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UNIVALENT HARMONIC MAPPINGS OF ANNULI

Abdallah Lyzzaik

Dedicated to the memory of Professor Walter Hengartner

ABSTRACT. This paper is a survey of the author's recent results on univalent harmonic mappings of annuli.

1. Introduction

A *harmonic mapping* f of a region $\mathcal{D}$ is a complex-valued function of the form $f = h + \overline{g}$, where h and g are analytic functions in $\mathcal{D}$, unique up to an additive constant, that are single-valued if $\mathcal{D}$ is simply connected and possibly multiple-valued otherwise. We call h and g the *analytic* and *co-analytic* parts of f, respectively. If f is (locally) injective, then f is called *(locally) univalent*. Note that every conformal and anti-conformal function is a univalent harmonic mapping. The *Jacobian* and *second complex dilatation* of f are given by the functions $J(z) = |h'(z)|^2 - |g'(z)|^2$ and $\omega(z) = g'(z)/h'(z)$, $z \in \mathcal{D}$, respectively. Note that ω is either a nonconstant meromorphic function or a (possibly infinite) constant. A result of Lewy **[10]** states that if f is a locally univalent mapping, then its Jacobian J is never zero; namely, for $z \in \mathcal{D}$, either $J(z) > 0$ or $J(z) < 0$. In the first case $|\omega(z)| < 1$ and f is sense-preserving, and in the second $|\omega(z)| > 1$ and f is sense-reversing.

Throughout the paper we shall use the following notation: $\mathbb{C}$ for the complex plane, $\widehat{\mathbb{C}}$ for the extended complex plane, $\mathbb{D}$ for the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{T}$ for the unit circle $\{z \in \mathbb{C} : |z| = 1\}$, $0 < \rho < 1$, $\mathbb{T}_\rho$ for the circle $\{z \in \mathbb{C} : |z| = \rho\}$, $\mathbb{A}(\rho, 1)$ for the annulus $\{z \in \mathbb{C} : \rho < |z| < 1\}$, $\mathbb{G}$ for a bounded convex domain unless otherwise is specified, and ∂S and $\overline{S}$, $S \subset \mathbb{C}$, for the boundary and closure of S respectively. We shall call the *diameter* of S the least upper bound of the Euclidean distances between any two points of S, a Jordan curve *convex* if it is the boundary of a bounded convex domain, and a *ring domain* is a doubly-connected open subset of the plane. We shall need the notion of the *module* of a ring domain **[16]**. It is known that a ring domain R is conformally equivalent to a unique annulus

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$A(\rho, 1)$. The module of R, denoted by $M(R)$, is defined by $\log(1/\rho)$ if $\rho \neq 0$ and by ∞ if $\rho = 0$. It is known that M is a conformal invariant and that if $R \subset R'$, where R' is also a ring domain, then $M(R) \leqslant M(R')$ with equality if and only if $R = R'$. The *Grötzsch ring domain*, $B(s)$, $0 < s < 1$, is the ring domain whose boundary components are $\mathbb{T}$ and the segment $\{x : 0 \leqslant x \leqslant s\}$. Observe that $B(s)$ is unique. The module of $B(s)$ is usually denoted by $\mu(s)$. Thus $\mu(s) = \log(1/\rho)$. It is known that μ is a strictly decreasing function of $[0, 1)$.

The purpose of this article is to survey the author's recent results on harmonic univalent mappings of annuli.

2. Boundary Functions

The boundary functions of univalent harmonic mappings onto punctured convex domains are characterized by the following notion introduced by Bshouty, Hengartner and Naghibi-Beidokhti [**3**].

Definition 2.1. Let f be a function of $\mathbb{T}$ into a Jordan curve C of $\mathbb{C}$. We say f is a sense-preserving quasihomeomorphism of $\mathbb{T}$ into C if it is a pointwise limit of a sequence of sense-preserving homeomorphisms of $\mathbb{T}$ onto C. If in addition f is a continuous function onto C, then f is called a *sense-preserving weak homeomorphism.*

Sense-preserving quasihomeomorphisms and sense-preserving weak homeomorphisms are characterized as follows [**12**].

Proposition 2.1. *Let f be a function of $\mathbb{T}$ into a Jordan curve C, and let F be a sense-preserving homeomorphism of $\mathbb{T}$ onto C.*

(i) *If f is a sense-preserving quasihomeomorphism of $\mathbb{T}$ onto C, then there is a real-valued nondecreasing function φ on $\mathbb{R}$ such that $\varphi(t+2\pi) = \varphi(t)+2\pi$ and $f(e^{it}) = F(e^{i\varphi(t)})$.*

(ii) *If $f(e^{it}) = F(e^{i\varphi(t)})$, where φ is a real-valued nondecreasing function on $\mathbb{R}$ such that $\varphi(t + 2\pi) = \varphi(t) + 2\pi$, and if E is the countable set of points $e^{i\varphi(t)}$ where φ is discontinuous, then f coincides on $\mathbb{T} \setminus E$ with a sense-preserving quasihomeomorphism of $\mathbb{T}$. In this case, f is the pointwise limit in $\mathbb{T} \setminus E$ of a sequence of sense-preserving homeomorphisms $f_n(e^{it}) = F(e^{i\varphi_n(t)})$ of $\mathbb{T}$ onto C, where each φ_n is a real-valued infinite differentiable function on $\mathbb{R}$ such that $\varphi_n(t + 2\pi) = \varphi_n(t) + 2\pi$ and ${\varphi_n}'(t)$ is always positive.*

(iii) *f is a sense-preserving weak homeomorphism of $\mathbb{T}$ onto C if and only if there is a real-valued continuous nondecreasing function φ on $\mathbb{R}$ such that $\varphi(t + 2\pi) = \varphi(t) + 2\pi$ and $f(e^{it}) = F(e^{i\varphi(t)})$. In this case, f is the uniform limit of a sequence of sense-preserving homeomorphisms $f_n(e^{it}) = F(e^{i\varphi_n(t)})$ of $\mathbb{T}$ onto C, where each $\{\varphi_n\}$ is a real-valued infinite differentiable function on $\mathbb{R}$ such that $\varphi_n(t + 2\pi) = \varphi_n(t) + 2\pi$ and ${\varphi_n}'(t)$ is always positive.*

Let f be a function of $\mathbb{A}(\rho, 1)$ into $\widehat{\mathbb{C}}$, and let $\xi \in \mathbb{T}$. We say that f has the *unrestricted limit* $a \in \widehat{\mathbb{C}}$ at ξ if

$$f(z) \to a \quad z \to \xi, \quad z \in \mathbb{A}(\rho, 1);$$

by defining $f(\xi) = a$ the function f becomes continuous at ξ as a function in $\mathbb{A}(\rho, 1) \cup \{\xi\}$. We shall use $f(\xi)$ to denote the unrestricted limit whenever it exists, and call the resulting function, on its domain of definition in $\mathbb{T}$, the *unrestricted limit* function f. We also define the *cluster set* $C(f, \xi)$ of f at ξ as the set of all $b \in \widehat{\mathbb{C}}$ for which there are sequences $\{z_n\}$ such that

$$z_n \in \mathbb{A}(\rho, 1), \quad z_n \to \xi, \quad f(z_n) \to b \quad \text{as } n \to \infty.$$

Moreover, If F is a subset of $\mathbb{T}$, then we define the *cluster set* $C(f, F)$ of f at F as the set-union of the cluster sets $C(f, \xi)$ for $\xi \in E$.

Sense-preserving quasihomeomorphisms are essential for describing the boundary behavior of univalent harmonic mappings of ring domains onto bounded convex domains. Suppose f is a univalent harmonic mapping of $\mathbb{A}(\rho, 1)$ onto a ring domain $\mathbb{G} \setminus \{\zeta\}$, $\zeta \in \mathbb{G}$. Then either $\lim_{|z| \uparrow 1} f(z) = \zeta$ and $C(f, \mathbb{T}_\rho) = \partial\mathbb{G}$, or $\lim_{|z| \downarrow \rho} f(z) = \zeta$ and $C(f, \mathbb{T}) = \partial\mathbb{G}$. In the first case, $f(\rho/z)$ becomes a univalent harmonic mapping of $\mathbb{A}(\rho, 1)$ onto $\mathbb{G} \setminus \{\zeta\}$ with $\lim_{|z| \downarrow \rho} f(\rho/z) = \zeta$ and $C(f(\rho/z), \mathbb{T}) = \partial\mathbb{G}$. This leads us to consider, without loss of generality, only univalent harmonic mappings of $\mathbb{A}(\rho, 1)$ onto ring domains $\mathbb{G} \setminus \{\zeta\}$, $\zeta \in \mathbb{G}$, with $\lim_{|z| \downarrow \rho} f(z) = \zeta$.

Definition 2.2. Let $\mathcal{H}_u(\rho, \mathbb{G})$ be the class of all univalent harmonic mappings f of $\mathbb{A}(\rho, 1)$ onto a ring domain $\mathbb{G} \setminus \{\zeta\}$, $\zeta \in \mathbb{G}$, with $f(\mathbb{T}_\rho) = \zeta$.

The boundary behavior of functions $f \in \mathcal{H}_u(\rho, \mathbb{G})$ is given as follows [**12**].

Theorem 2.1. *Let $f \in \mathcal{H}_u(\rho, \mathbb{G})$. Then there is a countable set $E \subset \mathbb{T}$ such that the following hold:*

(i) *For each $e^{i\theta} \in \mathbb{T} \setminus E$, the unrestricted limit $f(e^{i\theta})$ exists and belongs to $\partial\mathbb{G}$. Furthermore, f is continuous in $\overline{\mathbb{A}(\rho, 1)} \setminus E$.*
(ii) *For each $e^{i\theta_0} \in E$, the side-limits $\lim_{\theta \uparrow \theta_0} f(e^{i\theta})$ and $\lim_{\theta \downarrow \theta_0} f(e^{i\theta})$ exist in $\partial\mathbb{G}$ and are distinct.*
(iii) *For each $e^{i\theta_0} \in E$, the cluster set $C(f, e^{i\theta_0})$ lies in $\partial\mathbb{G}$ and is the straight-line segment joining the side-limits $\lim_{\theta \uparrow \theta_0} f(e^{i\theta})$ and $\lim_{\theta \downarrow \theta_0} f(e^{i\theta})$.*
(iv) *$\overline{\mathrm{co}}(f(\mathbb{T} \setminus E)) = \overline{\mathbb{G}}$; $\overline{\mathrm{co}}(-)$ is the closed convex hull of $-$.*
(v) *There is a sense-preserving quasihomeomorphism of $\mathbb{T}$ into $\partial\mathbb{G}$ that coincides with the unrestricted limit function f on $\mathbb{T} \setminus E$.*
(vi) *f is the Dirichlet solution in $\mathbb{A}(\rho, 1)$ of the function f^* defined by the unrestricted limit function of f on $\mathbb{T}$ and the value of f on $\mathbb{T}_\rho$.*

The fact that f^* is not defined on E in (vi) is insignificant. Indeed, Dirichlet solutions in multiply connected domains coincide whenever their boundary functions coincide almost everywhere.

3. A Representation Theorem and Univalence Criteria

Hengartner and Szynal [**7**] and Bshouty and Hengartner [**1**] gave the following useful representation for harmonic mappings f defined on an annulus $\mathbb{A}(\rho,1)$ and constant on the inner circle.

THEOREM 3.1. *Let f be a harmonic mapping of $\mathbb{A}(\rho,1)$ that extends continuously across $\mathbb{T}_\rho$ with f identically ζ there. Then there exist a constant c and a function h analytic in $\mathbb{A}(\rho^2,1)$ such that*

$$f(z) = h(z) - h(\rho^2/\overline{z}) + \zeta + 2c\log(|z|/\rho). \tag{3.1}$$

Further, if f extends continuously across $\mathbb{T}$ and f^* is the restriction of f on $\mathbb{T}$, then $c=0$ if and only if ζ equals

$$\zeta_0 = \frac{1}{2\pi}\int_0^{2\pi} f^*(e^{it})\,dt. \tag{3.2}$$

Using Theorem 3.1, Bshouty and Hengartner [**1**] obtained the following result.

THEOREM 3.2. *Let f^* be a sense-preserving homeomorphism between $\mathbb{T}$ and $\partial\mathbb{G}$ that assumes on $\mathbb{T}_\rho$ the constant $\zeta_0\in\mathbb{G}$ given by* (3.2)*, and let f be the Dirichlet solution of f^* in $\mathbb{A}(\rho,1)$. Then $f\in\mathcal{H}_u(\rho,\mathbb{G})$.*

Theorem 3.2 was extended by the author [**12**] as follows.

THEOREM 3.3. *Let f^* be a sense-preserving quasihomeomorphism of $\mathbb{T}$ into $\partial\mathbb{G}$ such that $\overline{\mathrm{co}}(f(\mathbb{T}\setminus E)) = \overline{\mathbb{G}}$, and let f^* be defined on $\mathbb{T}_\rho$ by the constant ζ_0 given in* (3.2)*. Also, let f be the Dirichlet solution of f^* in $\mathbb{A}(\rho,1)$. Then $\zeta_0\in\mathbb{G}$ and $f\in\mathcal{H}_u(\rho,\mathbb{G})$.*

Further, the author [**11**, Theorem 2] showed that, without using Theorem 3.1, Theorem 3.2 remains true under the weaker condition $f(\mathbb{A}(\rho,1))\subset\mathbb{G}$ rather than the convexity of $\mathbb{G}$. In fact, in view of Theorem 2.1 and the proof of the previous theorem, the following more general result can be obtained.

THEOREM 3.4. *Let f^* be a sense-preserving quasihomeomorphism of $\mathbb{T}$ into the boundary of a bounded Jordan domain $\mathbb{G}$ such that $\overline{\mathrm{co}}(f(\mathbb{T}\setminus E)) = \overline{\mathbb{G}}$, and let f^* be defined on $\mathbb{T}_\rho$ by the constant ζ_0 given in* (3.2)*. Also, let f be the Dirichlet solution of f^* in $\mathbb{A}(\rho,1)$. Then $\zeta_0\in\mathbb{G}$ and $f\in\mathcal{H}_u(\rho,\mathbb{G})$.*

We introduce here the following subclass of $\mathcal{H}_u(\rho,\mathbb{G})$.

DEFINITION 3.1. Denote by $\mathcal{H}_0(\rho,\mathbb{G})$ the class of all Dirichlet solutions f satisfying the hypotheses of Theorem 3.3.

The classes $\mathcal{H}_u(\rho,\mathbb{G})$ and $\mathcal{H}_0(\rho,\mathbb{G})$ are related as follows [**12**].

PROPOSITION 3.1. *Suppose that the following are true:*

(i) *f^* is a sense-preserving quasihomeomorphism of $\mathbb{T}$ into $\partial\mathbb{G}$ such that $\overline{\mathrm{co}}(f(\mathbb{T}\setminus E)) = \overline{\mathbb{G}}$.*

(ii) *f is the Dirichlet solution in $\mathbb{A}(\rho,1)$ of the function defined on $\mathbb{T}$ by f^* and on $\mathbb{T}_\rho$ by a constant $\zeta\in\mathbb{G}$.*

(iii) $f_0 \in \mathcal{H}_0(\rho, \mathbb{G})$ *is the Dirichlet solution of the function defined on* $\mathbb{T}$ *by* f^* *and on* $\mathbb{T}$ *by the average* ζ_0 *of* f^*.

Then there is an analytic function h *in* $\mathbb{A}(\rho^2, 1)$ *such that* f *has form* (3.1) *or the equivalent form*

$$f(z) = f_0(z) + 2c_\zeta \log|z| \tag{3.3}$$

where

$$f_0(z) = h(z) - h(\rho^2/\overline{z}) + \zeta_0, \quad (z \in \mathbb{A}(\rho, 1)), \tag{3.4}$$

and

$$c_\zeta = \frac{\zeta - \zeta_0}{2\log\rho}. \tag{3.5}$$

The function f_0 is called the *average associate of* f.

In what follows we use $f'(e^{i\theta})$ for $df(e^{i\theta})/d\theta$. Note that, according to Proposition 3.1, f may not belong to $\mathcal{H}_u(\rho, \mathbb{G})$ even though it shares with its average associate f_0 the same analytic and co-analytic part h. Thus only the functions $f \in \mathcal{H}_0(\rho, \mathbb{G})$ of the form (3.4) are used in stating the following two results [**12**].

Theorem 3.5. *Let* $f \in \mathcal{H}_0(\rho, \mathbb{G})$ *be of form* (3.4). *Then*

(a) h' *is nonvanishing on* $\mathbb{T}_\rho$ *and* h *maps* $\mathbb{T}_\rho$ *homeomorphically onto a convex curve whose diameter is bounded above by*

$$D = (4d/\pi)\tanh^{-1}\left(\mu^{-1}(\log(1/\rho)\right).$$

(b) *If* $h(z) = \sum_{-\infty}^{\infty} a_n z^n$, $z \in \mathbb{A}(\rho^2, 1)$, *then*

$$\sum_{n=1}^{\infty} n|a_{-n}|^2\rho^{-2n} < \sum_{n=1}^{\infty} n|a_n|^2\rho^{2n} \leqslant D^2/4 + \sum_{n=1}^{\infty} n|a_{-n}|^2\rho^{-2n}.$$

Theorem 3.6. *Let* $f \in \mathcal{H}_0(\rho, \mathbb{G})$ *be of form* (3.4). *Then there is a univalent close-to-convex function* H *of the unit disc* $\mathbb{D}$ *and a homeomorphism* ϕ *of* $\mathbb{A}(\rho, 1) \cup \mathbb{T}$ *into* $\overline{\mathbb{D}}$ *with* $\phi(\mathbb{T}) = \mathbb{T}$ *such that* $h = H \circ \phi$.

The author conjectures that the function H is convex.

4. A Hengartner's Problem Regarding Univalent Harmonic Mappings

Let f be the Dirichlet solution in $\mathbb{A}(\rho, 1)$ of a function f^* of $\partial\mathbb{A}(\rho, 1)$ defined by a sense-preserving quasihomeomorphism of $\mathbb{T}$ into $\partial\mathbb{G}$ satisfying $\overline{\mathrm{co}}(f(\mathbb{T} \setminus E)) = \overline{\mathbb{G}}$, and by a constant $\zeta \in \mathbb{G}$ on $\mathbb{T}_\rho$. Theorem 3.3 asserts that f belongs to $\mathcal{H}_u(\rho, \mathbb{G})$ if $\zeta = \zeta_0$, where ζ_0 is the average of f^* on $\mathbb{T}_\rho$ given by (3.2). Hengartner and Schober [**6**] showed that this condition is not necessary, and recently Duren and Hengartner [**4**, Example 1] gave the harmonic mapping

$$F(z) = (z - \rho^2/\overline{z})/(1 - \rho^2) + 2c\log|z|, \quad (z \in \mathbb{A}(\rho, 1)),$$

which belongs to $\mathcal{H}_u(\rho, \mathbb{D})$, with $\zeta_0 = 0$, whenever $|c| < \rho/(1 - \rho^2)$; note that $F(e^{it}) = e^{it}$ and $F(\mathbb{T}_\rho) = 2c\log\rho$. This concludes a negative answer to the following question of Nitsche [**15**, §879]:

QUESTION (Nitsche). *Are all univalent harmonic mappings of* $\mathbb{A}(\rho,1)$ *onto* $\mathbb{A}(0,1)$*, up to a rotation, of the form*

$$f(z) = (z - \rho^2/\overline{z})/(1-\rho^2)? \tag{4.1}$$

In this connection, Hengartner [**2**, Problem 15] raised the following problem:

PROBLEM (Hengartner). *Let* f^* *be a sense-preserving quasihomeomorphism of* $\mathbb{T}$ *into* $\partial\mathbb{G}$ *with* $\overline{\text{co}}(f(\mathbb{T}\setminus E)) = \overline{\mathbb{G}}$*, and let* f *be the harmonic extension in* $\mathbb{A}(\rho,1)$ *of the function defined by* f^* *on* $\mathbb{T}$ *and by a constant* $\zeta \in \mathbb{G}$ *on* $\mathbb{T}_\rho$*. Find the set of all* ζ *for which* $f : \mathbb{A}(\rho,1) \to \mathbb{G}\setminus\{\zeta\}$ *a homeomorphism.*

Denote by $\mathcal{H}(\rho, f^*)$ the class of Dirichlet solutions in $\mathbb{A}(\rho,1)$ of functions of $\partial\mathbb{A}(\rho,1)$ defined on $\mathbb{T}$ by f^* and on $\mathbb{T}_\rho$ by some constant $\zeta\in\mathbb{G}$, by $\mathcal{H}_u(\rho,f^*)$ the subclass of $\mathcal{H}(\rho,f^*)$ of univalent mappings, and by $K(\rho,f^*)$ the set of values $\zeta\in\mathbb{G}$ for which a function $f\in\mathcal{H}(\rho,f^*)$ belongs to $\mathcal{H}_u(\rho,f^*)$.

The results of this section were obtained in an attempt by the author to give a satisfactory answer to Hengartner's problem. The first result [**12**] states as follows.

THEOREM 4.1. $K(\rho,f^*)$ *is a nonempty compact subset of* $\mathbb{G}$.

By Proposition 3.1, the class $\mathcal{H}(\rho,f^*)$ yields an analytic function h in $\mathbb{A}(\rho^2,1)$, unique up to an additive constant, such that every $f\in\mathcal{H}_u(\rho,f^*)$ is of form(3.3). The second result [**12**] characterizes the boundary points of $K(\rho,f^*)$ in terms of h and f^* in a manner leading to a univalence criterion for functions $f\in\mathcal{H}(\rho,f^*)$. The result states as follows.

THEOREM 4.2. *Let* $f\in\mathcal{H}_u(\rho,f^*)$ *be of form* (3.3)*, where* $f^*:\mathbb{T}\to\partial\mathbb{G}$ *is a twice-differentiable function with nonvanishing derivative and absolutely continuous second derivative. Then the dilatation* ω *of* f *and* $zh'(z)+c_\zeta$ *extend continuously to* $\mathbb{A}(\rho,1)\cup\mathbb{T}$ *such that* $e^{i\theta}h'(e^{i\theta})+c_\zeta\neq 0$ *for all* θ*. Moreover, we have:*

(a) *If* $\zeta\in\partial K(\rho,f^*)$*, then either* $\rho e^{i\theta_1}h'(\rho e^{i\theta_1})+c_\zeta = 0$ *for some* θ_1*, or* $|\omega(e^{i\theta_2})| = 1$ *for some* θ_2*.*

(b) *If* $|\omega(e^{i\theta})|=1$ *for some* θ*, then* $\zeta\in\partial K(\rho,f^*)$*.*

(c) *If in* (a) *and* (b) *the function* $|\omega(e^{i\theta})|$ *is replaced by the function*

$$2\operatorname{Re}\left\{\frac{e^{i\theta}h'(e^{i\theta})+c_\zeta}{e^{i\theta}f'(e^{i\theta})}\right\},$$

then (a) *and* (b) *continue to hold.*

Regarding (a), Hengartner and Szynal [**7**, Theorem 3.1] asserted that if $\zeta\in\partial K(\rho,f^*)$, then $\rho e^{i\theta_1}h'(\rho e^{i\theta_1})+c_\zeta$ has at most one zero and that this zero is of order one.

Applying Theorem 4.2 to functions $f\in\mathcal{H}_u(\rho,f^*)$ of form (3.4), where ζ is the average ζ_0 of f^* on $\mathbb{T}$, $c_\zeta=0$, $\rho e^{i\theta}h'(\rho e^{i\theta})\neq 0$ for all θ by Theorem 3.5(a), and $|\omega(e^{i\theta})|=1$ for some θ if and only if $\rho^2|h'(\rho^2e^{i\theta})| = |h'(e^{i\theta})|$, the following result is obtained [**12**].

COROLLARY 4.1. *Let* $f\in\mathcal{H}_u(\rho,f^*)$ *be of form* (3.4)*, where* f^* *is as in Theorem* 4.2*. Then the following statements are equivalent:*

(a) $\zeta_0 \in \partial K(\rho, f^*)$.
(b) $\rho^2|h'(\rho^2 e^{i\theta})| = |h'(e^{i\theta})|$ *for some* θ.
(c) $2\operatorname{Re}\{h'(e^{i\theta})/f'(e^{i\theta})\} = 1$ *for some* θ.

The next result [**12**] provides sufficient conditions for the univalence of functions $f \in \mathcal{H}(\rho, f^*)$ whose f^* are as given in Theorem 4.2.

THEOREM 4.3. *Let* $f \in \mathcal{H}(\rho, f^*)$ *be of form* (3.3), *where* f^* *be smooth as in Theorem* 4.2. *Then* $f \in \mathcal{H}_u(\rho, f^*)$ *if* $zh'(z) + c_\zeta \neq 0$ *for* $z \in \overline{\mathbb{A}(\rho,1)}$, *and if one of the following two inequalities holds for all* θ:

(a) $|\omega(e^{i\theta})| \leqslant 1$. (b) $2\operatorname{Re}\left\{\dfrac{e^{i\theta}h'(e^{i\theta}) + c_\zeta}{e^{i\theta}f'(e^{i\theta})}\right\} \geqslant 1$.

We remark that f^* as defined in Theorem 4.2 yields $zh'(z) \neq 0$ for $z \in \overline{\mathbb{A}(\rho,1)}$ which makes the above hypothesis, $zh'(z) + c_\zeta \neq 0$ for $z \in \overline{\mathbb{A}(\rho,1)}$, easily achievable for functions $f \in \mathcal{H}(\rho, f^*)$ with sufficiently small c_ζ.

The next result [**12**] asserts the existence of a large family of triplets, $0 < \rho < 1$, $\mathbb{G}_\rho$, f^*, where $\mathbb{G}_\rho$ is a bounded convex domain and $f^*_\rho : \mathbb{T} \to \partial\mathbb{G}_\rho$ is a sense-preserving homeomorphism, such that $K(\rho, f^*)$ has a nonempty interior containing the average of f^*.

THEOREM 4.4. *Let* Ω *be a bounded convex domain, and let* h *be a homeomorphism of* $\overline{\mathbb{D}}$ *onto* $\overline{\Omega}$ *that maps* $\mathbb{D}$ *conformally onto* Ω. *Suppose that* h'' *is continuous on* $\overline{\mathbb{D}}$, $h''(e^{i\theta})$ *is absolutely continuous, and*

$$\operatorname{Re}\left\{1 + e^{i\theta}\frac{h''(e^{i\theta})}{h'(e^{i\theta})}\right\} > 0 \tag{4.2}$$

for all θ. *Then there exists* $\delta > 0$ *such that for each* $0 < \rho < \delta$ *we can find a bounded convex domain* $\mathbb{G}_\rho$ *such that the harmonic mapping*

$$f_\rho(z) = h(z) - h(\rho^2/\overline{z}), \quad (z \in \overline{\mathbb{A}(\rho,1)}),$$

satisfies the following properties:

(i) $f_\rho : \mathbb{T} \to \partial\mathbb{G}_\rho$ *is a sense-preserving homeomorphism.*
(ii) f_ρ *is continuously twice-differentiable on* $\overline{\mathbb{A}(\rho,1)}$.
(iii) $f_\rho \in \mathcal{H}_0(\rho, \mathbb{G}_\rho)$.
(iv) *There is* $\sigma > 0$, *depending on* ρ, *such that for any* $|\zeta| < \sigma$ *the function*

$$f_\zeta(z) = h(z) - h(\rho^2/\overline{z}) + \zeta + 2c_\zeta \log(|z|/\rho)$$

belongs to $\mathcal{H}_u(\rho, \mathbb{G}_\rho)$.

REMARK 4.1. (i) Without (4.2), the hypothesis of the theorem yields the following weaker form of(4.2):

$$\operatorname{Re}\left\{1 + e^{i\theta}\frac{h''(e^{i\theta})}{h'(e^{i\theta})}\right\} \geqslant 0. \tag{4.3}$$

To see this, observe that $zh'(z)$ is a univalent starlike function in $\mathbb{D}$ which gives

$$\operatorname{Re}\left\{1 + z\frac{h''(z)}{h'(z)}\right\} > 0, \quad (z = re^{i\theta} \in \mathbb{D}). \tag{4.4}$$

Now, because h'' extends continuously to $\overline{\mathbb{D}}$, the integral

$$\int_0^z h''(\zeta)\,d\zeta, \quad (z \in \overline{\mathbb{D}}),$$

where the differentiable path of integration from 0 to z lies in $\overline{\mathbb{D}}$, yields, by Cauchy's theorem, the continuous extension of $h'(z)$ to $\overline{\mathbb{D}}$. On the other hand, since $zh'(z)$ is univalent in $\mathbb{D}$ and maps the origin to itself, $zh'(z) \neq 0$ for $z \in \overline{\mathbb{D}}$. Then (4.3) follows at once by letting $r \to 1$ in (4.4).

(ii) Using Kellogg and Warschawski [**16**, Theorem 3.6, p. 49], the hypothesis that $h''(z)$ admits a continuous extension to $\overline{\mathbb{D}}$ with absolutely continuous $h''(e^{i\theta})$ follows if $\partial\mathbb{G}$ has a parameterization $w(t)$, $0 \leqslant t \leqslant 2\pi$, whose first derivative is nonvanishing and second derivative is Lipschitz of order α, $0 < \alpha < 1$.

5. Nitsche's Question Revisited

In this section all harmonic mappings $f \in \mathcal{H}_u(\rho, \mathbb{G})$ whose analytic parts extend analytically throughout $\mathbb{D}$ are determined explicitly. It follows that the function f defined by (4.1) is the only harmonic mapping, up to rotation, in $\mathcal{H}_0(\rho, \mathbb{D})$, (here $\mathbb{G}$ is taken as $\mathbb{D}$), of $\mathbb{A}(\rho, 1)$ onto $\mathbb{A}(0, 1)$ whose analytic part is analytic in $\mathbb{D}$. This somehow justifies Nitsche's Question above. The result of this section states as follows [**12**].

Theorem 5.1. *Let $f \in \mathcal{H}_u(\rho, \mathbb{G})$ be of form (3.3) with h analytic in $\mathbb{D}$. Then*

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} \frac{\lambda^n b_n}{1-\rho^{2n}}[z^n - (\rho^2/\overline{z})^n] + \zeta + 2c_\zeta \log(|z|/\rho) \\ &= \sum_{n=1}^{\infty} \frac{\lambda^n b_n}{1-\rho^{2n}}[z^n - (\rho^2/\overline{z})^n] + \zeta_0 + 2c_\zeta \log|z|, \end{aligned}$$

where b_n, $n = 1, 2, \cdots$, is the n-th coefficient of the conformal map

$$F(z) = \zeta_0 + \sum_{n=1}^{\infty} b_n z^n$$

of $\mathbb{D}$ onto $\mathbb{G}$ satisfying $F(0) = \zeta_0$ and c_ζ is as given in (3.5).

As an application of Theorem 5.1, if $\mathbb{G} = \mathbb{D}$, then

$$F(z) = \frac{z+\zeta_0}{1+\overline{\zeta}_0 z} = \zeta_0 + (1-|\zeta_0|^2)\sum_{n=2}^{\infty}(-\zeta_0)^{n-1}z^n$$

and the following result holds.

Corollary 5.1. *Let $f \in \mathcal{H}_u(\rho, \mathbb{D})$ be of the form* (3.3) *with h analytic in $\mathbb{D}$. Then there is a unimodular constant λ such that*

$$\begin{aligned} f(z) = \lambda(1-|\zeta_0|^2)&\left\{\frac{z-\rho^2/\overline{z}}{1-\rho^2} + \sum_{n=2}^{\infty}\frac{(-\lambda\zeta_0)^{n-1}}{1-\rho^{2n}}[z^n - (\rho^2/\overline{z})^n]\right\} \\ &+ \zeta + 2c_\zeta \log(|z|/\rho), \quad (z \in \mathbb{A}(\rho, 1)). \end{aligned}$$

In particular, if $\zeta_0 = 0$*, then*

$$f(z) = \lambda \frac{z - \rho^2/\overline{z}}{1 - \rho^2} + 2c_\zeta \log|z|, \quad (z \in \mathbb{A}(\rho, 1)).$$

Also, if $\zeta_0 = 0$ *and* $f(\mathbb{T}_\rho) = 0$*, then*

$$f(z) = \lambda \frac{z - \rho^2/\overline{z}}{1 - \rho^2}, \quad (z \in \mathbb{A}(\rho, 1)).$$

6. The Modulus of the Image Annuli under Univalent Harmonic Mappings and a Conjecture of Nitsche

For harmonic mappings $f : \mathbb{A}(\rho, 1) \to A(R, 1)$, R can possibly be zero as with (4.1) which maps $\mathbb{A}(\rho, 1)$ univalently onto the punctured disc $A(0, 1)$. On the other hand, R admits a universal upper bound (less than 1) as was shown in 1962 by Nitsche [**14**]. To state this result, let $\mathcal{K}(\rho)$ be the class of univalent harmonic mappings of the annulus $\mathbb{A}(\rho, 1)$ onto some annulus $A(R, 1)$, and let $\kappa(\rho)$ be the supremum of R as f ranges over all $f \in \mathcal{K}(\rho)$. Using Harnack's inequality, Nitsche proved the following result [**14**]:

THEOREM 6.1. *The value* $\kappa(\rho)$ *is less than 1.*

Consider now the class of harmonic mappings

$$f_t(z) = tz + (1 - t)/\overline{z} = [t + (1 - t)/\sigma]e^{i\theta} \quad (z = \sigma e^{i\theta}).$$

Each f_t maps concentric circles onto concentric circles, and maps $\mathbb{A}(\rho, 1)$ univalently onto $A(R(t), 1)$, $R(t) = t\rho + (1 - t)/\rho$, if, and only if, $1/(1 + \rho^2) \leqslant t \leqslant 1/(1 - \rho^2)$. Restricted to these values of t, Nitsche [**14**] observed that $R(t)$ admits its maximum value $2\rho/(1 + \rho^2)$ at $t = 1/(1 + \rho^2)$. This led him to suggest the following:

CONJECTURE (Nitsche). $\kappa(\rho) = 2\rho/(1 + \rho^2)$.

The conjecture was raised again in 1989 by Schober [**17**] as "an intriguing open problem", and subsequently in 1994 by Bshouty and Hengartner [**2**] as "open problem 3.1". Looking closer at Nitsche's proof of the above theorem, the latter authors observed that the proof also applies to the wider class of harmonic mappings of $\mathbb{A}(\rho, 1)$ that are not necessarily univalent and that admit a point in each of the vertical strips $\{w : R < \operatorname{Re} w < 1\}$ and $\{w : -1 < \operatorname{Re} w < -R\}$. Consequently, they remarked that $\kappa(\rho)$ is unlikely to be found by parlaying Nitsche's proof of his Theorem 6.1.

Until recently, it was believed that no quantitative upper bound for $\kappa(\rho)$ was found. However, in a personal communication dated December 1999, Nitsche wrote that he had "developed the estimate ($\kappa(\rho) \leqslant \tanh[\pi(1 + \rho)/(1 - \rho)] \approx 0.9926$) at the time (of his article [**14**])", but refrained from publishing the "poor bound" in order "not to detract from the impact of the conjecture".

The author, being unaware of Nitsche's result, gave a substantial upper bound of $\kappa(\rho)$ in terms of the *Grötzsch's ring domain* $B(s)$ of $\mathbb{A}(\rho, 1)$ [**13**].

Theorem 6.2. *Let f be a univalent harmonic mapping of the annulus $\mathbb{A}(\rho, 1)$ onto the annulus $A(R, 1)$, and let $B(s(\rho))$ be the Grötzsch's ring domain that is conformally equivalent to $\mathbb{A}(\rho, 1)$. Then $R \leqslant s(\rho)$.*

Further, it was conjectured in [**13**] that the inequality $R \leqslant s$ is sharp, but the conjecture was subsequently disproved by Weitsman [**18**] in the following result.

Theorem 6.3. *Let f be a univalent harmonic mapping of the annulus $\mathbb{A}(\rho, 1)$ onto the annulus $A(R, 1)$. Then*

$$R \leqslant \sigma(\rho) = \frac{1}{1 + (\rho \log \rho)^2/2}.$$

Computations reveal that for a value $\rho_0 \approx 0.36$, $\sigma(r\rho) < s(\rho)$ if $\rho_0 < \rho < 1$, $\sigma(\rho)$ is substantially smaller that $s(\rho)$ if ρ is close to 1, $\sigma(\rho) > s(\rho)$ if $0 < \rho < \rho_0$, and $\sigma(\rho)$ is of no value when ρ is small. Further, if $\tau(\rho) = 2\rho/(1 + \rho^2)$, which is the upper bound conjectured by Nitsche, then $\lim_{\rho \to 1^-} (1 - \tau(\rho))/(1 - \sigma(\rho)) = 1$.

It was noted by the referee that Kalaj [**8**] has recently improved Weitsman's result above as follows.

Theorem 6.4. *Let f be a univalent harmonic mapping of the annulus $\mathbb{A}(\rho, 1)$, $0 < \rho < 1$, onto the annulus $A(R, 1)$. Then*

$$R \leqslant \eta(\rho) = \frac{1}{1 + (\log \rho)^2/2}.$$

Obviously, $\eta(\rho) < \sigma(\rho)$ and $\eta(\rho)$, like $\sigma(\rho)$, is of no value when ρ is small.

In conclusion, Nitsche's conjecture remains an unsettled interesting problem.

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Department of Mathematics
American University of Beirut
Beirut, Lebanon
lyzzaik@aub.edu.lb

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