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CONFORMAL MAPPING OF RIEMANN SURFACES AND THE CLASSICAL THEORY OF UNIVALENT FUNCTIONS

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Dedicated to Professor Dr. Horst Tietz on his 80th Birthday

Abstract. Analytic mappings between Riemann surfaces are very natural objects in complex analysis. Corresponding to the classical univalent functions we have the class of injective holomorphic mappings — i.e., conformal embeddings — of a Riemann surface into another. We find indeed a number of analogies between them. On the other hand, because of the non-planarity of the domain surface, we face some new problems which we have never encountered in the classical theory. We discuss various problems concerning the conformal embeddings.

1. Introduction

The classical univalent function theory has been systematically producing a number of deep and interesting results (cf. e.g., [**2**], [**6**], [**16**] and [**37**]). It is still one of the fascinating and active fields in complex analysis. As is well known, it is naturally and closely connected with the theory of conformal mapping (cf. e.g., [**6**], [**8**] and [**16**]). Not only qualitative but also quantitative results have been obtained in these fields; in most cases they are largely based on the planarity of the domains of definition. That is, the univalent function theory as well as conformal mapping theory — both in the classical sense — belong to the complex analysis on the complex plane $\mathbb{C}$ or on the Riemann sphere $\hat{\mathbb{C}}$.

In the present expository article we are concerned with "theory of univalent functions on Riemann surfaces". The readers will easily notice that this is obviously a contradictory expression. Indeed, there exist *no* univalent functions — injective (analytic) mappings into $\hat{\mathbb{C}}$ — on a Riemann surface of positive genus, as an easy topological argument shows. Our research objects are actually "Riemann surfaces of positive finite genus" and "conformal embedding of one such surface into another".

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In other words, we study "the one-to-one holomorphic mappings of a Riemann surface into another". We show on the one hand some results which easily remind us of the classical ones, and find on the other hand some which contrast with the classical. The study on conformal mapping between Riemann surfaces therefore are important for its own sake. Sometimes one can reach a good understanding of analyticity only when its properties — univalence and conformality in our present case — can be well explained in the framework of Riemann surfaces. One of the results below can be regarded as a straightforward generalization of the Riemann Mapping Theorem and the Koebe generalized Uniformization Theorem.

A noteworthy difference between the classical univalent function theory and ours is as follows: one cannot prescribe the target surface in our case, while the complex plane $\mathbb{C}$ or the Riemann sphere $\hat{\mathbb{C}}$ is always the target in the classical case. This observation brings us a new interesting problem: determine all the possible compact Riemann surfaces into which a given (noncompact) Riemann surface can be conformally embedded. In fact this is one of the main topics discussed in this article.

The author tries to give the readers a comprehensive survey of this field. Not a theorem in this article has a proof at length, but it will be of ease to trace the detailed discussion in the bibliography at the end of the article. The bibliography, though not exhaustive in any sense, contains sufficiently many of the classical sources as well as recent papers and books. In the last section we mention several problems and survey some further topics related with our study, among which are hydrodynamics of an ideal liquid flow on a surface, analytic continuation beyond the ideal boundary, the role of the hyperbolic metric in our problem, a new construction of a fundamental domain for a Fuchsian group.

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2. Embedding Theorem

We begin with the following theorem, which is another version of the result stated and proved in [**51**] and [**59**].

THEOREM 1. *Let R_0 be a noncompact Riemann surface of finite genus g. Then there is a compact Riemann surface R of genus g, a conformal mapping $\iota : R_0 \to R$, and a nonconstant meromorphic function f on R such that*

i) *$R \setminus \iota(R_0)$ is a null set*[1] *(in the sense of Lebesgue),*
ii) *f is holomorphic on $R \setminus \iota(R_0)$, and*
iii) *$\operatorname{Im} f$ is constant on each component of $R \setminus \iota(R_0)$.*

The composed mapping $f_0 := f \circ \iota$ is a meromorphic function on R_0 which shows a typical boundary behavior. That is, $\operatorname{Im} f_0$ takes in a certain wider sense a constant value on each ideal boundary component of R_0. It should be noted,

[1] It makes sense to say that a set on a Riemann surface is measurable and is a null set.

however, that this description is not rigorous at all, since the ideal boundary of R_0 has only an abstract meaning and the function f_0 has not been defined on it. On the other hand, the value of f on $R \smallsetminus \iota(R_0)$ makes sense; in fact f is holomorphic on $R \smallsetminus \iota(R_0)$ and $\operatorname{Im} f$ is equal to a constant on each component in a strict sense.

$$\begin{tikzcd} R_0 \arrow[d, "\iota"'] \arrow[dr, "f_0"] & \\ R \arrow[r, "f"'] & \hat{\mathbb{C}} \end{tikzcd}$$

The function f_0 can be regarded as a generalization of a complex potential function (cf. [**62**]).[2] This kind of functions or their differentials, on which our proof ([**51**] and [**59**]) heavily depends, are defined on general noncompact Riemann surfaces and studied in great detail around 1960 by L. V. Ahlfors [**1**], Y. Kusunoki [**20**] and L. Sario [**38**] independently. Although different names — distinguished differentials, canonical semiexact differentials, or principal functions — are used, they share essential properties in common. They all have a finite Dirichlet integral near the ideal boundary of R_0 — outside a compact subset of R_0. In particular the number of poles is in all cases finite. Roughly speaking, a canonical semiexact differential is a meromorphic differential whose real part is distinguished, and a $(Q)L_1$-principal function is a singlevalued harmonic function (with singularities) whose differential is the real part of a canonical semiexact differential. See also [**40**, p. 41, Remark].

In our proof ([**51**] and [**59**]) we start with a nonconstant meromorphic function f_0 such that df_0 is a canonical semiexact differential and construct R and ι. To complete the proof new insight about the covering property of such functions is substantially required in addition to the deep results on df_0 as an Abelian differential, and indeed it is the generalized Riemann-Hurwitz relation $g = 1 - m + V/2 + W/2$ that plays an essential role in our proof (see [**51**]). In the above formula,

$$\begin{aligned} m &:= \text{the number of poles of } f_0, \\ V &:= \text{the total order of branch points in } R_0, \text{ and} \\ W &:= \lim_{n\to\infty}\left(-\frac{1}{2\pi}\int_{\partial R_n} d\arg df_0 - h_n\right), \end{aligned}$$

where $(R_n)_{n=1,2,\dots}$ is an arbitrarily fixed canonical exhaustion (for the definition, see [**1**]) of R_0, ∂R_n the (positively oriented) contour of R_n, and h_n the number of the boundary components of ∂R_n.[3] The key to the Riemann-Hurwitz relation is to show that W represents the total order of branch points *on the ideal boundary.*

[2] Sometimes referred to as a "complex velocity potential". It is "Strömungsfunktion" observed by F. Klein [**17**]. Cf. also [**32**] for hydrodynamical background.

[3] We can show that the number m is equal to the (maximal) number of sheets of the covering $f : R_0 \to \hat{\mathbb{C}}$

The location of poles of the function f_0 can be rather arbitrarily prescribed. Indeed, for any $g+1$ distinct points $p_1, p_2, \ldots, p_{g+1}$ on R_0, we can find a nonconstant meromorphic function f_0 such that

i) f_0 is holomorphic on $R_0 \setminus \{p_1, p_2, \ldots, p_{g+1}\}$,
ii) f_0 has at worst a simple pole at each of $p_1, p_2, \ldots, p_{g+1}$, and
iii) df_0 is a canonical semiexact differential.

The condition that the prescribed points $p_1, p_2, \ldots, p_{g+1}$ on R_0 be distinct is not essential. If some of them coincide, then the obtained function has a pole of higher order at that point. The really important quantity here is the total order of the function f_0. For the existence of such a nonconstant meromorphic f_0 on R_0, see [**20**] and [**46**], for example.

The surface R in the above theorem is not unique but there are in general infinitely many distinct surfaces with the required properties, even if we fix f_0 on R_0 in advance. Nonuniqueness of R can be easily shown by using an example given in [**19**]. See [**59**], where it is also shown that for a fixed R_0 and a fixed f_0 on R_0 the compact surfaces R onto which f_0 extends R_0 depend on W number of real parameters.

The uniqueness of the meromorphic function f_0 on R_0 with a required boundary behavior and that of the compact Riemann surface R are often confused but they are completely different notions — even if R_0 is a finitely connected Riemann surface. The same can be said in the case of a quadratic differential on R_0 which has a holomorphic extension onto a compact Riemann surface. Indeed, our theorem easily yields

THEOREM 2. *Let R_0 be a noncompact Riemann surface of finite genus g. Then there is a compact Riemann surface R of genus g, a conformal mapping $\iota : R_0 \to R$, and a meromorphic quadratic differential φ on R such that*

i) *$R \setminus \iota(R_0)$ is a null set,*
ii) *φ has at most $2g+2$ poles on R,*
iii) *φ is holomorphic on $R \setminus \iota(R_0)$, and*
iv) *each component of $R \setminus \iota(R_0)$ is realized as a (possibly branched) trajectory arc[4] of φ.*

If there actually exists[5] a component of $R \setminus \iota(R_0)$ which is realized as a branched trajectory arc of φ, there are infinitely many distinct[6] pairs (R, φ) with the required properties. Note that the pull-back $\varphi_0 := \iota^*(\varphi)$ is the same for these distinct φ.

Compare the theorem with [**4**] and Chapter 13 of [**5**]. If a quadratic differential has critical points on the ideal boundary — that is, if $W > 0$ — the surface R can be constructed in infinitely many distinct ways. They determine distinct points in the Teichmülles space.

[4]For the definition and properties of trajectory arcs, see [**63**].

[5]This is to say that $W > 0$.

[6]The word "distinct" here means that any two of them are not mutually conformally equivalent.

To explain the situation more clearly we restrict ourselves for a while to finitely connected Riemann surfaces. The reason for confusion is then a unnatural restriction on the boundary. One often starts with the assumption that the boundary is realized as analytic Jordan curves, but this is to say that the given surface is from the outset embedded in a larger surface.[7] This is absurd when we consider noncompact Riemann surfaces as independent objects. One can neither introduce nor fix a particular conformal structure on the boundary. Otherwise, the Riemann Mapping Theorem would lose its essence.[8] Consider, for example, the Joukowski transformation $z \to z + 1/z$ on the open unit disk.

If $g = 0$, then R is conformally equivalent to the Riemann sphere $\hat{\mathbb{C}}$. For simplicity we assume that $R = \hat{\mathbb{C}}$. We can choose $\varphi = dz^2$ with a suitable coordinate z on R. There are no branch points on each trajectory arc, so that each component of $R \setminus \iota(R_0)$ is a vertical segment which is traced exactly twice — go and back. We can indeed show that the image domain $\iota(R_0)$ in Theorem 1 is a so-called extremal parallel slit domain. For the definition of an extremal slit domain, see [**16**] and [**40**]. This shows that Theorem 1 is a natural generalization of the classical uniformization theorem due to Koebe (see [**6**], [**8**], [**16**], and [**40**]).

In case $g > 0$ our theorem gives a general and exact formulation of the results due to Nehari [**35**], Kusunoki [**20**], Mori [**34**] and Mizumoto [**33**]. They are concerned primarily with the range of functions and little with the covering properties. As a consequence the notion of "slits" is very ambiguous, although Mizumoto [**33**] implicitly pointed out the importance and difficulty of the problem.

Here are some typical questions: What is a "slit" in the first place? Is it the resulting boundary component obtained by the removal of a "closed segment" or a "closed arc" from a "surface"? If so, what kind of surface should be considered in advance? What kind of segment or arc should be removed? How simple (or complicated) can they be? How does the function f_0 behave *near* a slit — continuity, holomorphy, injectivity, and so on? How about the behavior of f_0 *on* a slit?

Our approach to this set of questions consists of

(1) starting with a noncompact Riemann surface R_0 of finite genus g and a generalized complex potential function f_0 on it.
(2) constructing a compact Riemann surface R of genus g and a conformal embedding $\iota : R_0 \to R$,
(3) extending the function $\iota^*(f_0) = f_0 \circ \iota^{-1}$ on $\iota(R_0)$ onto R as a meromorphic function f on R,
(4) showing that f is holomorphic on $R \setminus \iota(R_0)$ and is univalent on a neighborhood of every — but a finite number of exceptional ones — connected component of $R \setminus \iota(R_0)$, and finally

[7]In other words, the interior of a compact bordered Riemann surface is not generic enough in this context. The conformal structure of the border is fixed in advance.

[8]As is well known, even the continuity of a mapping function on the boundary is not guaranteed.

(5) verifying that df vanishes somewhere on each exceptional connected component of $R \setminus \iota(R_0)$, while $\operatorname{Im} f$ is constant on every connected component of $R \setminus \iota(R_0)$.

In our argument the study of $f_0 : R_0 \to \hat{\mathbb{C}}$ as a covering surface (not as a complex-valued function) plays an essential role. See [**51**] for further details. We note that a connected component of $R \setminus \iota(R_0)$ is a realization of a Kerékjártó-Stoïlow ideal boundary component of R_0.

3. Compact continuations of a Riemann surface of finite genus

According to the classical usage a Riemann surface R is a continuation[9] of R_0, if R_0 is a (proper) subregion of R. If R is compact, it is called a compact continuation. See, e.g., [**40**]. In the present article we always assume that the genus g of R_0 is positive and finite[10], so that compact continuations play an important role. We are particularly interested in *compact continuations of the same genus*, compact Riemann surfaces which are continuations of R_0 and are of genus g. In fact, one of the central problems is to determine, for a fixed noncompact Riemann surface R_0, the set of its compact continuations of the same genus.

For a more detailed study of compact continuations of the same genus, we have to consider the homology groups. The definition and the argument below can be applied to general g (> 0), but to simplify the notation and the statements we confine ourselves to the simplest case $g = 1$. If this is the case, we can explicitly describe the set of compact continuations of the same genus, which is another advantage of our restriction.

Now, let R_0 be a noncompact Riemann surface of genus one. For the abuse of language, we sometimes call it a *noncompact torus*. Let $\{a_0, b_0\}$ be a canonical homology basis of R_0 modulo dividing cycles.[11] The pair $(R_0, \{a_0, b_0\})$ is referred to as a homologically marked noncompact torus. For simplicity we often omit "homologically". Our terminology is just an extension of the following classical usage: a compact Riemann surface of genus one is called a torus and a torus with a canonical homology basis $\{a, b\}$ is called a (homologically) marked torus.

We say that a triple $(R, \{a, b\}, \iota)$ — a marked torus $(R, \{a, b\})$ together with a conformal embedding $\iota : R_0 \to R$ — is a compact realization of a marked noncompact torus $(R_0, \{a_0, b_0\})$, if $\iota(a_0)$ (resp. $\iota(b_0)$) is homologous to a (resp. b).

$$\begin{tikzcd} & (R', \{a', b'\}) \arrow[dd, "h"] \\ (R_0, \{a_0, b_0\}) \arrow[ur, "\iota'"] \arrow[dr, "\iota''"'] & \\ & (R'', \{a'', b''\}) \end{tikzcd}$$

Two compact realizations $(R', \{a', b'\}, \iota')$ and $(R'', \{a'', b''\}, \iota'')$ of a marked noncompact torus $(R_0, \{a_0, b_0\})$ are said to be equivalent, if there is a (surjective)

[9] The terms "prolongation" and "extension" are also used.

[10] For the case of infinite genus, see e.g., [**41**], where the situation is quite different.

[11] For the precise definition of "dividing cycles" see [**1**].

conformal mapping $h : R' \to R''$ such that $h(a')$ (resp. $h(b')$) is homologous to a'' (resp. b'') and $h \circ \iota' = \iota''$. Each equivalence class $[R, \{a, b\}, \iota]$ is called a *compact continuation* of $(R_0, \{a_0, b_0\})$. We will use a simplified notation $[R]$ instead of $[R, \{a, b\}, \iota]$ if there is no serious confusion. We denote by $C(R_0, \{a_0, b_0\})$ the set of compact continuations of $(R_0, \{a_0, b_0\})$. Note that $C(R_0, \{a_0, b_0\})$ is considered in the Torelli space.

To describe the set $C(R_0, \{a_0, b_0\})$ quantitatively, we make use of the moduli of tori. To each compact continuation $[R, \{a, b\}, \iota]$ of $(R_0, \{a_0, b_0\})$ we can associate a complex number τ with positive imaginary part. In fact, using a compact realization $(R, \{a, b\}, \iota)$ which represents $[R, \{a, b\}, \iota]$ and the holomorphic differential ω on R with $\int_a \omega = 1$, we set[12] $\tau := \int_b \omega$. As is easily verified, τ does not depend on the choice of a representative $(R, \{a, b\}, \iota)$. Hence we have the correspondence

$$C(R_0, \{a_0, b_0\}) \ni [R] \to \tau = \tau([R]) \in \mathbb{H},$$

where $\mathbb{H}$ denotes the upper half plane $\{\tau \in \mathbb{C} \mid \operatorname{Im} \tau > 0\}$. We call $\tau([R])$ the modulus of $[R] = [R, \{a, b\}, \iota]$. We finally set

$$\mathfrak{M}(R_0, \{a_0, b_0\}) = \{\tau = \tau([R]) \mid [R] \in C(R_0, \{a_0, b_0\})\}.$$

The following result is called the moduli disk theorem:

THEOREM 3. *Suppose that a marked noncompact torus $(R_0, \{a_0, b_0\})$ is given. Then, the following hold:*

(1) *$\mathfrak{M}(R_0, \{a_0, b_0\})$ is a nonempty closed disk in $\mathbb{H}$, and it reduces to a single point if and only if $R_0 \in O_{AD}$.*[13]

(2) *To each boundary point τ of $\mathfrak{M}(R_0, \{a_0, b_0\})$ there exists a* unique *$[R, \{a, b\}, \iota] \in C(R_0, \{a_0, b_0\})$ with $\tau = \tau[R, \{a, b\}, \iota]$. Furthermore, $R \setminus \iota(R_0)$ is a null set consisting of horizontal trajectory arcs of $-ie^{-i\theta}\omega^2$, where $\theta := \arg(\tau - \tau_E) \in (-\pi, \pi]$ with τ_E the center of the disk $\mathfrak{M}(R_0, \{a_0, b_0\})$.*

(3) *To any interior point τ of $\mathfrak{M}(R_0, \{a_0, b_0\})$ there are infinitely many distinct*[14] *$[R] \in C(R, \{a_0, b_0\})$ with $\tau = \tau([R])$.*

The differential ω induces a natural metric on $(R, \{a, b\})$. We may and do assume that a is a geodesic curve on $(R, \{a, b\})$ with respect to this metric. If $\tau \in \partial\mathfrak{M}(R_0, \{a_0, b_0\})$ and $(R, \{a, b\}, \iota)$ is the marked torus with $\tau = \tau[R_0, \{a_0, b_0\}, \iota]$, then the marked noncompact torus $(\iota(R_0), \{a, b\})$ is a parallel slit torus; the boundary components of R_0 realized on R are linear segments which are geodesic parallel to each other. This property, together with the extremality of the associated Abelian differential of the first kind, allows us to say that the compact continuation $[R]$ of $(R_0, \{a_0, b_0\})$ for $\theta = -\pi/2$ (resp. $\theta = \pi/2$) is "an extremal geodesic horizontal (resp. vertical) slit torus". The embedding of R_0 for any $\tau \in \partial\mathfrak{M}(R_0, \{a_0, b_0\})$ thus yields a natural generalization of the classical results due to Koebe.

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[12] This ω is often referred to as a normal Abelian differential of the first kind.

[13] The statement "$R_0 \in O_{AD}$" means that R_0 carries no nonconstant analytic functions with a finite Dirichlet integral. See [**1**] or [**39**], for example.

[14] These continuations will be studied in the next section.

Here are some historical remarks. The above theorem refines a theorem due to M. Heins [**9**], who proved the boundedness of the principal moduli of tori. Our theorem gives another proof of H. Grötzsch [**7**].

For a proof of the preceding theorem see [**52**]. One of the key steps toward the theorem is the construction of an Abelian differential of the first kind with a special kind of periods and boundary behavior. This is achieved in [**46**] and [**47**]. Another key is the generalized Riemann-Hurwitz formula stated in the last section (cf. [**51**]). Indeed, suppose that $g = 1$ and take a holomorphic differential φ. Then $n = 0$, which yields $V = W = 0$, so that there are critical points of φ^2 neither in R nor on the boundary.

The above theorem gives a good explanation for the total energy and the boundary behavior of an ideal fluid flow on a noncompact torus. See [**52**], [**58**], [**15**]. A clear and intuitive account on the close relationship between complex analysis and hydrodynamics can be found in [**17**], [**18**], and [**62**].

For higher genera we can similarly discuss with some necessary modification. For example, the homology basis $\{a_0, b_0\}$ of R_0 (resp. $\{a, b\}$ of R) should be replaced by $\{a_0^{(1)}, a_0^{(2)}, \dots, a_0^{(g)}; b_0^{(1)}, b_0^{(2)}, \dots, b_0^{(g)}\}$ (resp. $\{a^{(1)}, a^{(2)}, \dots, a^{(g)}; b^{(1)}, b^{(2)}, \dots, b^{(g)}\}$), and we consider conformal mapping f such that $f(a_0^{(k)})$ (resp. $f(b_0^{(k)})$) is homologous to $a^{(k)}$ (resp. $b^{(k)}$), $k = 1, 2, \dots, g$. The description of the set corresponding to $\mathfrak{M}(R_0, \{a_0, b_0\})$ is not completely known. As for partial results, see, for example, [**53**], [**54**], and also [**29**].

4. Classical theory of univalent functions and the theory of conformal embeddings

The readers will be easily aware of the affinity between the results in the last section and those in the classical theory of univalent functions. In fact, it suffices to recall the theorems of Koebe, de Possel, and Grötzsch. Cf. e.g., [**8**], [**16**] and [**40**]. To state the classical results, let G be a plane domain and let $\zeta\,(\neq \infty)$ be a fixed point in G. Let $F(G, \zeta)$ be the family of univalent meromorphic functions f on G whose Laurent expansion about ζ is

$$f(z) = \frac{1}{z - \zeta} + \kappa_f (z - \zeta) + \cdots$$

and set $\mathfrak{K}(G, \zeta) = \{\kappa \in \mathbb{C} \mid \kappa = \kappa_f \text{ for some } f \in F(G, \zeta)\}$.

The theorem below gathers the classical results and is a prototype of Theorem 3.

THEOREM 4. (1) *$\mathfrak{K}(G, \zeta)$ is a nonempty closed disk in $\mathbb{C}$. It reduces to a single point if and only if G belongs to O_{AD}.*

(2) *To each boundary point of $\mathfrak{K}(G, \zeta)$ there corresponds a unique function in $F(G, \zeta)$. Furthermore, if $\kappa(\theta) = \kappa_E + \rho_E e^{i\theta}$, $-\pi < \theta \leqslant \pi$, parametrizes $\partial \mathfrak{K}(G, \zeta)$ and if $f_{\kappa(\theta)}$ is the element of $\mathfrak{K}(G, \zeta)$ corresponding to $\kappa(\theta)$, then $\hat{\mathbb{C}} \setminus f_{\kappa(\theta)}(G)$ is a plane null set which consists of parallel segments making an angle $\theta/2$ with the real axis.*

The following result due to Grunsky is also remarkable:

THEOREM 5. *The area of* $\hat{\mathbb{C}} \setminus f(G)$ *assumes its maximum in* $F(G,\zeta)$ *for the function* $f_E := \dfrac{1}{2}(f_{\kappa(0)} + f_{\kappa(\pi)}) \in F(G,\zeta)$ *only. The function* f_E *is expanded near* ζ *as*

$$f_E(z) = \frac{1}{z-\zeta} + \kappa_E(z-\zeta) + o(|z-\zeta|)$$

and the maximum area is equal to $\pi\rho_E$.

An analogous theorem for the conformal embeddings of a noncompact torus into tori holds. To state it we recall the unique Abelian differential ω of the first kind on $[R,\{a,b\},\iota] \in C(R_0,\{a_0,b_0\})$ with $\int_a \omega = 1$ (see section 3). Let $\alpha([R]) = \alpha([R,\{a,b\},\iota])$ be the area of $R \setminus \iota(R_0)$ measured by ω:

$$\alpha([R]) := \iint_{R\setminus\iota(R_0)} \omega \wedge \bar{\omega}^*.$$

We are now ready to generalize the Grunsky theorem to the case of genus one.

THEOREM 6. *The area* $\alpha([R])$ *assumes its maximum in* $C(R_0,\{a_0,b_0\})$ *for a unique conformal embedding. The embedding is associated with the arithmetic mean of the holomorphic differentials on* R_0 *which respectively determine the extremal geodesic horizontal and the extremal geodesic vertical slit tori. The maximum area is equal to* $\rho_E/2$.

The Grunsky theorem shows that the area of the complement of the image in $\hat{\mathbb{C}}$ is maximized at the euclidean center of the disk $\mathfrak{K}(G,\zeta)$. Similarly, our theorem shows — just as its prototype due to Grunsky — shows that the area of the complement of the image in the target torus[15] is maximized at the euclidean center of the moduli disk. The euclidean centers of the disks $\mathfrak{K}(G,\zeta)$ and $\mathfrak{M}(R_0,\{a_0,b_0\})$ thus play an important role.

On the other hand, we can regard the (euclidean) disk $\mathfrak{M}(R_0,\{a_0,b_0\})$ as a disk with respect to the spherical metric on $\hat{\mathbb{C}}$. Since it lies entirely in $\mathbb{H}$, we can regard it also as a disk with respect to the hyperbolic metric on $\mathbb{H}$. Hence, it makes sense to ask what kind of role the center of the hyperbolic or spherical disk $\mathfrak{M}(R_0,\{a_0.b_0\})$ plays.

To solve this problem we study the areal property of conformal embeddings in further detail. For each $[R] = [R,\{a,b\},\iota] \in C(R_0,\{a_0,b_0\})$ let $A([R])$ be the area of R measured by ω:

$$A([R]) = A([R,\{a,b\},\iota]) := \iint_R \omega \wedge \bar{\omega}^*.$$

Obviously $A([R]) = \operatorname{Im}\tau([R])$ holds. We also set

$$S([R]) = S([R,\{a,b\},\iota]) := \frac{\alpha([R])}{A([R])} \quad \text{and}$$

$$\Delta([R]) = \Delta([R,\{a,b\},\iota]) := \frac{\alpha([R])}{\pi(1+|\tau([R])|^2)}.$$

[15] Recall that this is now variable!

We consider the normalized conformal complete metrics

$$ds_{\mathbb{C}} := |d\tau|, \quad ds_{\hat{\mathbb{C}}} := \frac{2|d\tau|}{1+|\tau|^2} \quad \text{and} \quad ds_{\mathbb{H}} := \frac{|d\tau|}{\operatorname{Im}\tau}$$

on $\mathbb{C}$, $\hat{\mathbb{C}}$ and $\mathbb{H}$, respectively, where τ stands for the complex coordinate. By the normalization we mean that they are of constant curvature $0, +1$, and -1 respectively.[16]

We recall here that τ_E and ρ_E are respectively the euclidean center and radius of $\mathfrak{M}(R_0, \{a_0, b_0\})$. Denote now by τ_H and ρ_H (resp. τ_S and ρ_S) the hyperbolic (resp. spherical) center and radius of $\mathfrak{M}(R_0, \{a_0, b_0\})$. We call the radii ρ_E, ρ_H and ρ_S the euclidean, hyperbolic and spherical spans of $(R_0, \{a_0, b_0\})$, respectively. It can be easily shown that the hyperbolic span does not depend on the choice of $\{a_0, b_0\}$. Hence it is more intrinsic than the other spans; we can say that $\sigma_{\mathbb{H}}$ is the hyperbolic span of R (without referring to any particular canonical homology basis $\{a_0, b_0\}$ modulo dividing cycles). In any case, however, they vanish simultaneously.

We denote by $d_{\mathbb{H}}(\,\cdot\,,\,\cdot\,)$, $d_{\mathbb{C}}(\,\cdot\,,\,\cdot\,)$ and $d_{\hat{\mathbb{C}}}(\,\cdot\,,\,\cdot\,)$ the distance function induced by $ds_{\mathbb{H}}$, $ds_{\mathbb{C}}$ and $ds_{\hat{\mathbb{C}}}$ respectively. Furthermore we simplify the notation and set[17]

$$r_H(\tau) := d_{\mathbb{H}}(\tau_H, \tau), \quad r_E(\tau) := d_{\mathbb{C}}(\tau_E, \tau) \quad \text{and} \quad r_S(\tau) := d_{\hat{\mathbb{C}}}(\tau_S, \tau).$$

Then we can describe the set $\mathfrak{M}(R_0, \{a_0, b_0\})$ in three ways as follows:

$$\begin{aligned} \mathfrak{M}(R_0, \{a_0, b_0\}) &= \{\tau \in \mathbb{H} \mid r_H(\tau) \leqslant \rho_H\} \\ &= \{\tau \in \mathbb{C} \mid r_E(\tau) \leqslant \rho_E\} \\ &= \{\tau \in \hat{\mathbb{C}} \mid r_S(\tau) \leqslant \rho_S\}. \end{aligned}$$

The interior of the closed disk $\mathfrak{M}(R, \{a_0, b_0\})$ is itself a hyperbolic space, so that it carries its own normalized hyperbolic metric $ds_M := ds_{\mathfrak{M}(R,\{a_0,b_0\})}$ (of curvature -1). As is well known it is explicitly given by

$$ds_M = \frac{2\rho_E}{\rho_E^2 - r_E^2}|d\tau|, \qquad r_E = r_E(\tau) = |\tau - \tau_E|.$$

For each point $\tau \in \mathfrak{M}(R_0, \{a_0, b_0\})$ we consider

$$C_\tau(R_0, \{a_0, b_0\}) := \{[R, \{a, b\}, \iota] \in C(R_0, \{a_0, b_0\}) \mid \tau[R, \{a, b\}, \iota] = \tau\},$$

the set of compact continuations of modulus τ.

The following result is a refinement[18] of the preceding theorem.

Theorem 7. *Suppose that $\tau \in \mathfrak{M}(R_0, \{a_0, b_0\})$ is given. Then, for any $[R] \in C_\tau(R_0, \{a_0, b_0\})$, the following three inequalities hold:*

$$(1) \quad \alpha([R]) \leqslant \left.\frac{ds_{\mathbb{C}}}{ds_M}\right|_\tau; \qquad (2) \quad S([R]) \leqslant \left.\frac{ds_{\mathbb{H}}}{ds_M}\right|_\tau; \qquad (3) \quad \Delta([R]) \leqslant \left.\frac{ds_{\hat{\mathbb{C}}}}{ds_M}\right|_\tau.$$

---

[16]The hyperbolic and spherical metrics are uniquely determined by this normalization of curvature, while the euclidean is not. By convention $ds_{\mathbb{C}} := |d\tau|$ is the normalized metric in the case of zero curvature.

[17]If there is no ambiguity we will drop even τ.

[18]One can similarly give a refinement of Theorem 5. See [**55**].

Equality holds either for none of (1), (2) *and* (3), *or for all of them. The latter case occurs for one and only one* $[R] \in C_\tau(R_0, \{a_0, b_0\})$.

We now set $\alpha_\tau := \sup \alpha([R])$, $S_\tau := \sup S([R])$ and $\Delta_\tau := \sup \Delta([R])$ respectively, where the supremum is taken over all $[R] \in C_\tau(R_0, \{a_0, b_0\})\}$. Then we have

$$\begin{aligned}
\alpha_\tau &= \frac{\rho_E^2 - r_E^2}{2\rho_E}, \\
S_\tau &= \frac{\cosh^2(\rho_H/2) - \cosh^2(r_H/2)}{\sinh(\rho_H/2)\cosh(\rho_H/2)} = \frac{\tanh^2(\rho_H/2) - \tanh^2(r_H/2)}{\tanh(\rho_H/2)\cdot\bigl(1 - \tanh^2(r_H/2)\bigr)}, \\
\Delta_\tau &= \frac{\tan^2(\rho_S/2) - \tan^2(r_S/2)}{\tan(\rho_S/2)\cdot\bigl(1 + \tan^2(r_S/2)\bigr)} = \frac{\tan^2(\rho_S/2) - \tan^2(r_S/2)}{\tan(\rho_S/2)\cdot\bigl(1 + \tan^2(r_S/2)\bigr)}.
\end{aligned}$$

In particular, α_τ (resp. S_τ or Δ_τ) is constant on each euclidean (resp. hyperbolic or spherical) concentric circle.

By letting the Riemann surface R_0 move continuously and by showing the continuity of span as a function of R_0, we can prove the following theorem ([**60**], [**44**] and [**23**]; cf. also [**16**]).

THEOREM 8. (1) *The moduli disk* $\mathfrak{M}(R_0, \{a_0, b_0\})$ *is exhausted by compact continuations* $[R, \{a, b\}, \iota]$ *such that* $R \smallsetminus \iota(R_0)$ *has a vanishing area.*

(2) *The range of* α *on* $C_\tau(R_0, \{a_0, b_0\})$ *is exactly the finite closed interval* $[0, \alpha_\tau]$, *and the range of* S *on* $C_\tau(R_0, \{a_0, b_0\})$ *is exactly the finite closed interval* $[0, S_\tau]$.

5. Some related topics

In this section we mention some applications, related problems, and further development.

A. Some new theorems in the theory of univalent functions. The results in the preceding sections suggest theorems for univalent functions on a plane domain. Some of them are indeed new; for example, the estimate of the area of the omitted set by a normalized conformal mapping refines classical results. The role and meaning of the spherical center is not yet known completely, however.

To show another application, let G be a plane domain as before and f_0 (resp. f_1) be the normalized extremal horizontal (resp. vertical) mapping function of G. It is well known (see, e.g., [**8**], [**16**]) that any convex combination of these functions yields a univalent function. That is, $tf_0 + (1-t)f_1$ with $0 \leqslant t \leqslant 1$ is again univalent. F. Maitani asked the following question in [**21**]: When is a complex combination $c_0 f_0 + c_1 f_1$, $c_0, c_1 \in \mathbb{C}$, a univalent function? He gave an answer to this problem for a finitely connected G. We can use Theorem 4, for example, to deal with general G (cf. [**56**]). Namely we have

THEOREM 9. *Let* G *be a plane domain and* $\zeta \in G$ *be a fixed point. Let* f_0, f_1 *be the normalized extremal horizontal (resp. vertical) mapping of* G *with the pole at* ζ. *Then,* $c_0 f_0 + c_1 f_1$, $c_0, c_1 \in \mathbb{C}$, *is a univalent function if and only if* $\mathrm{Re}(c_0/c_1) \geqslant 0$.

B. Rankine ovoid — another kind of continuation. The compact continuations with which we have been concerned realize the original noncompact Riemann surface $(R_0, \{a_0, b_0\})$ as a dense subset of a compact Riemann surface $(R, \{a, b\})$. Thereby the ideal boundary are realized as a set of stream lines of an ideal liquid flow on R_0; in other words, the ideal boundary is supposed to be impenetrable for the flow. The complex potential function $f_0 \circ \iota^{-1}$ (Strömungsfunktion) on $\iota(R_0)$ can be holomorphically extended onto $R \setminus \iota(R_0)$.

There is another kind of compact continuations, where $R \setminus \iota(R_0)$ has an interior point, and the extension of the complex potential function $f_0 \circ \iota^{-1}$ on $\iota(R_0)$ onto R is meromorphic (but not holomorphic[19]) on $R \setminus \iota(R_0)$. The hydrodynamical phenomenon on the plane which describe this situation is known as a Rankine ovoid (cf. [**32**]). Concerning the Rankine ovoids on a torus, see [**14**] and [**15**].

The Rankine ovoid for a torus can be also interpreted as a conformal sewing of a slit torus and a slit Riemann sphere (crosswise along the slits). Our problem is to study the change of modulus of the torus under the conformal sewing. This is a rather algebraic problem and was studied in [**10**], [**11**] and [**12**].

C. Riemann surfaces of higher genera. It is natural to ask what happens for Riemann surfaces of genus greater than one. Let R_0 be a noncompact Riemann surface of genus $g > 1$. If we fix a canonical homology basis χ_0 of R_0 modulo dividing cycles, the definition of a compact continuation of (R_0, χ_0) is easily obtained. Instead of the moduli set $\mathfrak{M}(R_0, \chi_0)$ we can consider the set of period matrices of $[R, \chi, \iota] \in C(R_0, \chi_0)$. Although some partial results have been obtained (see, for example, [**53**], [**54**], [**44**]; [**26**], [**27**], and [**29**], there still remain a number of unsolved problems.

It is also of interest to consider analytic mappings of a Riemann surface of genus (> 1) to a torus. For example, we can apply such observation to prove a classical theorem of Hurwitz concerning the degeneration of a conformal automorphism of a compact Riemann surface of genus > 1 and its generalizations. See [**50**].

D. Analytic continuation beyond the ideal boundary. A conformal embedding of a Riemann surface into another yields a realization of the ideal boundary. This is one of the significant features of the classical Riemann Mapping Theorem and its generalizations. Analytic continuation of a meromorphic function, together with the extension of conformal structure beyond the ideal boundary, deserves to be studied in further detail. Indeed, this kind of problem is regarded as a natural generalization of the classical theorem of Abel to noncompact surfaces. One of the results in this direction is shown in [**57**], where the previous work ([**42**] and [**45**]) concerning the size of the realized ideal boundary plays an important role.

Algebraic theorems such as the Riemann-Roch theorem and Abel's theorem for noncompact Riemann surfaces are studied in detail around 1960. The key to the generalizations is the uniqueness of an Abelian differential — e. g., a canonical semiexact differential or a distinguished differential — with prescribed periods and singularities. The analytic extension property of the meromorphic functions and

[19] Indeed, no holomorphic extension is possible by the maximum modulus principle.

differentials on a noncompact surface, which we give in section 2, well explains why the uniqueness is guaranteed on noncompact surfaces.

E. Hyperbolically maximal domains and fundamental domains for a discontinuous group. Our theorems suggest an interesting property of the quotient of conformal metrics. For example, from Theorem 7 we know that the ratio of the hyperbolic metric on $\mathbb{H}$ to that on a (hyperbolic) disk D in $\mathbb{H}$ is constant on each concentric circle of D (see [**55**]).[20] This simple property itself was later studied in [**30**] as a general theorem, which does not imply the results in [**55**], however.

Along this line and also as a generalization of the Riemann Mapping Theorem, we consider an extremal problem for simply connected domains on a Riemann surface. Suppose that a Riemann surface R is conformally equivalent neither to $\hat{\mathbb{C}}$ nor to $\mathbb{C}$, and p is a point of R. Let $\mathcal{D}$ be the class of simply connected domains D on R which contains p. Our extremal problem is to minimize ds_D at p, where ds_D denotes the hyperbolic metric on D with constant curvature -1.[21] A hyperbolically maximal domain is, by definition, the unique solution to this kind of extremal problem (for some p). It has a number of interesting properties; for example, a hyperbolically maximal domain has a characteristic geometric structure described in terms of the closed horizontal trajectories of a certain quadratic differential on R. This will be discussed in a forthcoming joint paper with Masumoto.[22]

We furthermore observe a similar problem under the properly discontinuous action of a conformal automorphism group Γ of R. We assume now that $\gamma(D) \cap D = \emptyset$ holds for any $D \in \mathcal{D}$ and for any nontrivial $\gamma \in \Gamma$. This assumption necessarily requires that p is not fixed by any nontrivial element $\gamma \in \Gamma$. A hyperbolically maximal domain is defined as above. It turns out that a hyperbolically maximal domain for Γ is a locally finite fundamental domain for Γ. In a special case where R is the open unit disk and Γ is a Fuchsian group, we obtain a new type of a fundamental domain, which is neither of Dirichlet nor of Ford. See [**31**] for further details.

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[20] The same holds also for a hyperbolic disk in the euclidean plane or in the sphere, and a spherical disk in the euclidean plane or in the sphere.

[21] More precisely, we introduce a quasi-order relation in $\mathcal{D}$ by the point evaluation of the differential quotient $ds_{D'}/ds_{D''}$ at p. We say that D'' is larger than D' if $(ds_{D'}/ds_{D''})_p \geqslant 1$ holds.

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