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THE COMPUTATION OF CAPACITY OF PLANAR CONDENSERS

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ABSTRACT. We describe a new adaptive finite element method for the numerical computation of the capacity of planar (possibly multiply-connected) condensers. We compare this method with other numerical methods and we give several examples that illustrate its effectiveness.

1. Introduction

The capacity of condensers has been studied because of its physical importance and its close relation with the theory of conformal and quasiconformal mapping. The analytic computation of capacity is possible only for very few types of condensers and for this reason several methods have been developed for the numerical computation of capacity. In this paper we give a brief description of such a method, an adaptive finite element method (AFEM in brief), which was recently devised by one of the authors (Samuelsson). We then give examples of computations of several condenser capacities and we compare the results with those obtained by other methods.

It turns out that AFEM has many advantages: it gives results both for bounded and unbounded condensers, both for ring domains (doubly-connected) and for multiply-connected condensers, both for polygonal and curved boundaries. The capacity of a great variaty of condensers can be computed by one and the same computer program based on AFEM. The results are very accurate and the programs run reasonably fast.

In Section 2 we recall the main properties of the capacity of planar condensers. Section 3 contains a brief review of numerical methods applied to the computation of capacities. A description of AFEM and of the corresponding software appears in Section 4. Finally, in Section 5, we consider several examples of condensers and compute their capacities. Some of these capacities have been computed earlier and

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the results agree with those given by AFEM. We also compute capacities of some condensers not previously investigated in the literature as far as we know.

2. Definition and main properties of capacity

Let E and F be two disjoint compact sets in the extended complex plane $\mathbb{C}_\infty$. We assume that each of E and F is the union of a finite number of nondegenerate disjoint continua, and that the open set $R = \mathbb{C}_\infty \setminus (E \cup F)$ is connected. Without any loss of generality, we also assume that $\infty \notin E$. The domain R is a *condenser*. The complementary compact sets E and F are the *plates* of the condenser. The *capacity* of R is defined by

$$\operatorname{cap} R = \inf_u \iint_R |\nabla u|^2 \, dx \, dy, \tag{2.1}$$

where the infimum is taken over all nonnegative, piecewise differentiable functions u with compact support in $R \cup E$ such that $u = 1$ on E. It is well-known that under the assumptions we made above, R is regular for the Dirichlet problem and the harmonic function on R with boundary values 1 on E and 0 on F is the unique function that minimizes the integral in (2.1). This function is called the *potential function* of the condenser.

Capacity is a conformal invariant: Suppose that f maps R conformally onto R'. Let E and F correspond to E' and F' respectively (in the sense of the boundary correspondence under conformal mapping). Then $\operatorname{cap} R = \operatorname{cap} R'$. This property can be used for the analytic computation of capacity provided that the capacity of some 'canonical' condensers is known and the corresponding conformal mappings can be constructed. Unfortunately such an analytic computation can be made only for very few doubly-connected condensers; see [**IvTr**]. If E is fixed and F is a circle centered at 0 with radius $t > 0$, then $\operatorname{cap} R$ depends analytically on t [**EM**].

If both E and F are connected (and hence R is doubly-connected), R is called a *ring domain*. A ring domain R can be mapped conformally onto the annulus $\{z : 1 < |z| < e^M\}$, where $M = \operatorname{Mod} R$ is the *conformal modulus* of the ring domain R, defined by $\operatorname{Mod} R = 2\pi / \operatorname{cap} R$. There are several equivalent definitions of the capacity of ring domains in terms of Green's function, extremal length, or various 'energy integrals'. The reader is referred to [**Tsu**, pp. 94–100], [**Ahl**, pp. 65–70], [**Bag**], [**Wei1**], [**Gai1**], [**BaFl**] for more details and further references. Only some of these definitions provide equivalent characterizations of the capacity of general condensers. The theory of capacity extends in various (not equivalent) ways to higher dimensions but we will not consider such extensions in this paper.

3. Review of some numerical methods

The paper [**Gai2**] of Gaier includes a review of the various methods applied to the computation of the capacity of planar ring domains. We are not aware of any condenser of connectivity $n > 2$ (with nondegenerate boundary components), whose capacity has been *analytically or numerically* computed. The book [**IvTr**] contains several examples of conformal mappings of doubly-connected regions and a great number of references. For the numerical conformal mapping of multiply-connected domains we refer to [**Gai1**, ch. 5] and [**May**]. In Section 5 we present several examples of multiply-connected condensers whose capacities are computed by AFEM.

The finite element method was first applied to the computation of capacity by Opfer [**Opf**]. Several numerical experiments are reported by Weisel [**Wei2**]. Another numerical method is based on the Gauss–Thompson principle which implies a formula for the capacity involving Green's function. Numerical computations are given in [**Wei1**]. In the work of Bagby [**Bag**] the capacity of a condenser is proven to be equal to its *discrete module*, a quantity that generalizes the transfinite diameter of compact sets. Numerical experiments based on discrete module appear in [**Men**]. Papamichael and his collaborators [**PaKo**], [**PPSS**], [**PaWa**] have developed an orthonormalization technique for the approximation of the conformal mapping of doubly-connected domains. This technique gives, in particular, approximations of capacity. A great number of numerical computations is reported in the above papers. The above numerical methods have some limitations in their applicability, limitations related to the geometry of the ring domains considered.

In addition to the orthonormalization method mentioned above, some other methods have been developed recently for the numerical computation of conformal mapping. The paper [**Del**] contains a survey and comparison of these methods (mainly for simply-connected domains). We also refer to [**DePf**] which contains results for doubly-connected domains and references to earlier works of B. Fornberg and R. Wegmann. An overview of the development of numerical conformal mapping during the past fifty years can be obtained from the following books and proceedings of conferences: [**Bec**], [**Gai1**], [**Tre**], [**PaSa**], [**PRS**].

The capacity of a polygonal ring domain can be also computed by the Schwarz–Christoffel transformation which provides a semi-explicit formula for the conformal mapping of the domain to an annulus (see [**Hen**]). This formula contains unknown parameters; the problem of their numerical computation is the *parameter problem* for the Schwarz–Christoffel transformation. For simply-connected domains this subject has been brought to a very satisfactory form by Trefethen and others, (see [**Hen**], [**TrDr**]). It seems that for doubly-connected polygonal domains the only related works are those of Daeppen [**Dae**] and Hu [**Hu**]. Hu's method has been tested successfully in several computations, (see [**Hu**], [**BeVu**]). It is partially based on the wise choice of certain points on the complementary sets E, F of the ring domain.

Another numerical-analytic method that can be used for the computation of capacity is the multipole method. The potential function is written as a linear combination of explicit basic functions (multipoles) with unknown coefficients. The coefficients are then computed numerically. The multipoles constitute a complete, minimal system in a certain Hardy-type space of functions. Their construction is based on the theory of conformal mapping. This method has been developed by Vlasov (see [**Vla**] and references therein) as a general method for numerical solution of a wide class of boundary value problems. He has applied this approach to find the potential function of condensers. This method has been tested in some of the examples of the present paper and in those cases the results agree with the results reported here.

4. Description of AFEM

Let R be a condenser with plates E and F. We assume first that R is bounded and that each of E and F is the union of rectilinear segments. So R is a bounded polygonal condenser; (we will later comment on infinite and curved boundaries).

The adaptive algorithm. The algorithm of AFEM can be described as follows:
Step 1: Generate an initial triangulation of R for the application of the finite element method.
Step 2: Using the finite element method approximate the potential function u of R by a function which is equal to a linear or quadratic polynomial on each triangle of the triangulation.
Step 3: Use *a posteriori* inequalities to estimate the error of the approximation.
Step 4: Check the termination criteria. If they are met then stop. Otherwise create a new triangulation by refining the triangles which give large contributions in the error estimate and go to Step 2.

The initial triangulation of Step 1 as well as its later refinements must have certain properties so that the *a posteriori* estimates work well. For example there must be a lower bound for the smallest angle of the triangulation; in many examples this bound is taken to be around 25°. The algorithm that constructs the successive refinements of the triangulation has the property that the minimal angle of the refined triangulations has a lower bound depending on the angles of the triangles in the initial triangulation. This property is known as stability of the refinement algorithm.

Regarding Step 4, the procedure is terminated when the error in the approximation is small enough. One may add some other termination criteria related to the number of triangles and the number of refinements to limit the amount of memory usage.

The software. Samuelsson's software consists of three parts. At first, a short program generates the boundaries of the condenser. The user inserts the coordinates of the vertices of R, indicates which vertices are joined with a segment and provides a code for the boundary condition (Dirichlet or Neumann) at each such segment.

The second part of the software is used for the generation of the initial grid. Here `triangle` [**Sch**] is used. This is a grid generation program for two dimensions.

The third part contains the main program, an adaptive finite element solver. The linear system of equations associated with the finite elements is solved by a multigrid algorithm. This program may handle much more general problems, but for efficiency a specialized version for the computation of condenser capacities has been made.

The output of the program includes pictures of the condenser, of the initial triangulation, of the refined triangulations, and of contour plots of the potential function. It also contains the approximating value of the capacity and error estimates for the capacity.

Unbounded condensers. If R is an unbounded condenser, by applying a suitable Möbius transformation if necessary, we may assume that $\infty \in F$. We then truncate R by a large square $[-S,S]^2 = [-S,S] \times [-S,S]$ and use AFEM as in the bounded case. The boundary value at the boundary of the square is set to be of Dirichlet type with value 0. Because of this approximation an additional error term appears which decreases as the size of the box increases. If $\infty \notin F$ it is also possible to do computations without applying a Möbius transformation. In this case homogeneous Neumann boundary conditions on the boundary of the large square are used.

Curved boundaries. In the case of curved boundaries AFEM uses curved elements. The equation of the curved boundary is inserted in a parametric form in the first part of the software. In the refined triangulations the shape of the curved boundaries is respected.

Symmetries. The program runs faster if one exploits the possible symmetries of the condenser. This can be done easily: homogeneous Neumann boundary conditions are posed on the lines of symmetry.

Computational time. The computational time is 1–3 minutes to get 6–8 accurate digits of the capacity. The numerical experiments have been performed with Sun Ultra10 workstation and a PC with PentiumII, 350 Mhz, with the workstation being slightly faster. In the computations, approximately 200,000 nodal points (with quadratic polynomials in the finite element method) have been used. The required amount of memory for problems of this size is approximately 100 Mbytes.

5. Numerical examples

In the numerical examples of this section we present the error bound of the computed capacity as given by an *a posteriori* error estimate implemented in the solver. In the tables in this section the absolute error bound is presented in the column `Error`. The exact value of the capacity then lies in the interval [cap $-$ error, cap $+$ error], where cap and error are the presented approximate value of the capacity and the presented error bound, respectively.

5.1. Ring domains treated by other methods. In this subsection we present five examples of ring domains whose capacities have been computed numerically by various methods.

Example 1. *Square in square.* We compute here the capacity of the ring domain with plates $E = [-a,a] \times [-a,a]$ and $F = \mathbb{C}_\infty \setminus ((-1,1) \times (-1,1))$, $0 < a < 1$.

Let $c(a)$ denote the capacity of the condenser with plates E and F. In the second column of Tables 1–5 we list the values of e^M where M is the conformal

TABLE 1. Table for Example 1

a	exp(Mod)	cap	Error	Exact value
0.1	9.139106	2.8397774	5e-8	2.8397774191
0.2	4.5708597	4.1344870	5e-8	4.1344870242
0.5	1.8477090	10.2340926	9e-8	10.2340925694
0.7	1.35067994	20.9015817	2e-7	20.9015816794
0.8	1.20145281	34.2349152	2e-7	34.2349151988
0.9	1.088324350	74.2349152	2e-7	74.2349151988
0.99	1.00794236562	794.2349152	8e-8	794.2349151988
0.999	1.000786273508517	7994.234915199	2e-9	7994.2349151988
0.9999	1.000078548561383	79994.234915202	3e-9	79994.2349151988

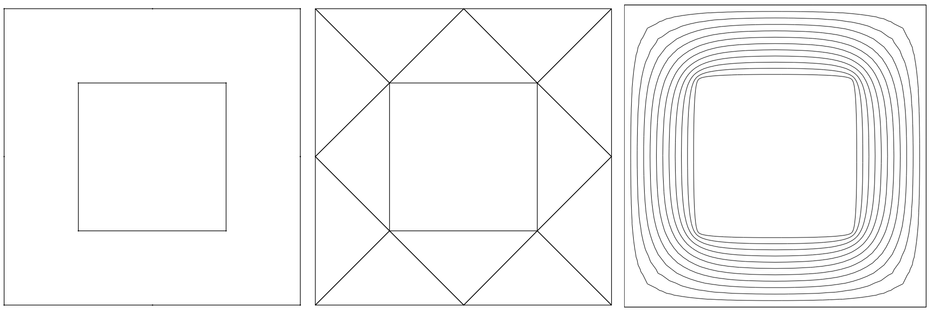

FIGURE 1. The condenser, initial triangulation and level lines of the potential function of Example 1 ($a = 0.5$).

modulus of the corresponding ring domain; the quantity e^M is often used in the literature instead of the capacity, see e.g. [**PaKo**] or [**PaWa**]. We used *Mathematica* to compute the exact values of $c(a)$ in the fifth column of Table 1; the computation was based on the following formulae due to Bowman [**Bow**, pp. 99–104]:

$$c(a) = \frac{1}{\mu(k)}, \quad k = \Big(\frac{l-l'}{l+l'}\Big)^2, \quad l' = \sqrt{1-l^2}, \quad l = \mu^{-1}\Big(\frac{2}{\pi}\,\frac{a-1}{a+1}\Big),$$

where

$$\mu(r) = \frac{\pi}{2}\frac{\mathcal{K}(\sqrt{1-r^2})}{\mathcal{K}(r)}, \quad \mathcal{K}(r) = \int_0^1 \frac{dx}{(1-x^2)^{1/2}(1-r^2x^2)^{1/2}}, \quad r \in (0,1),$$

is the complete elliptic integral. The computation of the function μ^{-1} by *Mathematica* is based in the following formula (see [**Akh**, p. 77]).

$$\mu^{-1}(y) = \lambda(\tau) = \frac{\vartheta_2^2(0\,|\,\tau)}{\vartheta_3^2(0\,|\,\tau)},$$

where λ is the elliptic modular function, ϑ_2, ϑ_3 are the elliptic Theta functions and $\tau = 2iy/\pi$, $y > 0$. Note that the complete elliptic integrals and the elliptic Theta functions are built in *Mathematica*. An equivalent formula is

$$c(a)=\frac{4\pi}{\mu\left(\dfrac{r-s}{r+s}\right)};\quad r=\mu^{-1}\Big(\frac{\pi(1-t)}{2(1+t)}\Big),\quad s=\mu^{-1}\Big(\frac{\pi(1+t)}{2(1-t)}\Big).$$

This example has been considered in [**Wei1**] for $a=0.5$. The computed value of e^M in [**Wei1**] is 1.847698. In [**PaKo**] the authors compute $c(a)$ for $a=0.2,\,0.5,\,0.8$. Their values agree with the exact values in 9 or 10 decimal places.

The computation of $c(a)$ for a close to 1. When $a\gtrsim 0.9999$, AFEM cannot be applied directly because the number of triangles is very large already in the original triangulation; recall that there is a lower bound ($\approx 25^o$) for the minimal angle of the triangles. To compute $c(a)$ in this case, we apply the domain decomposition method which was studied by Papamichael-Stylianopoulos and Gaier–Hayman; see [**Gai2**], [**PaSt**] and references therein. The application of this method for Example 1 is described as follows:

First by symmetry, we decompose the condenser into eight congruent *quadrilaterals.* Let $G=[0,a]\times[a,1]\cup\{z=x+iy: a\leqslant x\leqslant 1,\ x\leqslant y\leqslant 1\}$ be one of them. The *module* $m(G)$ of G is defined by

$$m(G)=\iint_G|\nabla u|^2,$$

where u is the harmonic function in G with mixed boundary conditions $u=1$ on $\{x+ia: 0<x<a\}$, $u=0$ on $\{x+i: 0<x<1\}$, and $\partial u/\partial n=0$ (normal derivative) on the rest of the boundary. Then by symmetry

$$c(a)=8m(G). \tag{5.1}$$

To compute $m(G)$, we decompose G into two quadrilaterals G_1, G_2:

$$G_1=[0,1-k(1-a)]\times[a,1],\quad G_2=G\smallsetminus G_1,$$

where $k<1/(1-a)$ is a positive integer to be determined below and with $\partial u/\partial n=0$ at $G_1\cap G_2$. The quadrilateral G_1 is a rectangle and hence its module is

$$m(G_1)=\frac{1}{1-a}-k. \tag{5.2}$$

The module of G_2 can be computed by AFEM, provided that k is small enough so that AFEM does not use too many triangles.

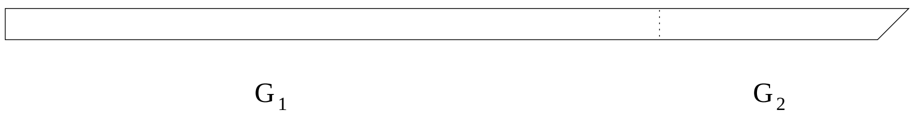

FIGURE 2. The decomposition $G=G_1\cup G_2$ of the quadrilateral G.

Now the decomposition method [**PaSt**, Theorem 2.3 and Remark 2.4] (which cites [**GaHa**, Theorem 5]) asserts that

$$0\leqslant m(G)-[m(G_1)+m(G_2)]\leqslant\frac{0.381}{2}\,e^{-2\pi(k-1)}, \tag{5.3}$$

when

$$k \geqslant 2. \tag{5.4}$$

From (5.3), (5.4) we see that we can choose, for example $k = 10$. With this choice AFEM works, the approximation (5.3) is very good, and (5.4) holds.

Finally, by (5.1), (5.2) and (5.3), we have the approximation

$$0 \leqslant c(a) - \Big[m(G_2) + 8\Big(\frac{1}{1-a} - 10\Big)\Big] \leqslant 5 \cdot 10^{-25}, \tag{5.5}$$

for $k = 10$ and $a > 0.9$.

The error in this approximation comes essentially only from the approximation of $m(G_2)$ by AFEM, because (5.3) shows that the approximation $m(G) \approx m(G_1) + m(G_2)$ is excellent.

The above discussion explains the impressive numerical coincidence in the last four rows of Table 1: Let a take values close to 1 ($a > 0.9$), and let $k = 10$. By the invariance of modules under scaling, $m(G_2)$ does not depend on a. Moreover, by (5.2), $m(G_1)$ is an integer when a takes the values $0.9, 0.99, \ldots$. Hence $c(a)$ is approximately the sum of an integer (that depends on a) and a constant (independent of a).

TABLE 2. Table for Example 2

a	exp(Mod)	cap	Error
0.1	8.4721312	2.9404895	1e-7
0.3	2.823966	6.0523354	6e-8
0.5	1.6915649	11.9530801	1e-7
0.7	1.1393578	48.1600237	1e-6

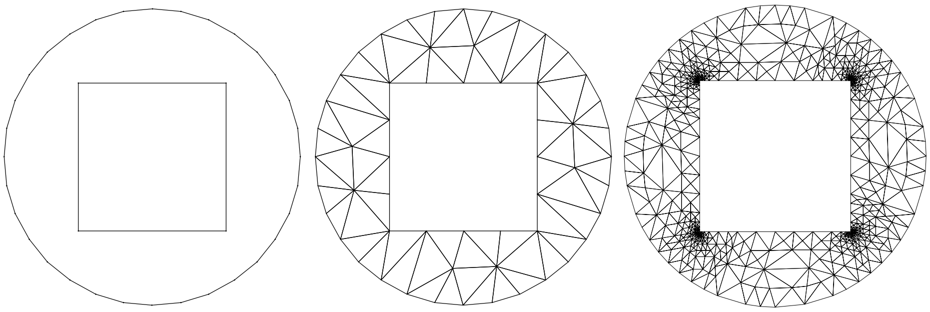

FIGURE 3. The condenser, the initial and an adaptively refined triangulation of Example 2 ($a = 0.5$).

EXAMPLE 2. *Square in disk.* In this Example we consider the ring domain with plates $E = [-a, a] \times [-a, a]$ and $F = \{z : |z| \geqslant 1\}$, $0 < a < 1/\sqrt{2}$.

This example has been considered in [**Men**], [**Wei1**] for $a = 0.5$. Their values of e^M are 1.672934 and 1.69203, respectively.

TABLE 3. Table for Example 3

a	exp(Mod)	[**PaKo**]	cap	Error
0.1	10.7876523		2.64182917	7e-8
0.2	5.3935358	5.39353525710616	3.72845236	6e-8
0.4	2.6967244	2.69672443123	6.33361439	4e-8
0.5	2.1572262		8.17247082	5e-8
0.8	1.3429904	1.3429903655992	21.30624642	3e-8
0.9	1.184091		37.18403286	5e-8
0.99	1.0404121		158.5985299	2e-7

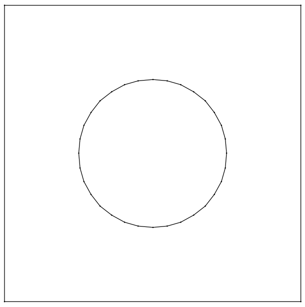

FIGURE 4. The condenser of Example 3 ($a = 0.5$).

EXAMPLE 3. *Disk in square.* Here $E = \{z : |z| \geqslant a\}$ and $F = \mathbb{C}_\infty \setminus ((-1,1) \times (-1,1))$, $0 < a < 1$.

This example has been considered in [**Wei1**], [**PaKo**]. The values of e^M in the third column come from [**PaKo**].

EXAMPLE 4. *Cross in square.* Let $G_{ab} = \{(x,y) : |x| \leqslant a,\ |y| \leqslant b\} \cup \{(x,y) : |x| \leqslant b,\ |y| \leqslant a\}$ and $G_c = \{(x,y) : |x| < c,\ |y| < c\}$, where $a < c$ and $b < c$. We compute the capacity of the ring domain $R = G_c \setminus G_{ab}$.

TABLE 4. Table for Example 4

a	b	c	exp(Mod)	[**PaKo**]	cap	Error
0.5	1.2	1.5	1.3314734	1.331473449	21.9472192	9e-7
0.5	1.0	1.5	1.5662892	1.566289179	14.0027989	4e-7
0.2	0.7	1.2	1.981644	1.9816441	9.1869265	3e-7
0.1	0.8	1.1	1.747487	1.7474925	11.2565821	7e-7
0.5	0.6	1.5	2.3583812		7.3232695	2e-7
0.1	1.2	1.3	1.311995		23.1386139	2e-6

The values of e^M in the fifth column come from [**PaKo**].

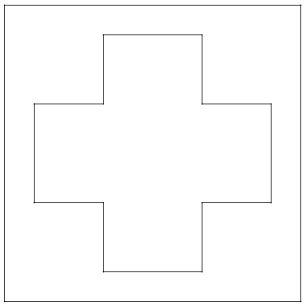

FIGURE 5. The condenser of Example 4 ($a = 0.5, b = 1.2, c = 1.5$).

EXAMPLE 5. *Disk in regular pentagon.* Let F be the unbounded complementary component of a regular pentagon centred at the origin and having short radius (apothem) equal to unity. Let $E = \{z : |z| \leqslant a\}$. We compute the capacity of the ring domain with plates E and F.

TABLE 5. Table for Example 5

a	exp(Mod)	[**PaWa**]	cap	Error
0.1	10.5246525		2.669469753	2e-9
0.4	2.631159439		6.494754531	2e-9
0.9	1.1626499972	1.162649997	41.692813032	2e-9
0.99	1.03331141431	1.033311414	191.74402525	2e-8
0.999	1.00939037579	1.009390376	672.2457359	2e-7

The values of e^M in the third column come from [**PaWa**].

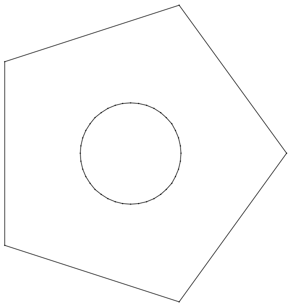

FIGURE 6. The condenser of Example 5 ($a = 0.4$).

5.2. Multiply-connected condensers. Here we present two examples of triply-connected condensers. The next subsection contains some more such examples.

TABLE 6. Table for Example 6

Case A

a	b	c	cap	Error
-0.9	0	2	1.7086693	6e-7
-0.5	0.5	2	2.0953263	7e-7
-0.9	0.9	2	3.0676361	9e-7
0	0.9	2	3.0332745	1e-6
-0.5	0.5	3	2.4125750	8e-7
-0.7	0.2	3	2.1318391	6e-7
0.5	0.8	3	2.8071236	1e-6

Case B

a	b	c	cap	Error
-0.9	0	2	3.4537720	1e-6
-0.5	0.5	2	2.9410234	9e-7
-0.9	0.9	2	5.1877511	3e-6
0	0.9	2	3.4537719	2e-6
-0.5	0.5	3	3.0486876	1e-6
-0.7	0.2	3	3.0172100	8e-7
0.5	0.8	3	2.3121085	6e-7

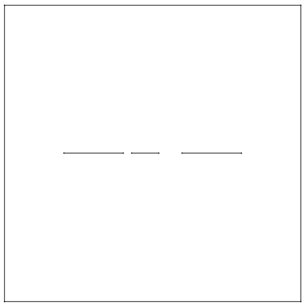

FIGURE 7. The condenser of Example 6.

EXAMPLE 6. *Three slits on a line.* Let $I_1 = [-c, -1]$, $I_2 = [a, b]$ and $I_3 = [1, c]$, where $-1 < a < b < 1 < c$. We consider two condensers bounded by $I_1 \cup I_2 \cup I_3$. In Case A the condenser has plates $E = I_1 \cup I_2$ and $F = I_3$. In Case B the condenser has plates $E = I_2$ and $F = I_1 \cup I_3$. The condenser of Example 6 is unbounded and ∞ is not on its plates. We truncate the condenser by a large square and use homogeneous Neumann boundary conditions on the boundary of the square. The size of the square is $S = 1e9$ except for the parameter values in Case B which give a symmetric solution (rows 2, 3 and 5 of Table 7). For these parameter values we use square of size $S = 1e5$.

EXAMPLE 7. *Square with two equilateral triangles removed.* In this example $F = \mathbb{C}_\infty \setminus ((-1,1) \times (-1,1))$ and E is the union of an equilateral triangle T and its reflection in the real axis. The vertices of T are the points $(0, a)$, $(b-a)/\sqrt{3}, b)$ and $(-(b-a)/\sqrt{3}, b)$, where $0 < a < b < 1$.

5.3. Examples related to polarization and symmetrization. Polarization and Symmetrization are geometric transformations that reduce the capacity of condensers. We illustrate this fact with the three following examples.

EXAMPLES 8,9,10. *Square with two slits.* Let $F = \mathbb{C}_\infty \setminus ((-1,1) \times (-1,1))$. Inside the square $[-1,1] \times [-1,1]$ we consider the points: $A = (-2/3, -1/4)$, $B =$

TABLE 7. Table for Example 7

a	b	cap	Error
0.1	0.3	3.9324143	2e-7
0.2	0.4	4.4119861	3e-7
0.2	0.7	9.4930811	4e-7
0.3	0.8	12.1180117	6e-7
0.3	0.9	21.6586487	9e-7

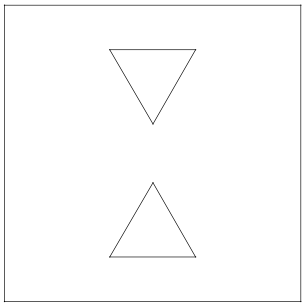

FIGURE 8. The condenser of Example 7 ($a = 0.2, b = 0.7$).

$(-2/3, 3/4)$, $C = (1/2, -1/3)$, $D = (1/2, 1/6)$, $\bar{C} = (1/2, 1/3)$, $\bar{D} = (1/2, -1/6)$, $A_s = (-2/3, -1/2)$, $B_s = (-2/3, 1/2)$, $C_s = (1/2, -1/4)$, $D_s = (1/2, 1/4)$.

TABLE 8. Table for Examples 8,9 and 10

Example	cap	Error
8	8.7576166	5e-7
9	8.7369062	7e-7
10	8.4701600	5e-7

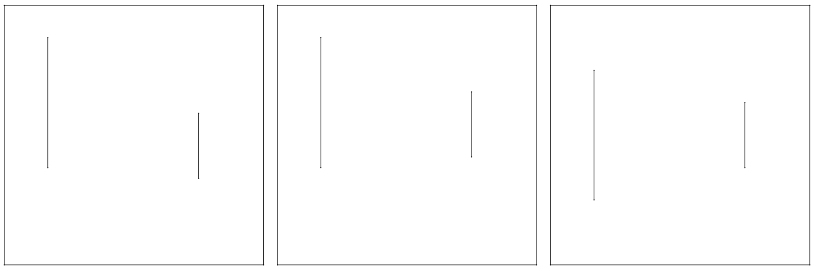

FIGURE 9. From left to right the condensers of Examples 8, 9 and 10.

In Example 8 the condenser has plates F and $E = AB \cup CD$. In Example 9 the condenser has plates F and $E = AB \cup \bar{D}\bar{C}$. In Example 10 the condenser has plates F and $E = A_sB_s \cup C_sD_s$.

Note that the condenser R_9 of Example 9 is the polarization (with respect to the real axis) of the condenser R_8 of Example 8. Also, the condenser R_{10} of Example 10 is the (Steiner) symmetrization of both R_8 and R_9. According to polarization and symmetrization theorems of Wolontis and Polya–Szegö (see [**Wol**]), we have

$$\operatorname{cap} R_{10} < \operatorname{cap} R_9 < \operatorname{cap} R_8.$$

5.4. Examples related to Teichmüller's modulus problem. Let $z \neq 1$ be a point in the plane with $\operatorname{Re} z \geqslant 1/2$ and $\operatorname{Im} z \geqslant 0$. O. Teichmüller posed the following extremal problem (see [**Kuz**, Ch. 5]): Find the minimal capacity $p(z)$ of all ring domains with complementary continua E, F such that $0, 1 \in E$ and $z, \infty \in F$. G. V. Kuz'mina expressed the function $p(z)$ in terms of elliptic integrals of complex argument:

$$p(z) = \frac{2}{\operatorname{Im}\left(i\frac{\mathcal{K}'(r)}{\mathcal{K}(r)}\right)}, \quad r^2 = \frac{1}{z}. \tag{5.6}$$

For the precise definition of the elliptic integrals $\mathcal{K}'(r)$ and $\mathcal{K}(r)$ for complex r we refer to [**Kuz**].

The ring domain $\mathbb{C} \setminus ([-1,0] \cup [t,\infty))$, $t > 0$ is called *Teichmüller's ring*. Its capacity is denoted by $\tau(t)$ and can be computed in terms of elliptic integrals. We refer to [**AVV**] for more information about Teichmüller's problem and ring.

TABLE 9. Table for Example 11

t	p(t)	cap	Error
0.5	4	4.000000	2e-6
0.6	4.0170835643	4.020605	2e-6
0.7	4.0734693962	4.088615	2e-6
0.8	4.1906531335	4.229908	2e-6
0.9	4.4486294884	4.540293	2e-6
0.99	5.5458897502	5.836042	3e-6

EXAMPLE 11. *Semi-circle and half-line.* In this example the condenser has plates $E = \{z : |z - 1/2| = 1/2, \ \operatorname{Im} z \leqslant 0\}$ and $F = \{t + iy : y \geqslant 0\}$, where $t \in [1/2, 1)$. For $t = 1/2$ we have *Mori's ring* which is extremal for Teichmüller's problem (see [**Kuz**]).

In the second column of Table 9 we have values of $p(t)$ computed using (5.6) and the software of [**AVV**].

EXAMPLE 12. *Segment and half-line.* Here the plates of the condenser are $E = [-1, 1]$, $F = \{iy : y \geqslant 1\}$.

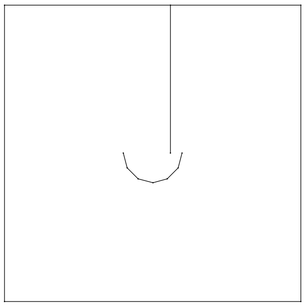

FIGURE 10. The condenser of Example 11 ($a = 0.8$).

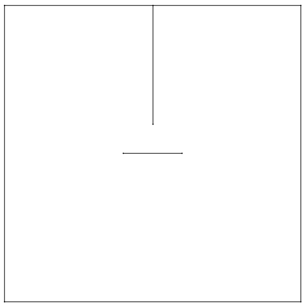

FIGURE 11. The condenser of Example 12.

This condenser is conformally equivalent to Teichmüller's ring and therefore its capacity can be computed in terms of elliptic integrals. Our computations led to the conjecture

$$\tau\left(\frac{1}{2+2\sqrt{2}}\right) = 2\sqrt{2}.$$

We communicated this conjecture to M. K. Vamanamurthy who quickly proved it analytically using identities for complete elliptic integrals. The computed value of the capacity is 2.8284269 with error $7e-7$. (So the real error to $2\sqrt{2}$ is $2.25e-7$.)

EXAMPLE 13. *Two semi-circles.* Let $0 < a < 1$ and $\theta \in [\pi/2, \pi]$. Let E be the semi-circle that lies in the upper half-plane and joins the points 1 and $ae^{i\theta}$. Let F be the semi-circle that lies in the lower half-plane and joins -1 with $ae^{i(\pi+\theta)}$. We compute the capacity of the ring domain with plates E and F.

The third column of Table 10 contains lower bounds of the capacity. These bounds are obtained as follows: We first apply the Möbius transformation T that maps -1 to 0, $-ae^{i\theta}$ to 1, and 1 to ∞. Let $z = T(ae^{i\theta})$. Then $p(z)$, by its very definition, is a lower bound for the capacity of the condenser with plates E and F. The numerical values of $p(z)$ in the third column are obtained by using (5.6) and the software of [**AVV**].

TABLE 10. Table for Example 13, computed with box size 1e9.

θ	a	LB	cap	Error
$\pi/2$	0.2	2.4360353	2.5977547	7e-7
$2\pi/3$	0.2	2.8334104	2.9342954	8e-7
$5\pi/6$	0.2	3.3531278	3.3915666	9e-7
π	0.2	4.0051910	4.0062608	2e-6
$\pi/2$	0.6	2.0529462	2.17444770	5e-7
$2\pi/3$	0.6	2.5835228	2.6711758	9e-7
$5\pi/6$	0.6	3.3729826	3.4193335	2e-6
π	0.6	4.6494327	4.7808008	2e-6

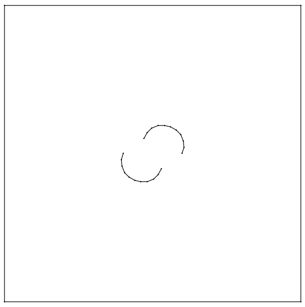

FIGURE 12. The condenser of Example 13 ($a = 0.6$, $\theta = 2\pi/3$).

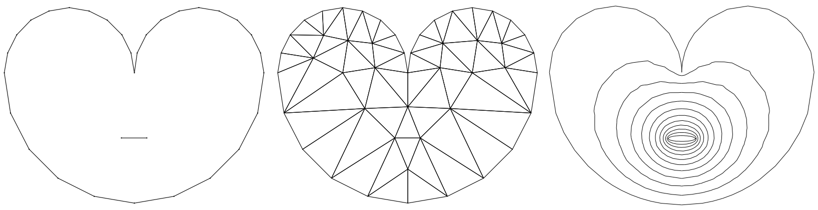

FIGURE 13. The condenser, inital triangulation and equi-potential lines of Example 14.

5.5. More examples. In this subsection we present some more computational results.

EXAMPLE 14. *A cardioid type domain with a slit.* Consider the disks: $D_1 = \{|z+0.5| < 0.5\}$, $D_2 = \{|z-0.5| < 0.5\}$ and the half-disk $D_3 = \{|z| < 1, \operatorname{Im} z > 0\}$. The cardioid-type domain is $C = D_1 \cup D_2 \cup D_3$. Let F be the complement of C and E be the slit from the point $-0.1 - i0.5$ to the point $0.1 - i0.5$. The computed value of the capacity of the ring domain with plates E, F is 2.4269776 with error bound $2e-7$.

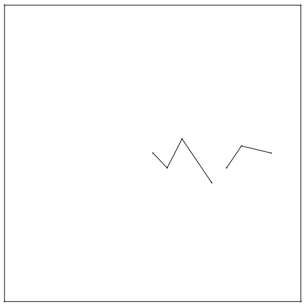

FIGURE 14. The condenser of Example 15.

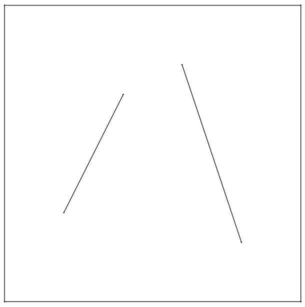

FIGURE 15. The condenser of Example 16.

EXAMPLE 15. *Irregular polygons.* Here we consider the ring domain whose plates are the polygonal lines $E = ABCD$, $F = XYZ$, where the vertices are: $A = (0,0)$, $B = (1,-1)$, $C = (2,1)$, $D = (4,-2)$, $X = (5,-1)$, $Y = (6,0.5)$, $Z = (8,0)$. The computed value of the capacity is 2.6134742 with error bound $6e-7$.

EXAMPLE 16. *Extremal distance.* AFEM can compute the *extremal distance* $\lambda(E,F,\Omega)$ [**Ahl**, ch. 4] between two compact sets E and F with respect to a closed domain Ω that contains E and F. Let $A = 0.2+i0.3$, $B = 0.4+i0.7$, $C = 0.8+i0.2$, $D = 0.6+i0.8$, $E = AB$, $F = CD$, and $\Omega = [0,1]\times[0,1]$. Homogeneous Neumann condition is used on the boundary of the unit square. The Dirichlet integral is 2.1988250 and its inverse is 0.45478835 with absolute error bound $8e-8$ for the Dirichlet integral.

EXAMPLE 17. *Bounded ring with polygonal boundary.* The boundary of the condenser consists of a closed polygonal line with vertices

$$\{(1.0,0.1),(0.4,1.0),(-0.5,0.5),(-0.6,-0.5),(0.6,-0.7)\}$$

and another one with vertices

$$\{(1.7,0.1),(1.0,1.5),(-0.5,1.3),(-1.6,0.6),(-1.4,-0.9),(-0.3,-1.7),(1.2,-1.4)\}.$$

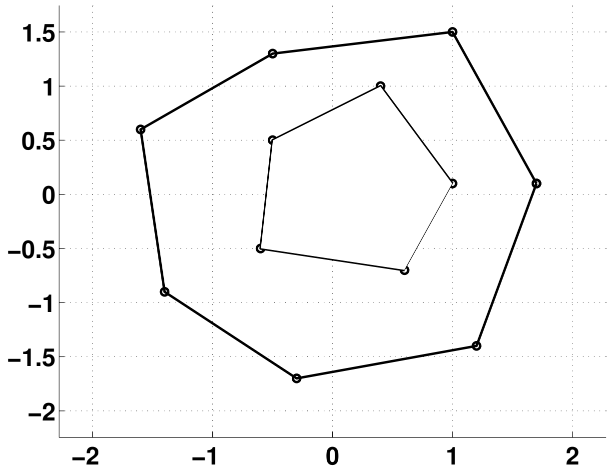

FIGURE 16. Condenser with polygonal boundaries.

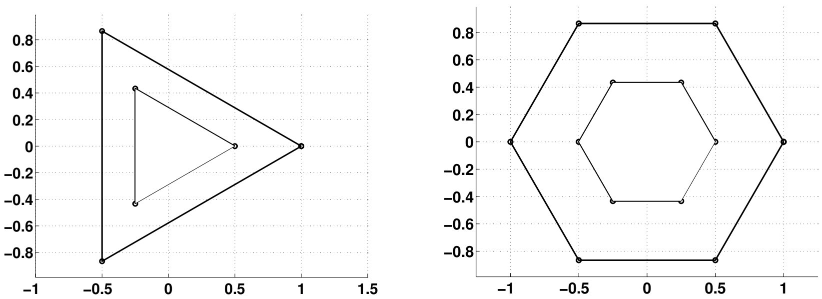

FIGURE 17. Regular n-gon in regular n-gon

The capacity obtained with AFEM is 9.5219.

EXAMPLE 18. *Regular n-gon in regular n-gon.* The outer boundary consists of the nth roots of unity whereas the inner boundary is obtained from the outer boundary by scaling with a factor $t \in (0, 1)$. Example 1 is a particular case of this general case when $n = 4$. With $n = 3$, $t = 0.5$ and $n = 6$, $t = 0.5$ the capacities are 12.4412 and 9.3804, respectively.

We have explored the behavior of the capacity when $n = 3, 5, 7, 9$ and $t = 0.1 : 0.1 : 0.9$. The results are summarized in the following table.

The results of this computation are summarized in the following figure. The topmost curve corresponds to the case $n = 3$. The lowest curve labelled with dash-dot markers is the curve $f(t) = 2\pi/\log(1/t)$, which represents the capacity of a circular annulus with inner and outer radii t and 1, respectively. It seems that for a fixed $t \in (0, 1)$ the capacites decrease toward $f(t)$ as n increases to infinity. For

TABLE 11. Table for Example 18

t	cap (n=3)	cap (n=5)	cap (n=7)	cap (n=9)	cap (annulus)
0.1	3.0676	2.7807	2.7464	2.7370	2.7288
0.2	4.6201	4.0110	3.9404	3.9206	3.9040
0.3	6.4989	5.4106	5.2836	5.2482	5.2187
0.4	8.9768	7.1874	6.9697	6.9085	6.8572
0.5	12.4412	9.6266	9.2598	9.1541	9.0647
0.6	17.6373	13.2634	12.6496	12.4636	12.3001
0.7	26.2975	19.3159	18.2712	17.9370	17.6160
0.8	43.6181	31.4211	29.5099	28.8559	28.1576
0.9	95.5796	67.7548	63.2096	61.6069	59.6351

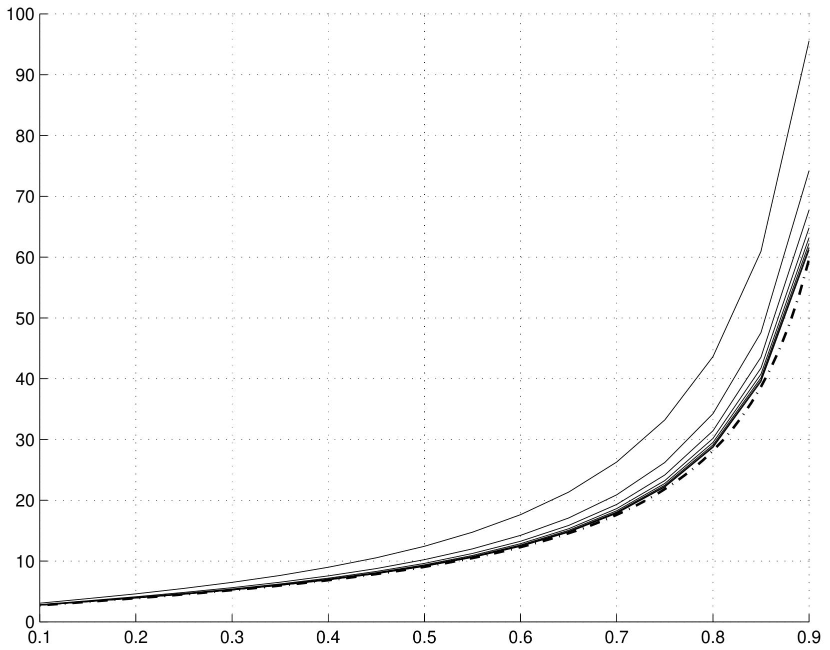

FIGURE 18. The capacity of Example 18 as a function of t for various values of n.

small values of the parameter t we can easily verify this visual observation if we use circular annuli to find upper and lower bounds for the capacity.

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