, elements of mentioned group are represented by reflections in 7-dimensional planes and their polars, 120 other involutive elements of this group are represented by symmetries in paratactic congruences of straight lines in space S^{15} [1, p 269].

Equations (2) imply that coordinates x^i of all common points of two lines in mentioned planes belong to one associative subalgebra of the tensor product of two algebras O.

The coincidence of the number 135 of common points of two straight lines in Hermitian elliptic plane over the tensor product of two algebras O with the number $27+108$ of rectilinear generators and trisecants on variety V^2 shows that it is possible to establish the connection between variety V^2 and mentioned Hermitian elliptic plane, and that the groups of symmetries of the configuration of 135 common points of two straight lines in mentioned plane and of variety V^2 are isomorphic. Since maximal associative subalgebra in algebra O is isomorphic to algebra H of quaternions, the maximal associative subalgebra of the tensor product of two algebras O is isomorphic to the tensor product of two algebras H which is isomorphic to the algebra of real (4×4)-matrices and is 16-dimensional. Therefore straight lines over this tensor product are also 16-dimensional.

The 7-dimensional planes in space S^{15} can be determined by affine matrix coordinates (see [1, p. 133]) which are real (8×8)-matrices. These matrices determining 7-dimensional planes representing 135 common points with coordinates 0, 0, x^2 consist of four (4×4)-submatrices: the upper left submatrix is the unit

(4×4)-matrix, the upper right and lower left submatrices are zero (4×4)- matrices, the lower right submatrix is real (4×4)-matrix representing element x^2 of tensor product of two algebras H.

Dimension N of manifold of m-dimensional planes in n-dimensional projective space P^n is equal to

$$N = (m+1)(n-m) \tag{3}$$

(see [1, p. 134]). According to equality (3), 135 7-dimensional planes in space S^{15} representing 135 common points of a straight line in Hermitian elliptic plane over the tensor product of two algebras O with another straight line in this plane lie in 9-dimensional plane in space S^{15}, because $16 = (7+1)(9-7)$. If this 9-dimensional plane is regarded as space S^9, for every 7-dimensional plane in this space there is a polar straight line.

According to equality (1), the intersection of every two 7-dimensional planes in space S^9 is 5-dimensional. Since common points of two straight lines in a Hermitian elliptic plane over the tensor product of two algebras O lie on two straight lines, every two of these points are adjacent (see [1, p. 118]). Therefore all 7-dimensional planes in space S^9 representing 135 common points of two straight lines in mentioned plane pass through the same 4-dimensional plane corresponding to right (4×4)-submatrices of the affine matrix coordinates of 7-dimensional planes representing common points of two straight lines.

The polar straight lines of 135 7-dimensional planes lie in the 4-dimensional plane polar to mentioned 4-dimensional plane. Thus we obtain in this 4-dimensional plane the configuration of 135 straight lines.

Since the group of symmetries of this configuration is isomorphic to the group of symmetries of variety V^2, the lines of this configuration are rectilinear generators and trisecants of variety V^2.

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