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LOEWNER CHAINS AND BIHOLOMORPHIC MAPPINGS IN $\mathbb{C}^n$ AND REFLEXIVE COMPLEX BANACH SPACES

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Abstract. This paper is a survey of very recent results about biholomorphic mappings of the ball in $\mathbb{C}^n$ and in reflexive complex Banach spaces. After recalling existence and regularity results in $\mathbb{C}^n$, we present certain applications including univalence criteria and quasiconformal extension results. We also consider nonuniqueness phenomena for solutions of the Loewner differential equation, and a geometric characterization of Loewner chains which satisfy a growth condition in t based on a generalization of the Carathéodory convergence theorem. Finally we describe some properties of Loewner chains and the Loewner equation on the unit ball of a reflexive complex Banach space.

1. Introduction

In this paper we survey some of the most recent advances in geometric function theory of several variables that have emerged since the publication of the book [12] earlier this year. In our presentation we shall describe results and their connections to other work. We give examples that illustrate the work, and in some cases, suggest questions for further research. One of the new results, Theorem 3.4, is a generalization to $\mathbb{C}^n$ for $n > 1$ of the Carathéodory kernel convergence theorem that gives the geometric characterization of local uniform convergence of a sequence of biholomorphic mappings of the unit ball in terms of convergence of the sequence of image domains to their kernel (see [21]). Then this result is applied in Theorem 3.6 to give a geometric characterization of a large class of Loewner chains. Also new are the results in Section 4 (see [16]) that generalize to dimension $n > 1$ the fundamental work of Becker [3] on existence and uniqueness of univalent solutions of the Loewner differential equation. Our work for $n > 1$ reveals a nonuniqueness

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of univalent solutions that comes from the fact that in contrast to $\mathbb{C}$ the group of automorphisms of $\mathbb{C}^n$ is much larger than the group of linear transformations. Our survey ends with the very recent study of the Loewner theory in a complex reflexive Banach space (see [19]).

The Loewner method has been applied with great success in the study of extremal problems for complex valued univalent functions on the unit disc. With good reason, subordination chains, Loewner chains and the Loewner differential equation in $\mathbb{C}^n$ are the central theme of this paper. It is natural to try to extend to higher dimensions those parts of the method that do not depend upon the Riemann mapping theorem. One cannot expect to embed an arbitrary biholomorphic mapping of the unit ball in $\mathbb{C}^n$ in a well-behaved Loewner chain, but for those mappings which permit such an embedding there are many types of problems which should be accessible.

Indeed, many aspects of the theory of Loewner chains and the Loewner differential equation in higher dimensions have now been studied, beginning with the work of Pfaltzgraff in the 1970's (see [23,24]). Pfaltzgraff formulated the generalization to higher dimensions of the Loewner differential equation and subordination chains and developed existence and uniqueness theorems for its solution on B, the unit ball in $\mathbb{C}^n$. His work also generalized to higher dimensions the one variable univalence criteria and quasiconformal extension results of Becker [2-4]. Since then, a number of authors have considered applications to characterizing subclasses of univalent mappings, growth theorems, coefficient estimates and quasiconformal extensions (see [10], [11], [12], [13–15], [17,18], [21], [25], [28]).

Within the last few years, the existence theory for the Loewner differential equation in several variables has been improved as a consequence of the discovery that the several variables analog of the Carathéodory class is compact [10]. Also, regularity properties of arbitrary Loewner chains in several variables have been studied (see [14,15]). Significant differences between the one variable and the several variables Loewner theory have been discovered (see [10], [14]). A self-contained account of Loewner theory in several variables may be found in the recent book by Graham and Kohr [12].

In 1989, Poreda [28] began the study of the Loewner differential equation on the unit ball of a complex Banach space. It has recently been shown by Hamada and Kohr that Poreda's regularity assumptions can be weakened when the Banach space is reflexive [19]. The existence and regularity theory in this situation is now basically the same as in finite dimensions. We discuss some of these results in the last section of the paper.

2. Loewner chains on the unit ball in $\mathbb{C}^n$. Applications

In this section we shall present some results related to Loewner chains and the Loewner differential equation in several complex variables. For a detailed discussion of this material and additional references, the reader may consult the book [12].

2.1. The Loewner differential equation. Let $\mathbb{C}^n$ denote the space of n complex variables $z = (z_1, \dots, z_n)$ with the Euclidean inner product $\langle \cdot, \cdot \rangle$ and the

Euclidean norm $\|z\| = \langle z, z\rangle^{1/2}$. Let $B_r = \{z \in \mathbb{C}^n : \|z\| < r\}$ and let $B = B_1$. In the case of one variable, B_r is denoted by U_r and U_1 by U. The topological closure of a subset A of $\mathbb{C}^n$ is denoted by $\overline{A}$. If $\Omega \subset \mathbb{C}^n$ is an open set, let $H(\Omega)$ be the set of holomorphic mappings from Ω into $\mathbb{C}^n$. $H(\Omega)$ will be given the topology of locally uniform convergence (or uniform convergence on compact subsets). Let $L(\mathbb{C}^n, \mathbb{C}^m)$ be the space of continuous linear operators from $\mathbb{C}^n$ into $\mathbb{C}^m$ with the standard operator norm. Let I be the identity in $L(\mathbb{C}^n, \mathbb{C}^n)$.

If $f \in H(B)$, we say that f is normalized if $f(0) = 0$ and $Df(0) = I$. Also if $f \in H(B)$, let $D^k f(z)$ be the k-th Fréchet derivative of f at $z \in B$ and let

$$D^k f(z)(w^k) = D^k f(z)(\underbrace{w, \ldots, w}_{k-times}), \quad w \in \mathbb{C}^n.$$

We say that $f \in H(B)$ is locally biholomorphic if f has a local holomorphic inverse at each $z \in B$. This is equivalent to the condition that $Df(z)$ is invertible at each point in B. A biholomorphic mapping on B will also be called univalent. Let $S(B)$ be the subset of $H(B)$ consisting of normalized biholomorphic mappings on B. In the case of one variable, $S(B)$ is denoted by S. Let $S^*(B)$ and $K(B)$ be the subsets of $S(B)$ consisting respectively of normalized starlike and convex mappings on B.

Recall that if $f : B \to \mathbb{C}^n$ is a locally biholomorphic mapping, then f is starlike if and only if (see [30])

$$\mathrm{Re}\langle [Df(z)]^{-1} f(z), z\rangle > 0, \quad z \in B \setminus \{0\}.$$

On the other hand, according to [25, Definition 1], a normalized locally biholomorphic mapping f on B is close-to-starlike if there exists a mapping $g \in S^*(B)$ such that

$$\mathrm{Re}\langle [Df(z)]^{-1} g(z), z\rangle > 0, \quad z \in B \setminus \{0\}.$$

Close-to-starlike mappings are biholomorphic on B as shown in [25].

If $f, g \in H(B)$, we say that f is subordinate to g, and write $f \prec g$, if there is a Schwarz mapping v (i.e., $v \in H(B)$, $v(0) = 0$, $\|v(z)\| < 1$, $z \in B$) such that $f(z) = g(v(z))$ for $z \in B$.

It is clear that if $f \prec g$ then $f(0) = g(0)$ and $f(B) \subseteq g(B)$. Moreover, if g is biholomorphic on B, then $f \prec g$ if and only if $f(0) = 0$ and $f(B) \subseteq g(B)$.

Definition 2.1. The mapping $f : B \times [0, \infty) \to \mathbb{C}^n$ is called a Loewner chain if the following conditions hold:

(i) $f(\cdot, t)$ is biholomorphic on B, $f(0, t) = 0$ and $Df(0, t) = e^t I$, for each $t \geqslant 0$;
(ii) $f(\cdot, s) \prec f(\cdot, t)$ whenever $0 \leqslant s \leqslant t < \infty$.

The condition (ii) is equivalent to the fact that there is a unique biholomorphic Schwarz mapping $v = v(z, s, t)$, called the transition mapping associated to $f(z, t)$, such that

$$f(z, s) = f(v(z, s, t), t), \quad z \in B,\ 0 \leqslant s \leqslant t < \infty. \tag{2.1}$$

Note that $Dv(0, s, t) = e^{s-t} I$, $0 \leqslant s \leqslant t < \infty$, in view of the normalization of $f(z, t)$.

From the equality (2.1) and the univalence of $f(\cdot,t)$, $t \geqslant 0$, we deduce the semigroup property of the transition mapping $v(z,s,t)$, i.e.,

$$v(z,s,u) = v(v(z,s,t),t,u), \quad z \in B,\ 0 \leqslant s \leqslant t \leqslant u < \infty. \tag{2.2}$$

We next give simple examples of Loewner chains generated by starlike mappings.

EXAMPLE 2.2. (i) Let $f \in S^*(B)$ and $f(z,t) = e^t f(z)$ for $z \in B$ and $t \geqslant 0$. The simple geometry of a starlike domain makes it easy to see that $f(z,t)$ is a Loewner chain [25].

(ii) For example, if $f : B \to \mathbb{C}^n$ is given by

$$f(z) = \left(\frac{z_1}{(1-z_1)^2}, \dots, \frac{z_n}{(1-z_n)^2}\right), \quad z \in B,$$

then clearly $f \in S^*(B)$ since each component $f_j(z_j) = z_j/(1-z_j)^2$ is starlike, $j = 1,\dots,n$. Hence $f(z,t) = e^t f(z)$ is a Loewner chain.

(iii) Let $n = 2$ and B be the unit ball in $\mathbb{C}^2$. Assume $|a| \leqslant 3\sqrt{3}/2$ and let $f(z) = (z_1 + az_2^2, z_2)$ for $z = (z_1,z_2) \in B$. Then f is starlike by [31, Example 3] (see also [29, Example 5]). Consequently, $f(z,t) = e^t f(z)$ is a Loewner chain.

(iv) Pfaltzgraff and Suffridge [25] proved that if f is close-to-starlike with respect to $g \in S^*(B)$ then

$$f(z,t) = f(z) + (e^t - 1)g(z) = e^t z + \cdots$$

is a Loewner chain. We will return to this idea in Example 2.9. This result enables one to deduce that for each r, $0 < r < 1$, the complement of $f(B_r)$ is the union of the nonintersecting rays

$$L(t,z;r) = \bigl\{ f(z) + tg(z) :\ t \geqslant 0,\ z \text{ fixed},\ \|z\| = r \bigr\},$$

which generalizes one of the familiar one variable geometric characterizations of close-to-starlike domains (cf. [25]).

If $f : B \times [0,\infty) \to \mathbb{C}^n$ is a mapping which is holomorphic on B for fixed t (not necessarily biholomorphic) and satisfies the second condition in Definition 2.1, then we say that $f(z,t)$ is a subordination chain. Thus $f(z,t)$ is a Loewner chain if and only if it is a subordination chain which is biholomorphic on B for fixed t and satisfies the normalization $f(0,t) = 0$ and $Df(0,t) = e^t I$ for $t \geqslant 0$.

A fundamental role in the study of the Loewner differential equation in higher dimensions (as well as in the study of certain classes of univalent mappings) is played by the n-dimensional version of the Carathéodory set

$$\mathcal{M} = \bigl\{ h \in H(B) :\ h(0) = 0,\ Dh(0) = I,\ \operatorname{Re}\langle h(z), z\rangle > 0,\ z \in B \setminus \{0\} \bigr\}.$$

It is well known that in the case of one variable this set is compact. Recently, Graham, Hamada and Kohr [10] have established the same result in the case of several complex variables. In fact, they proved that for each $r \in (0,1)$, there is some $M = M(r) \leqslant 4r/(1-r)^2$ such that $\|h(z)\| \leqslant M(r)$ for $\|z\| \leqslant r$ and $h \in \mathcal{M}$.

The basic existence theorem for the Loewner differential equation on B is due to Pfaltzgraff [23, Theorem 2.1]. In the original paper the author imposed a boundedness assumption on the mapping $h(z,t)$. In view of the compactness of the set $\mathcal{M}$, this assumption is not needed. Thus Pfaltzgraff's result can be simplified, as follows:

THEOREM 2.3. *Let $h : B \times [0,\infty) \to \mathbb{C}^n$ satisfy the following assumptions:*
(i) $h(\cdot,t) \in \mathcal{M}$, $t \geqslant 0$;
(ii) $h(z,\cdot)$ is measurable on $[0,\infty)$ for each $z \in B$.

Then there is a unique locally absolutely continuous solution $v(t) = v(z,s,t)$ of the initial value problem

$$\frac{\partial v}{\partial t} = -h(v,t), \text{ a.e. } t \geqslant s, \; v(s) = z. \tag{2.3}$$

The mapping $v(z,s,t) = e^{s-t}z + \dots$ is a biholomorphic Schwarz mapping on B and is Lipschitz continuous in $t \geqslant s$ locally uniformly with respect to $z \in B$.

As in one variable, the Schwarz mappings obtained by solving the initial value problem in the above result can be used to construct Loewner chains. Indeed, we have (see [28,Theorems 2 and 3]; see also [10] and [12])

THEOREM 2.4. *Under the hypotheses of Theorem 2.3, the limit*

$$\lim_{t\to\infty} e^t v(z,s,t) = f(z,s) \tag{2.4}$$

exists locally uniformly on B for each $s \geqslant 0$, $f(\cdot,s)$ is biholomorphic on B, $f(0,s) = 0$ and $Df(0,t) = e^s I$ and $f(z,s) = f(v(z,s,t),t)$, $z \in B$, $0 \leqslant s \leqslant t < \infty$. Thus $f(z,t)$ is a Loewner chain. Furthermore $f(z,\cdot)$ is a locally Lipschitz continuous function on $[0,\infty)$ locally uniformly with respect to $z \in B$, and for a.e. $t \geqslant 0$,

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t), \; \forall\, z \in B. \tag{2.5}$$

Moreover, $\{e^{-t}f(z,t)\}_{t\geqslant 0}$ is a normal family on B.

The Loewner chain given by (2.4) may be called the *canonical solution* of the Loewner differential equation (2.5).

An important question is to find conditions under which a solution of the equation (2.5) is a Loewner chain. It is sufficient to have $\{e^{-t}f(z,t)\}_{t\geqslant 0}$ a normal family on B. This result is due to Pfaltzgraff [23, Theorem 2.3]. Combining Pfaltzgraff's result with [28, Theorem 6], we deduce that in this case the mapping $f(z,t)$ which solves the differential equation (2.5) coincides with the mapping defined by (2.4).

THEOREM 2.5. *Let $f : B\times[0,\infty) \to \mathbb{C}^n$ be such that $f(\cdot,t) \in H(B)$, $f(0,t) = 0$, $Df(0,t) = e^t I$ for each $t \geqslant 0$, and $f(z,\cdot)$ is locally absolutely continuous on $[0,\infty)$ locally uniformly with respect to $z \in B$. Let $h : B \times [0,\infty) \to \mathbb{C}^n$ satisfy the assumptions (i) and (ii) of Theorem 2.3. Assume $f(z,t)$ satisfies the differential equation (2.5) for almost all $t \geqslant 0$ and for all $z \in B$.*

Further, assume there exists an increasing sequence $\{t_m\}_{m\in\mathbb{N}}$ such that $t_m > 0$, $t_m \to \infty$ and

$$\lim_{m\to\infty} e^{-t_m} f(z,t_m) = F(z)$$

locally uniformly on B. Then $f(z,t)$ is a Loewner chain and for each $s \geqslant 0$,

$$\lim_{t\to\infty} e^t v(z,s,t) = f(z,s)$$

locally uniformly on B, where $v(t) = v(z,s,t)$ is the unique locally absolutely continuous solution of the initial value problem (2.3).

REMARK 2.6. Graham, Kohr and Kohr [14] proved that if $f(z,t)$ is a Loewner chain, then $f(z,\cdot)$ is a locally Lipschitz continuous function on $[0,\infty)$ locally uniformly with respect to $z \in B$, and thus $(\partial f/\partial t)(z,t)$ exists for almost all $t \in [0,\infty)$. Moreover, an application of Vitali's theorem in several variables yields that the null set is independent of z and $(\partial f/\partial t)(\cdot,t) \in H(B)$ for almost all $t \geqslant 0$ (see [14] and [12]).

Combining this result with [10, Theorem 1.10], we deduce that any Loewner chain satisfies the Loewner differential equation (2.5) (cf. [14, Theorem 2.2]; see also [12]).

THEOREM 2.7. *Let $f(z,t)$ be a Loewner chain. Then there is a mapping $h = h(z,t)$ such that $h(\cdot,t) \in \mathcal{M}$ for each $t \geqslant 0$, $h(z,\cdot)$ is measurable on $[0,\infty)$ for each $z \in B$, and for almost all $t \geqslant 0$,*

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t),\ \forall\ z \in B.$$

Moreover, if there exists a sequence $\{t_m\}_{m\in\mathbb{N}}$ such that $t_m > 0$, $t_m \to \infty$, and

$$\lim_{m\to\infty} e^{-t_m} f(z,t_m) = F(z) \tag{2.6}$$

locally uniformly on B, then $f(z,s) = \lim_{t\to\infty} e^t v(z,s,t)$ locally uniformly on B for $s \geqslant 0$, where $v(t) = v(z,s,t)$ is the unique solution of the initial value problem

$$\frac{\partial v}{\partial t} = -h(v,t),\ \text{a.e. } t \geqslant s,\ v(s) = z,$$

for all $z \in B$.

REMARK 2.8. In the case of one variable, the condition (2.6) is always satisfied, as a consequence of the growth theorem for the class S. However, in higher dimensions (2.6) need not be satisfied, as shown in Example 3.7.

As an application of Theorem 2.5, we prove a particular case of [25, Theorem 1] concerning a characterization of close-to-starlike mappings by Loewner chains. For the mappings of the special form considered in this example, one can give a simpler proof of the characterization (see [12]).

EXAMPLE 2.9. Let ϕ and ψ be holomorphic functions from B into $\mathbb{C}$ such that $\phi(0) = \psi(0) = 1$ and $\phi(z) \neq 0$, $z \in B \setminus \{0\}$. Also let $f, g \in H(B)$ be given by $f(z) = z\phi(z)$, $g(z) = z\psi(z)$, $z \in B$, and assume that f is locally biholomorphic on B and close-to-starlike relative to $g \in S^*(B)$. Then

$$F(z,t) = f(z) + (e^t - 1)g(z), \quad z \in B,\ t \geqslant 0,$$

is a Loewner chain.

PROOF. A short computation (see also [26]) yields that g is starlike if and only if

$$\operatorname{Re}\left\{\frac{\psi(z)+D\psi(z)z}{\psi(z)}\right\}>0, \quad z\in B.$$

On the other hand, it is not difficult to see that the assumption that f be close-to-starlike with respect to the mapping g is equivalent to

$$\operatorname{Re}\left\{\frac{\phi(z)+D\phi(z)z}{\psi(z)}\right\}>0, \quad z\in B.$$

Taking into account the above inequalities, we obtain

$$\operatorname{Re}\left\{e^{-t}\frac{\phi(z)+D\phi(z)z}{\psi(z)}+(1-e^{-t})\frac{\psi(z)+D\psi(z)z}{\psi(z)}\right\}>0,$$

for all $z\in B$ and $t\geqslant 0$. Now let $h:B\times[0,\infty)\to\mathbb{C}^n$ be given by

$$h(z,t)=z\left\{e^{-t}\frac{\phi(z)+D\phi(z)z}{\psi(z)}+(1-e^{-t})\frac{\psi(z)+D\psi(z)z}{\psi(z)}\right\}^{-1}.$$

Then $h(\cdot,t)\in\mathcal{M}$, $t\geqslant 0$, $h(z,\cdot)$ is measurable on $[0,\infty)$, $z\in B$, and

$$\frac{\partial F}{\partial t}(z,t)=DF(z,t)h(z,t), \quad z\in B, \quad t\geqslant 0.$$

On the other hand, it is clear that $\lim_{t\to\infty} e^{-t}F(z,t)=g(z)$ locally uniformly on B. Hence in view of Theorem 2.5 we conclude that $F(z,t)$ is a Loewner chain. □

2.2. Quasiconformal extension results. We now discuss some applications of Theorem 2.5 in the study of univalence criteria and quasiconformal extension results.

DEFINITION 2.10. Let G be a domain in $\mathbb{C}^n$ and let $f:G\to\mathbb{C}^n$ be a holomorphic mapping. We say that f is K-quasiregular, $K\geqslant 1$, if

$$\|Df(z)\|^n\leqslant K|\det Df(z)|, \quad z\in G.$$

In addition, f is called quasiregular if there is $K\geqslant 1$ such that f is K-quasiregular.

A quasiregular (K-quasiregular) biholomorphic mapping is also called quasiconformal (K-quasiconformal).

One of the important applications of Theorem 2.5 is the following univalence condition and quasiconformal extension result due to Pfaltzgraff [23,24], which generalizes to higher dimensions a well known one variable result due to Becker [2].

THEOREM 2.11. *Let $c\leqslant 1$ and $f:B\to\mathbb{C}^n$ be a normalized locally biholomorphic mapping such that*

$$(1-\|z\|^2)\|[Df(z)]^{-1}D^2f(z)(z,\cdot)\|\leqslant c, \quad z\in B. \tag{2.7}$$

Then f is biholomorphic on B. In addition, if $c<1$ and f is quasiregular on B then f extends to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself.

We note that the idea of the proof is to show that under the stated assumptions,

$$f(z,t) = f(ze^{-t}) + (e^t - e^{-t})Df(ze^{-t})(z), \quad z \in B,\, t \geqslant 0,$$

is a Loewner chain which can be extended continuously to $\overline{B}$, and

$$F(z) = \begin{cases} f(z,0), & z \in \overline{B} \\ f\left(\frac{z}{\|z\|}, \log\|z\|\right), & z \notin \overline{B} \end{cases}$$

is the quasiconformal extension to $\mathbb{R}^{2n}$ of the mapping f.

REMARK 2.12. The constant $c = 1$ in Theorem 2.11 is sharp, i.e., f need not be univalent if $c > 1$. Let $f : B \to \mathbb{C}^n$ be given by $f(z) = (f_1(z_1), \ldots, f_n(z_n))$ for $z = (z_1, \ldots, z_n) \in B$, where $f_j(z_j)$ is a normalized locally univalent function on U, $j = 1, \ldots, n$. Then it is clear that f is a normalized locally biholomorphic mapping on B. Assume f satisfies (2.7). Note that f is biholomorphic on B if and only if each component f_j is univalent on U for $j = 1, \ldots, n$. On the other hand, a simple computation yields for $z = (z_1, 0, \ldots, 0) \in B$ that

$$(1 - \|z\|^2)\|[Df(z)]^{-1}D^2f(z)(z,\cdot)\| = (1 - |z_1|^2)\left|\frac{z_1 f_1''(z_1)}{f_1'(z_1)}\right| \leqslant c,$$

and if $c > 1$ then f_1 is not necessarily univalent on U by a result of Becker and Pommerenke [5]. This implies that the constant $c = 1$ is sharp, as claimed.

Recently, Hamada and Kohr [18] obtained the following quasiconformal extension result for a Loewner chain $f(z,t)$ under the assumption that $f(\cdot,0)$ is quasiconformal on B. Several applications of this result were obtained in [18]. See also [12].

THEOREM 2.13. *Let $f(z,t)$ be a Loewner chain which satisfies the assumptions of Theorem 2.5. Assume the following conditions hold:*

(i) $\|Df(z,t)\| \leqslant \dfrac{M(t)}{(1-\|z\|)^\alpha}$, *$z \in B$, $t \geqslant 0$, where $M(t)$ is a locally bounded function with respect to $t \in [0,\infty)$ and α is a constant with $0 \leqslant \alpha < 1$;*

(ii) there exists a constant $c_1 > 0$ such that $c_1\|z\|^2 \leqslant \mathrm{Re}\langle h(z,t), z\rangle$ for $z \in B \setminus \{0\}$, $t \geqslant 0$;

(iii) there exists a constant $c_2 > 0$ such that $\|h(z,t)\| \leqslant c_2$ for $z \in B$ and $t \geqslant 0$;

(iv) there is $K \geqslant 1$ such that $f(\cdot,t)$ is K-quasiconformal for each $t \geqslant 0$.

Then $f(\cdot,t)$ has a continuous extension to $\overline{B}$ (again denoted by $f(\cdot,t)$) for each $t \geqslant 0$, and

$$F(z) = \begin{cases} f(z,0), & z \in \overline{B} \\ f\left(\frac{z}{\|z\|}, \log\|z\|\right), & z \notin \overline{B} \end{cases}$$

is a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself.

Hamada and Kohr [18] gave the following example which shows that the condition (ii) in Theorem 2.13 cannot be omitted.

EXAMPLE 2.14. Let $n = 2$ and B be the unit ball in $\mathbb{C}^2$. Also let $f : B \to \mathbb{C}^2$ be given by

$$f(z) = (z_1 + az_2^2, z_2), \quad z = (z_1, z_2) \in B,$$

where $a = 3\sqrt{3}/2$. According to Example 2.2 (iii), $f(z,t) = e^t f(z)$ is a Loewner chain which satisfies the assumptions of Theorem 2.5. Moreover, since $Df(z,t) = e^t Df(z)$ and f is a polynomial, the condition (i) is satisfied. A simple computation yields that $h(z,t) = [Df(z)]^{-1} f(z) = (z_1 - az_2^2, z_2)$, and thus the condition (iii) is satisfied. Since $\|Df(z)\|$ is uniformly bounded in B, $\det Df(z) = 1$ and $Df(z,t) = e^t Df(z)$, the condition (iv) is satisfied. Since $\langle h(z,t), z\rangle = \|z\|^2 - a\overline{z}_1 z_2^2 \to 0$ as $(z_1, z_2) \to (1/\sqrt{3}, \sqrt{2/3})$, the condition (ii) is not satisfied. On the other hand, we can show that the mapping

$$F(z) = \begin{cases} f(z,0), & z \in \overline{B} \\ f\left(\frac{z}{\|z\|}, \log\|z\|\right), & z \notin \overline{B} \end{cases}$$

is not quasiconformal. Indeed, if $z \notin \overline{B}$ then $F(z) = (z_1 + az_2^2/\|z\|, z_2)$. By a direct computation, we have $\det DF(x,y) = 0$ for $z_1 = k/\sqrt{3}$, $z_2 = \sqrt{2}k/\sqrt{3}$ with $k > 1$, but $\|DF(x,y)\| \neq 0$. This implies that F is not quasiconformal.

We remark that Theorem 2.13 can be used to give another proof of Theorem 2.11, and also to prove the following simple sufficient condition for univalence and quasiconformal extension to $\mathbb{C}^n$ due to Brodskii [6] (see [18]).

THEOREM 2.15. *Let $c \in [0,1)$ and $f : B \to \mathbb{C}^n$ be a normalized holomorphic mapping such that*

$$\|Df(z) - I\| \leqslant c, \quad z \in B.$$

Then f is quasiconformal on B and extends to a quasiconformal homeomorphism of $\mathbb{R}^{2n}$ onto itself.

In this case the idea of the proof is to show that if we let

$$f(z,t) = f(ze^{-t}) + (e^t - e^{-t})z = e^t z + \dots, \quad z \in B,\ t \geqslant 0,$$

then $f(z,t)$ is a Loewner chain which satisfies the assumptions of Theorem 2.13.

Other applications of Theorem 2.13 can be found in [18] (see also [12]).

3. Kernel convergence and Loewner chains

In this section we consider the relation between Loewner chains which satisfy a growth condition in t and kernel convergence. In the case of one variable, kernel convergence was introduced and studied by Carathéodory [7]. He proved the fundamental convergence theorem that has turned out to be an extremely important tool in univalent function theory and the theory of conformal mappings, for example, in the study of Loewner chains and the Loewner differential equation. We note that Gehring [9] defined the notions of kernel convergence for domains in $\mathbb{R}^3$ and obtained an analogue of the Carathéodory kernel convergence theorem for K-quasiconformal mappings in $\mathbb{R}^3$.

We begin this section with the following definitions (see [21]):

DEFINITION 3.1. Let $\{G_k\}_{k\in\mathbb{N}}$ be a sequence of domains in $\mathbb{C}^n$ such that $0 \in G_k$, $k \in \mathbb{N}$. If 0 is an interior point of $\bigcap_{k\in\mathbb{N}} G_k$, we define the *kernel* G of $\{G_k\}_{k\in\mathbb{N}}$ to be the largest domain containing 0 with the property that if K is a compact subset of G, then there is a positive integer k_0 such that $K \subset G_k$ for $k \geqslant k_0$ (in other words, K is contained in all but finitely many of the sets G_k). If 0 is not an interior point of $\bigcap_{k\in\mathbb{N}} G_k$, we define the kernel to be $\{0\}$.

Let $\mathcal{G}$ be the set of all domains Ω in $\mathbb{C}^n$ such that $0 \in \Omega$ and each compact set K of Ω is contained in all but finitely many of the sets G_k. Suppose that 0 is an interior point of $\bigcap_{k\in\mathbb{N}} G_k$. An application of the Heine-Borel theorem shows that if $D = \bigcup_{\Omega\in\mathcal{G}} \Omega$, then D belongs to $\mathcal{G}$, and it is clear that no larger domain can belong to $\mathcal{G}$. This establishes the existence of the kernel of any sequence of domains $G_1, \dots, G_k, \dots$ such that 0 is an interior point of $\bigcap_{k\in\mathbb{N}} G_k$.

DEFINITION 3.2. We say that $\{G_k\}_{k\in\mathbb{N}}$ kernel converges to G, and write $G_k \to G$, if each subsequence of $\{G_k\}_{k\in\mathbb{N}}$ has the same kernel G.

It is not difficult to see that if $\{G_k\}_{k\in\mathbb{N}}$ is an increasing sequence of domains in $\mathbb{C}^n$, i.e., $G_k \subseteq G_{k+1}$, $k \in \mathbb{N}$, such that $0 \in G_k$, $k \in \mathbb{N}$, then $G = \bigcup_{k\in\mathbb{N}} G_k$ is the kernel of $\{G_k\}_{k\in\mathbb{N}}$ and $\{G_k\}_{k\in\mathbb{N}}$ converges to G in the sense of kernel convergence.

Let $S^c(B)$ be a compact subset of $S(B)$. Then it is not difficult to deduce that for each $r \in (0,1)$, there exist $m = m(r) > 0$ and $M = M(r) > 0$ such that

$$m(r) \leqslant \|f(z)\| \leqslant M(r), \quad \|z\| = r, \quad \forall f \in S^c(B). \tag{3.1}$$

REMARK 3.3. It is known that in the case of one variable the class S is compact. However, in several variables, the class $S(B)$ is not compact and for any positive functions $m(r) \leqslant M(r)$, $r \in (0,1)$, there exist mappings f in $S(B)$ which do not satisfy the above growth result. Thus $S^c(B) \subsetneqq S(B)$ in dimension $n \geqslant 2$ (see e.g., [10], [12]).

Indeed, if g is an arbitrary holomorphic function on the unit disc such that $g(0) = g'(0) = 0$, then the mapping f given by

$$f(z_1, z_2) = \big(z_1, z_2 + g(z_1)\big), \quad (z_1, z_2) \in B,$$

belongs to $S(B)$.

The next result is an analogue of the Carathéodory kernel convergence theorem on the convergence of conformal mappings of one variable, for biholomorphic mappings which satisfy the growth result (3.1) (see [21]).

THEOREM 3.4. *Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of biholomorphic mappings on B such that $f_k(0) = 0$, $Df_k(0) = \alpha_k I$, where $\alpha_k > 0$, $k \in \mathbb{N}$. Assume $f_k/\alpha_k \in S^c(B)$, $k \in \mathbb{N}$. Let $G_k = f_k(B)$, $k \in \mathbb{N}$, and let G be the kernel of $\{G_k\}_{k\in\mathbb{N}}$. Then $\{f_k\}_{k\in\mathbb{N}}$ converges locally uniformly on B to a mapping f if and only if $G_k \to G \neq \mathbb{C}^n$. In the case of convergence, either $f \equiv 0$ and $G = \{0\}$, or else f is biholomorphic on*

B, $f/\alpha \in S^c(B)$ where $\alpha = \lim_{k\to\infty} \alpha_k$, and $f(B) = G$. In the latter case, $f_k^{-1} \to f^{-1}$ locally uniformly on G.

Next, let $S^c_{1/4}(B)$ be the subset of $S(B)$ consisting of all mappings in $S(B)$ which satisfy the 1/4-growth result. That is, $f \in S^c_{1/4}(B)$ if and only if $f \in S(B)$ and

$$\frac{\|z\|}{(1+\|z\|)^2} \leqslant \|f(z)\| \leqslant \frac{\|z\|}{(1-\|z\|)^2},\ z \in B.$$

It is known that if $f \in S^*(B)$ then f satisfies the above 1/4-growth result (see [1]). In other words, $S^*(B) \subset S^c_{1/4}(B)$. Moreover, if $f \in K(B)$ then f satisfies the following growth result (see [8]):

$$\frac{\|z\|}{1+\|z\|} \leqslant \|f(z)\| \leqslant \frac{\|z\|}{1-\|z\|}, \quad z \in B.$$

Taking into account Theorem 3.4, Kohr [21] obtained the following connection between kernel convergence and convergence on compact sets of mappings in $S^*(B)$ and $K(B)$ respectively.

Corollary 3.5. *Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence of mappings in $S^*(B)$ (respectively in $K(B)$) and let $G_k = f_k(B)$. Also let G be the kernel of $\{G_k\}_{k\in\mathbb{N}}$. Then $\{f_k\}_{k\in\mathbb{N}}$ converges locally uniformly on B to a mapping f if and only if $G_k \to G \neq \mathbb{C}^n$. Moreover, $f \in S^*(B)$ (respectively $f \in K(B)$), $G = f(B)$, and $f_k^{-1} \to f^{-1}$ locally uniformly on G.*

For each $t \geqslant 0$, let $g_t(z) = g(z,t)$ be a biholomorphic mapping of B onto a domain $G(t)$ such that $g_t(0) = 0$, $Dg_t(0) = \alpha(t)I$, where $\alpha(t) > 0$ and $g_t/\alpha(t) \in S^c_{1/4}(B)$, $t \geqslant 0$. Also let $\alpha_0 = \alpha(0)$. Further, assume that the family $\{G(t)\}_{t\geqslant 0}$ satisfies the conditions

$$G(s) \subsetneqq G(t), \quad 0 \leqslant s < t < \infty \tag{3.2}$$

$$G(t_k) \to G(t_0) \text{ if } t_k \to t_0 < \infty, \text{ and } G(t_k) \to \mathbb{C}^n \text{ if } t_k \to \infty. \tag{3.3}$$

The convergence in question is kernel convergence. Then we obtain the following result (see [21]; compare with [27, Chapter 6]), which provides a geometric characterization of certain Loewner chains.

Theorem 3.6. *(i) Let g_t and $G(t)$ satisfy the conditions in the previous paragraph.*

(a) Then α is a strictly increasing function, continuous and $\alpha(t) \to \infty$ as $t \to \infty$.

(b) If $\beta(t) = \log[\alpha(t)/\alpha_0]$ then $f(z,t) = \alpha_0^{-1} g(z, \beta^{-1}(t))$ is a Loewner chain and $f(B,t) = \alpha_0^{-1} G(\beta^{-1}(t))$. Further, $\{e^{-t} f(z,t)\}_{t\geqslant 0}$ is a normal family on B.

(ii) Conversely, let $f(z,t)$ be a Loewner chain such that $\{e^{-t} f(z,t)\}_{t\geqslant 0}$ is a normal family on B. Also let $G(t) = f(B,t)$, $t \geqslant 0$. Then the family $\{G(t)\}_{t\geqslant 0}$ satisfies the conditions (3.2) and (3.3).

In the case of one variable, if $f(z,t)$ is a Loewner chain, then the function $e^{-t} f(\cdot,t)$ is in the class S, and thus satisfies the 1/4 growth result for each $t \geqslant 0$.

This implies that $\{e^{-t}f(z,t)\}_{t\geqslant 0}$ is a normal family on the unit disc U. However, in higher dimensions such a result is no longer true, as shown in the following example due to Graham, Hamada and Kohr [10].

EXAMPLE 3.7. First, we remark that if $f(z,t)$ is a Loewner chain and $\Phi : \mathbb{C}^n \to \mathbb{C}^n$ is an entire normalized biholomorphic mapping, not the identity, then it is easy to see that $(\Phi \circ f)(z,t)$ is also a Loewner chain. Such a mapping Φ can be either an automorphism of $\mathbb{C}^n$ or (when $n \geqslant 2$) a Fatou-Bieberbach map, i.e., a biholomorphic mapping of $\mathbb{C}^n$ onto a proper subset of $\mathbb{C}^n$.

Now, let $n = 2$ and $\Phi(z) = (z_1, z_2 + z_1^2)$, $z = (z_1, z_2) \in \mathbb{C}^2$. Then Φ is a normalized automorphism of $\mathbb{C}^2$. Also if

$$f(z,t) = \left(\frac{e^t z_1}{(1-z_1)^2}, \frac{e^t z_2}{(1-z_2)^2}\right), \quad z \in B,\ t \geqslant 0,$$

then $f(z,t)$ is a Loewner chain by Example 2.2 (ii), and thus

$$\Phi(f(z,t)) = \left(\frac{e^t z_1}{(1-z_1)^2}, \frac{e^t z_2}{(1-z_2)^2} + \frac{e^{2t} z_1^2}{(1-z_1)^4}\right), \quad z \in B, \quad t \geqslant 0,$$

is also a Loewner chain. However, since for each $r \in (0,1)$,

$$\|\Phi(f(r,0))\| = \frac{r}{(1-r)^2}\sqrt{1 + \frac{r^2}{(1-r)^4}} > \frac{r}{(1-r)^2},$$

the family $\{e^{-t}\Phi(f(z,t))\}_{t\geqslant 0}$ is not normal.

4. Solutions of the generalized Loewner differential equation

The main result of this section is a generalization of a one-variable theorem of Becker (see [3, Satz2]). It is easy to see that the Loewner chain $(\Phi \circ f)(z,t)$ in Example 3.7 satisfies the same partial differential equation as the Loewner chain $f(z,t)$ (see [10]). A natural question is the following: Let $h(z,t)$ satisfy the assumptions (i) and (ii) in Theorem 2.3. Then does any Loewner chain $g(z,t)$ which satisfies the partial differential equation

$$\frac{\partial g}{\partial t}(z,t) = Dg(z,t)h(z,t),\ a.e.\ t \geqslant 0,\ \forall\ z \in B,$$

have the form $g(z,t) = (\Phi \circ f)(z,t)$, where $f(z,t)$ is the canonical solution and Φ is a normalized univalent mapping from $\mathbb{C}^n$ into $\mathbb{C}^n$? Actually, one can prove a slightly more general result.

In studying the analogous question in one variable, it is useful to consider solutions of the Loewner differential equation which, for fixed t, are holomorphic on a punctured disc (rather than a disc) centered at 0. However, in higher dimensions point singularities of holomorphic functions are removable, so we shall assume that the solutions are holomorphic at 0 for fixed t. As in [3], we allow the radius of the ball on which the solution is initially defined in z to vary with t; this is potentially useful for applications (see [16]).

THEOREM 4.1. *Let $h(z,t)$ satisfy the assumptions (i) and (ii) of Theorem 2.3 and let $f(z,t)$ be given by (2.4). Also let $g(z,t)$ be a mapping such that for each $t \geqslant 0$, $g(\cdot,t) \in H(B_{r(t)})$ where $r(t) \in (0,1]$ and $\limsup_{t\to\infty} e^t r(t) = \infty$.*

Assume there exist two positive functions ρ and δ on $[0,\infty)$ such that $\rho(t) < 1$, $t \geqslant 0$, and for each $t_0 \geqslant 0$ the following conditions hold:

(a) $r(t) \geqslant \rho(t_0)$ for $t \in E_{\delta(t_0)} = [t_0 - \delta(t_0), t_0 + \delta(t_0)] \cap [0,\infty)$ (thus $g(\cdot,t)$ is holomorphic on $B_{\rho(t_0)}$ for $t \in E_{\delta(t_0)}$);

(b) $g(z,\cdot)$ is absolutely continuous on $E_{\delta(t_0)}$ for $z \in B_{\rho(t_0)}$, and for almost all $t \in E_{\delta(t_0)}$,

$$\frac{\partial g}{\partial t}(z,t) = Dg(z,t)h(z,t), \quad z \in B_{\rho(t_0)}.$$

Then $g(z,t)$ extends to a subordination chain on $B \times [0,\infty)$, again denoted by $g(z,t)$, and there exists a mapping $\Phi : \mathbb{C}^n \to \mathbb{C}^n$ which is holomorphic such that

$$g(z,t) = (\Phi \circ f)(z,t), \quad z \in B,\ t \geqslant 0.$$

Moreover, $g(z,t)$ is a univalent subordination chain if and only if Φ is univalent, i.e., an automorphism of $\mathbb{C}^n$ or a Fatou-Bieberbach map.

We next present the following particular cases of Theorem 4.1 (see [16]; in the case of one variable, see [3,4]):

COROLLARY 4.2. *Let $r \in (0,1]$ and $g = g(z,t) : B_r \times [0,\infty) \to \mathbb{C}^n$ be a mapping such that $g(\cdot,t) \in H(B_r)$, $t \geqslant 0$ and $g(z,\cdot)$ is locally absolutely continuous on $[0,\infty)$ locally uniformly with respect to $z \in B_r$. Let $h = h(z,t) : B \times [0,\infty) \to \mathbb{C}^n$ satisfy the assumptions (i) and (ii) of Theorem 2.3 and let $f(z,t)$ be given by (2.4). Assume that $g(z,t)$ satisfies the Loewner differential equation*

$$\frac{\partial g}{\partial t}(z,t) = Dg(z,t)h(z,t) \ \text{ a.e. } t \geqslant 0,\ \forall\ z \in B_r.$$

Then $g(z,t)$ can be extended to a subordination chain on $B \times [0,\infty)$, again denoted by $g(z,t)$, and there exists a holomorphic mapping Φ of $\mathbb{C}^n$ into $\mathbb{C}^n$ such that $g(z,t) = (\Phi \circ f)(z,t)$ for $z \in B$ and $0 \leqslant t < \infty$. Moreover, $g(z,t)$ is a univalent subordination chain if and only if Φ is univalent, i.e., an automorphism of $\mathbb{C}^n$ or a Fatou-Bieberbach map.

COROLLARY 4.3. *Let $g(z,t)$ and $h(z,t)$ satisfy the assumptions of Theorem 4.1. Also let $f(z,t)$ be given by (2.4) and let*

$$c_k(t) = \frac{1}{k!} D^k g(0,t), \quad t \geqslant 0,\ k \geqslant 0.$$

Assume that

$$\liminf_{t\to\infty} e^{-kt} \|c_k(t)\| = 0, \quad k \geqslant 2. \tag{4.1}$$

Then $g(z,t) = c_0(0) + c_1(0) f(z,t)$, $z \in B$, $t \geqslant 0$.

REMARK 4.4. As we have seen in this section, in higher dimensions univalent solutions of the generalized Loewner differential equation (2.5) need not be unique. This is a basic difference between the Loewner theory in one and higher dimensions.

For example, if $f(z,t)$ is a Loewner chain which satisfies (2.5) and $\Phi : \mathbb{C}^n \to \mathbb{C}^n$ is a normalized biholomorphic mapping, not the identity, then $g(z,t) = (\Phi \circ f)(z,t)$ is also a Loewner chain which satisfies (2.5).

5. Loewner chains and the Loewner differential equation in reflexive complex Banach spaces

In this section we shall present some recent results related to Loewner chains and the Loewner differential equation in reflexive complex Banach spaces. These results generalize to reflexive complex Banach spaces the results presented in the second section. To this end, let X be a complex Banach space with respect to a norm $\|\cdot\|$. Let B_r be the open ball centered at zero and of radius r, and let B be the open unit ball in X. If A is a subset of X, let $\overline{A}$ denote its closure. We denote by $L(X,Y)$ the set of continuous linear operators from X into Y with the standard operator norm. Let I be the identity in $L(X,X)$. Let Ω be a domain in X and $f : \Omega \to X$ be a mapping. We say that f is holomorphic if for each $z \in \Omega$ there is a mapping $Df(z) \in L(X,X)$ such that

$$\lim_{h\to 0} \frac{\|f(z+h) - f(z) - Df(z)(h)\|}{\|h\|} = 0.$$

Let $H(\Omega)$ be the set of holomorphic mappings from Ω into X. A mapping $f \in H(\Omega)$ is called biholomorphic if $f(\Omega)$ is a domain, and the inverse f^{-1} exists and is holomorphic on $f(\Omega)$. A mapping $f \in H(\Omega)$ is called locally biholomorphic if each $z \in \Omega$ has a neighbourhood V such that $f|_V$ is biholomorphic. By the inverse function theorem, f is locally biholomorphic if and only if $Df(z)$ has a bounded inverse at each $z \in \Omega$. A holomorphic and injective mapping on Ω will be called univalent. (Such a mapping need not be biholomorphic.) A mapping $f \in H(B)$ is called normalized if $f(0) = 0$ and $Df(0) = I$.

For $z \in X \setminus \{0\}$, we define $T(z) = \{l_z \in L(X,\mathbb{C}) : \ l_z(z) = \|z\|, \ \|l_z\| = 1\}$. It follows from the Hahn-Banach theorem that $T(z) \neq \emptyset$. Let

$$\mathcal{M} = \{f \in H(B) : \ f \text{ is normalized}, \ \mathrm{Re}[l_z(f(z))] > 0, \ z \in B \setminus \{0\}, \ l_z \in T(z)\}.$$

If $f, g \in H(B)$, we say that f is subordinate to g, and write $f \prec g$, if there is a Schwarz mapping v (i.e., $v \in H(B)$, $v(0) = 0$, $\|v(z)\| < 1$, $z \in B$) such that $f(z) = g(v(z))$, $z \in B$.

Definition 5.1. A mapping $f : B \times [0,\infty) \to X$ is called a Loewner chain if
(i) $f(\cdot,t)$ is univalent on B, $f(0,t) = 0$, $Df(0,t) = e^t I$, $t \geqslant 0$;
(ii) $f(z,s) \prec f(z,t)$, $z \in B$, $0 \leqslant s \leqslant t < \infty$.

As in the finite dimensional case, the latter condition is equivalent to the fact that there is a Schwarz mapping $v = v(z,s,t)$, called the transition mapping associated to $f(z,t)$, such that

$$f(z,s) = f(v(z,s,t),t), \quad z \in B, \ 0 \leqslant s \leqslant t < \infty. \tag{5.1}$$

Moreover, since $f(\cdot,t)$ is univalent, the mapping v is uniquely determined by (5.1), and furthermore it is univalent on B and satisfies the relations $Dv(0,s,t) = e^{s-t} I$ and

$$v(z,s,u)=v(v(z,s,t),t,u), \quad z\in B,\ 0\leqslant s\leqslant t\leqslant u<\infty.$$

Throughout this section we assume that X is a reflexive complex Banach space. The following result is due to Hamada and Kohr [19].

THEOREM 5.2. *Let $h: B\times[0,\infty)\to X$ be a mapping which satisfies the following conditions:*

(i) $h(\cdot,t)\in\mathcal{M}$ for $t\geqslant 0$;

(ii) $h(z,\cdot)$ is strongly measurable on $[0,\infty)$ for each $z\in B$.

Then for each $s\geqslant 0$ and $z\in B$, the initial value problem

$$\frac{\partial v}{\partial t}=-h(v,t)\ \text{a.e. } t\geqslant s,\ v(s)=z, \tag{5.2}$$

has a unique solution $v(t)=v(z,s,t)$ which is locally absolutely continuous in t uniformly with respect to $z\in\overline{B}_r$, $r\in(0,1)$. Moreover, for fixed s and t, $0\leqslant s\leqslant t$, $v(\cdot,s,t)$ is a univalent Schwarz mapping such that $Dv(0,s,t)=e^{s-t}I$.

We remark that Poreda [28] studied the notion of a Loewner chain and the Loewner differential equation on the unit ball of a complex Banach space. He obtained a version of [23, Theorem 2.1] in the context of complex Banach spaces which are not necessarily reflexive. However, he assumed a stronger regularity property on the mapping $h(z,t)$ than the above, namely continuity on $B\times[0,\infty)$.

The next result yields that the solution of the initial value problem (5.2) generates Loewner chains (see [19]; compare with [28, Theorem 4]).

THEOREM 5.3. *Let $h=h(z,t)$ be a mapping which satisfies the assumptions (i) and (ii) in Theorem 5.2, and let $v=v(z,s,t)$ be the solution of the initial value problem (5.2). Then for each $s\geqslant 0$, the limit*

$$\lim_{t\to\infty}e^t v(z,s,t)=f(z,s) \tag{5.3}$$

exists uniformly on each ball $\overline{B}_r$, $r\in(0,1)$. Moreover, $f(z,t)$ is a Loewner chain.

We now present a sufficient condition for a mapping to be a Loewner chain. In the finite dimensional case, this may be compared with [23, Theorem 2.3]; compare also with [28, Theorem 6] in the case of complex Banach spaces. We have (see [19])

THEOREM 5.4. *Let $f=f(z,t): B\times[0,\infty)\to X$ be a mapping such that $f(\cdot,t)\in H(B)$, $f(0,t)=0$, $Df(0,t)=e^tI$, $t\geqslant 0$, $f(z,\cdot)$ is strongly locally absolutely continuous on $[0,\infty)$ uniformly with respect to $z\in\overline{B}_r$ for $r\in(0,1)$. Suppose that $h: B\times[0,\infty)$ is a mapping which satisfies the assumptions (i) and (ii) in Theorem 5.2 and*

$$\frac{\partial f}{\partial t}(z,t)=Df(z,t)h(z,t)\ \text{a.e. } t\geqslant 0,\ \forall\ z\in B. \tag{5.4}$$

Moreover, suppose there exist $r_0\in(0,1)$ and $M=M(r_0)>0$ such that

$$\|f(z,t)\|\leqslant e^tM, \quad \|z\|\leqslant r_0,\ t\geqslant 0. \tag{5.5}$$

Then $f(z,t)$ is a Loewner chain.

REMARK 5.5. (i) It is known that any Hilbert space is reflexive. Therefore, the results presented in this section are true in the case of complex Hilbert spaces. Of course, any finite dimensional complex Banach space is also reflexive. Moreover, if $p \in (1, \infty)$ then the space ℓ_p of p-summable complex sequences is another example of a reflexive complex Banach space for which the results in this section remain true.

(ii) We do not know whether the above results remain true without assuming X is a reflexive complex Banach space. The usefulness of the reflexivity condition comes from the fact that any strongly absolutely continuous mapping g from an interval $E \subset \mathbb{R}$ into X has a strong derivative $(dg/dt)(t)$ for almost all $t \in E$, and furthermore this derivative is integrable on E and

$$g(t) - g(t_0) = \int_{t_0}^{t} \frac{dg}{d\tau}(\tau) d\tau, \quad t, t_0 \in E$$

(see [22]). It would be interesting to see whether the results of this section remain valid without the reflexivity assumption.

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