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PLURISUBHARMONIC FEATURES OF THE TEICHMÜLLER METRIC

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Abstract. The key result of this paper is a strengthened version for universal Teichmüller space of the fundamental Gardiner–Royden theorem on coincidence of the Kobayashi and Teichmüller metrics for Teichmüller spaces. Using the Grunsky coefficient inequalities for univalent functions, we show that the Teichmüller metric is logarithmically plurisubharmonic and has constant holomorphic sectional curvature $\kappa_{\mathcal{K}}(\psi, v) = -4$.

This result has various important applications in geometric function theory and geometry. Some applications to complex geometry of Teichmüller spaces are given.

1. Introduction. Main results

The purpose of this paper is to prove the following

Theorem 1.1. *The differential Kobayashi metric $\mathcal{K}_{\mathbf{T}}(\psi, v)$ on the tangent bundle $\mathcal{T}(\mathbf{T})$ of the universal Teichmüller space $\mathbf{T}$ is logarithmically plurisubharmonic in $\psi \in \mathbf{T}$, equals the canonical Finsler structure $F_{\mathbf{T}}(\psi, v)$ on $\mathcal{T}(\mathbf{T})$ generating the Teichmüller metric of $\mathbf{T}$ and has constant holomorphic sectional curvature $\kappa_{\mathcal{K}}(\psi, v) = -4$ on $\mathcal{T}(\mathbf{T})$.*

This theorem can be regarded as a strengthened version for the universal Teichmüller space of the fundamental Gardiner–Royden theorem on the coincidence of the Kobayashi and Teichmüller metrics for Teichmüller spaces, which is crucial for many results (cf. [**AP**], [**EKK**], [**EM**], [**GL**], [**Kr5**], [**Ro1**]).

The proof of Theorem 1.1 relies on the specific features of the space $\mathbf{T}$ and involves the technique of the Grunsky coefficient inequalities.

It has various applications. As its immediate consequences, one obtains the following two important statements.

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Corollary 1.2. *The Kobayashi metric $d_{\mathbf{T}}$ of the universal Teichmüller space $\mathbf{T}$ coincides with its Teichmüller metric $\tau_{\mathbf{T}}$, and*

$$d_{\mathbf{T}}(\psi_1,\psi_2) = \tau_{\mathbf{T}}(\psi_1,\psi_2) = \inf d_{\Delta}(h^{-1}(\psi_1), h^{-1}(\psi_2)), \tag{1.1}$$

where the infimum is taken over all holomorphic maps $h:\ \Delta \to \mathbf{T}$.

Corollary 1.3. *The Teichmüller metric $\tau_{\mathbf{T}}(\psi_1,\psi_2)$ is plurisubharmonic separately in each of its argument; hence, the pluricomplex Green function of $\mathbf{T}$ equals*

$$g_{\mathbf{T}}(\psi_1,\psi_2) = \log\tanh \tau_{\mathbf{T}}(\psi_1,\psi_2) = \log k(\psi_1,\psi_2), \tag{1.2}$$

where k is the norm of extremal Beltrami coefficient defining the Teichmüller distance between the points ψ_1,ψ_2 in $\mathbf{T}$.

Recall that the pluricomplex Green function of a domain X in a complex Banach space (or on a Banach manifold) E is defined as

$$g_X(x,y) = \sup u_y(x) \quad (x,y \in X),$$

where supremum is taken over all plurisubharmonic functions $u_y(x) : X \to [-\infty, 0)$ that have the representation

$$u_y(x) = \log\|x-y\| + O(1)$$

in a neighborhood of the pole y; here $\|\cdot\|$ is the norm on X and the remainder term $O(1)$ is bounded from above. If X is hyperbolic and its Kobayashi metric d_X is logarithmically plurisubharmonic, then the Green function relates to d_X by

$$g_X(x,y) = \log\tanh d_X(x,y)$$

(see, e.g., [**Kl2**]).

One deduces from (1.2) the following properties of the Green function $g_{\mathbf{T}}$:

(a) $g_{\mathbf{T}}$ is symmetric with respect to its arguments: $g_{\mathbf{T}}(\psi_1,\psi_2) = g_{\mathbf{T}}(\psi_2,\psi_1)$ for any pair $\psi_1,\psi_2 \in \mathbf{T}$ (in general, such a property does not hold even for the bounded domains in $\mathbb{C}^n$ $(n>1)$ with real analytic boundaries, see [**BD**]);

(b) for any fixed pole ψ_2 we have $\lim\limits_{\psi_1\to\partial\mathbf{T}} g_{\mathbf{T}}(\psi_1,\psi_2) = 0$;

(c) $g_{\mathbf{T}}(\psi_1,\psi_2) = \inf\{g_{\mathrm{Belt}(\Delta)_1}(\mu,\nu) : \phi(\mu) = \psi_1,\ \phi(\nu) = \psi_2\}$.

2. Some basic facts on universal Teichmüller space and on Grunsky coefficients

2.1. The universal Teichmüller space $\mathbf{T}$ is the space of quasisymmetric homeomorphisms h of the unit circle factorized by Möbius transformations. Its topology and real geometry are determined by the Teichmüller metric which naturally arises from extensions of those h to the unit disk. This space admits also the complex structure of a complex Banach manifold by means of the Bers embedding as a bounded subdomain of $\mathbf{B}$.

We shall identify the space $\mathbf{T}$ with this domain. In this model the points $\psi \in \mathbf{T}$ represent the Schwarzian derivatives S_f of univalent holomorphic functions f in $\Delta^* = \{z \in \widehat{\mathbb{C}} : |z| > 1\}$, which have quasiconformal extensions to the whole sphere $\widehat{\mathbb{C}}$ and $\|\psi\|_{\mathbf{B}} = \sup_{\Delta^*}(|z|^2-1)^2|\psi(z)|$.

To obtain the universal Teichmüller space $\mathbf{T}$, consider the Banach ball

$$\operatorname{Belt}(\Delta)_1 = \{\mu \in L_\infty(\mathbb{C}) : \mu|\Delta^* = 0,\ \|\mu\| < 1\}. \tag{2.1}$$

Each $\mu \in \operatorname{Belt}(\Delta)_1$ defines a conformal structure on the extended complex plane $\widehat{\mathbb{C}}$, i.e., a vector field of infinitesimal ellipses, or equivalently, a class of conformally equivalent Riemannian metrics $ds^2 = \lambda(z)|dz + \mu d\bar{z}|^2$, $\lambda(z) > 0$; these ellipses reduce to circles for $z \in \Delta^*$.

The universal Teichmüller space $\mathbf{T}$ is obtained from the ball (2.1) by natural identification, letting μ and ν in $\operatorname{Belt}(\Delta)_1$ be equivalent if $w^\mu|S^1 = w^\nu|S^1$, $S^1 = \partial\Delta$. We denote the equivalence classes by $[\mu]$.

There are certain natural intrinsic complete metrics on the space $\mathbf{T}$. The first one is the *Teichmüller metric*

$$\tau_{\mathbf{T}}(\phi(\mu), \phi(\nu)) = \frac{1}{2} \inf \left\{ \log K\big(w^{\mu_*} \circ \big(w^{\nu_*}\big)^{-1}\big) : \mu_* \in \phi(\mu), \nu_* \in \phi(\nu) \right\}, \tag{2.2}$$

where $\phi = \phi_{\mathbf{T}}$ is the canonical projection

$$\phi(\mu) = [\mu] : \operatorname{Belt}(\Delta)_1 \to \mathbf{T}.$$

This metric is generated by the *Finsler structure on* $\mathbf{T}$ (in fact, on the tangent bundle $\mathcal{T}(\mathbf{T}) = \mathbf{T} \times \mathbf{B}$ of $\mathbf{T}$); this structure is defined by

$$\begin{aligned} F_{\mathbf{T}}(\phi(\mu), \phi'(\mu)\nu) = \inf\big\{ &\big\|\nu_*(1 - |\mu|^2)^{-1}\big\|_\infty : \\ &\phi'(\mu)\nu_* = \phi'(\mu)\nu;\ \mu \in \operatorname{Belt}(\Delta)_1;\ \nu, \nu_* \in L_\infty(\mathbb{C})\big\}. \end{aligned} \tag{2.3}$$

The *Kobayashi metric* $d_{\mathbf{T}}$ on $\mathbf{T}$ is the largest pseudometric d on $\mathbf{T}$ contracted by holomorphic maps $h : \Delta \to \mathbf{T}$ so that for any two poins ψ_1, $\psi_2 \in \mathbf{T}$, we have

$$d_{\mathbf{T}}(\psi_1, \psi_2) \leqslant \inf\{d_\Delta(0, t) : h(0) = \psi_1,\ h(t) = \psi_2\}, \tag{2.4}$$

where d_Δ is the hyperbolic Poincaré metric on Δ of Gaussian curvature -4. The *Carathéodory metric* $c_{\mathbf{T}}$ is the least pseudometric on $\mathbf{T}$ with such property.

The infinitesimal (differential) Kobayashi metric $\mathcal{K}_{\mathbf{T}}(\psi, v)$ is a Finsler metric on the tangent bundle $\mathcal{T}(\mathbf{T})$ of $\mathbf{T}$ defined by

$$\begin{aligned} \mathcal{K}_T(\psi, v) &= \inf\{|t| : h \in \operatorname{Hol}(\Delta, \mathbf{T}),\ h(0) = \psi,\ dh(0)t = v\} \\ &= \inf\Big\{\frac{1}{r} : r > 0,\ h \in \operatorname{Hol}(\Delta_r, \mathbf{T}),\ h(0) = \psi,\ h'(0) = v\Big\}. \end{aligned} \tag{2.5}$$

Here v is a tangent vector at the point $\psi \in \mathbf{T}$ and Δ_r denotes the disk $\{|z| < r\}$ (cf. [**Ko**], [**FV**], [**Kl2**]).

As was observed in [**Ea**], [**Ga**], the Teichmüller contraction can be done in a more sharpened form than in (2.4). It will not be used here.

2.2. Consider quasiconformal maps f of $\widehat{\mathbb{C}}$ with $f(S^1) = L$, whose Beltrami coefficients $\mu_f(z) = \partial_{\bar{z}} f/\partial_z f$ are supported in the unit disk $\Delta = \{|z| < 1\}$, and

$$f(z) = z + b_0 + b_1 z^{-1} + \cdots, \quad |z| > 1. \tag{2.6}$$

The conformal maps $f : \Delta^* \to \widehat{\mathbb{C}} \setminus \{0\}$ normalized via (2.6) form the class Σ; its subclass of the maps with k-quasiconformal extensions to $\widehat{\mathbb{C}}$ is denoted by $\Sigma(k)$.

One defines for each $f \in \Sigma$ its *Grunsky coefficients* α_{mn} as the Taylor coefficients of the function

$$-\log \frac{f(z) - f(\zeta)}{z - \zeta} : (\Delta^*)^2 \to \widehat{\mathbb{C}},$$

where the branch of the logarithmic function is chosen which vanishes as $z = \zeta \to \infty$, and the *Grunsky constant*

(2.7)

$$\varkappa(f) = \sup \Big\{ \Big| \sum_{m,n=1}^{\infty} \sqrt{mn}\ \alpha_{mn} x_m x_n \Big| : \mathbf{x} = (x_n) \in l^2,\ \|\mathbf{x}\|^2 = \sum_1^{\infty} |x_n|^2 = 1 \Big\} \leqslant 1.$$

It is well known that for each $f^\mu \in \Sigma(k)$ we have $\varkappa(f^\mu) \leqslant k(f^\mu)$; on the other hand, if $f \in \Sigma$ satisfies this inequality with some $k < 1$, then it has a quasiconformal extension to $\widehat{\mathbb{C}}$ with a dilatation $k' \geqslant k$ (see, e.g., [**Ku1**], [**Po1**], [**Zh**]; [**KK**, pp. 82–84]).

A crucial point is that for generic maps f, we have here the strict inequality

$$\varkappa(f) < \inf k(f^\mu),$$

taking the infimum among all quasiconformal extensions of f (see, e.g., [**Ku2**], [**Kr2**]). The maps f with

$$\varkappa(f) = \inf k(f^\mu) \tag{2.8}$$

are intrisically related to holomorphic quadratic differentials $\varphi(z)\, dz^2$ in Δ having only zeros of even order in Δ. This property plays a crucial role in applying the Grunsky inequalities technique. It will be used here in an essential way.

The following theorem provides a complete characterization of the maps obeying the property (2.8). We shall use the following notations.

Let A_1 denote the subspace of $L_1(\Delta)$ formed by holomorphic functions in Δ, and let

$$A_1^2 = \{\psi \in A_1 : \psi = \omega^2\};$$

this set consists of the integrable holomorphic functions on Δ having only zeros of even order. Put

$$\langle \mu, \psi \rangle_\Delta = \frac{i}{2} \iint_D \mu(z)\psi(z)\, dz \wedge d\overline{z}, \quad \mu \in L_\infty(\Delta),\ \psi \in L_1(\Delta).$$

THEOREM A. [**Kr2**], [**Kr4**]. *Equality $\varkappa(f) = \inf\{\|\mu\|_\infty : w^\mu|\Delta^* = f\}$ holds if and only if the function f is the restriction to Δ^* of a quasiconformal self-map w^{μ_0} of $\widehat{\mathbb{C}}$ with the Beltrami coefficient μ_0 satisfying the condition*

$$\sup |\langle \mu_0, \varphi \rangle_\Delta| = \|\mu_0\|_\infty,$$

where the supremum is taken over holomorphic functions $\varphi \in A_1^2$ with $\|\varphi\|_{A_1} = 1$.

Geometrically, this means that the Carathéodory metric on the holomorphic extremal disk $\Delta_{\mu_0} = \{\phi_{\mathbf{T}}(t\mu_0/\|\mu_0\|) : t \in \Delta\}$ in the space $\mathbf{T}$ coincides with the intrinsic Teichmüller metric of this space.

There is an extension of Theorem A to differentials with zeros of odd order given in [**Kr7**]; this version will not be used here.

3. Complex Finsler metrics and holomorphic curvature

Let X be a domain or a Banach manifold in a complex Banach space E. Consider the upper semicontinuous functions $u : X \to [-\infty, +\infty)$ and define their *generalized Hessian* $\Delta_v u(x)$ *in a direction* v *at a point* $x \in X$ by

$$\Delta_v u(x) = 4 \liminf_{r \to 0} \frac{1}{r^2} \left\{ \frac{1}{2\pi} \int_0^{2\pi} u(x + vre^{i\theta}) d\theta - u(x) \right\} \quad (r > 0).$$

Similar to C^2 functions u on the domains in $\mathbb{C}^n$, for which Δ_v coincides with the usual Hessian, one obtains that u is plurisubharmonic on X if and only if $\Delta_v u(x) \geqslant 0$. Consequently, if x_0 is a point of a local maximum of an upper semicontinuous function u with $u(x_0) > -\infty$, then $\Delta_v u(x_0) \leqslant 0$ for all v. In particular, for $E = \mathbb{C}$ and $v = 1$, one obtains the *generalized Laplacian*

$$\Delta u(x) = 4 \liminf_{r \to 0} \frac{1}{r^2} \left\{ \frac{1}{2\pi} \int_0^{2\pi} u(x + re^{i\theta}) d\theta - u(x) \right\} \tag{3.1}$$

(again reducing to the usual Laplacian $4\partial\bar{\partial}$ for C^2 functions).

Let $F_X(x, v) \geqslant 0$ be a Finsler structure on the tangent bundle $\mathcal{T}(X) \subset X \times E$ of X. We suppose that the function F is upper semicontinuous on $\mathcal{T}(X)$.

Consider for a fixed $(x, v) \in \mathcal{T}(X)$ the holomorphic maps $h : \Delta_r \to X$ with $h(0) = x$, $h'(0) = v$ (for suitable $r > 0$). Any such h determines a conformal metric $ds = \lambda_h(t)|dt|$ on the disk Δ_r, with

$$\lambda_h(t) = F_X(h(t), h'(t)). \tag{3.2}$$

It is an infinitesimal Finsler metric whose density λ_h is upper semicontinuous on Δ_r. If $F_X(x, v)$ is plurisubharmonic (respectively, logarithmically plurisubharmonic) in $x \in X$, the function λ_h becomes subharmonic (respectively, logarithmically subharmonic) on the disk Δ_r.

The structure $F_X(x, v)$ can be regarded as an infinitesimal Finsler metric on $\mathcal{T}(X)$. Its *holomorphic sectional curvature* $\kappa_F(x, v)$ at a point (x, v), where $F_X(x, v) > 0$, is defined as the upper bound of Gaussian curvatures of metrics (3.2), using the generalized Laplacian (3.1), i.e,

$$\kappa_h(t) = -\frac{\Delta \log \lambda_h^2(t)}{2\lambda_h^2(t)} \quad (t \in \Delta_r) \tag{3.3}$$

and

$$\kappa_F(x, v) = \sup \frac{\Delta \log F_X^2(h(t), h'(t))}{-2F_X^2(h(t), h'(t))}\Big|_{t=0}, \tag{3.4}$$

where the supremum is taken over all holomorphic maps $h : \Delta_r \to X$ with $h(0) = x$, $h'(0) = v$ and all admissible $r > 0$ (or equivalently, over holomorphic maps $h : \Delta \to X$ and all $\xi \in \mathbb{C}$ so that $h(0) = x$, $dh(0)(1/\xi) = v$).

It is well known that the holomorphic curvature of the Kobayashi metric $\mathcal{K}_M(x, v)$ on a complete hyperbolic complex Banach manifold M satisfies $\kappa_{\mathcal{K}_M}(x, v) \geqslant -4$ for all $(x, v) \in T(X)$ (see, e.g., [**Di**], [**Ro2**]).

4. Proof of Theorem 1.1

The main idea of this proof is to show that the Finsler structure $F_{\mathbf{T}}$ of the universal Teichmüller space $\mathbf{T}$ determining its Teichmüller metric is equal to the plurisubharmonic differential Azukava metric. This will be accomplished in three stages.

Step 1: Quadratic differentials with zeros of even order.

Take a tangent vector $v_0 = \psi_0 \in \mathbf{B}$ to $\mathbf{T}$ at the basepoint $\psi = \mathbf{0}$ of such length $\|\psi_0\|_{\mathbf{B}}$ that it belongs to $\mathbf{T}$.

There is a unique $f_0 \in \bigcup_k \Sigma(k)$ whose Schwarzian $S_{f_0} = \psi_0$; this f_0 admits quasiconformal extensions w^μ to $\widehat{\mathbb{C}}$ with μ supported in Δ.

Take an extremal Beltrami coefficient μ_0 for f_0, i.e., such that

$$K(w^{\mu_0}) = \tau_{\mathbf{T}}(\phi(\mu_0), \mathbf{0}).$$

The Hamilton–Krushkal–Reich–Strebel theorem says that this equality is equivalent to

$$\sup\{|\langle \mu_0, \varphi\rangle_\Delta| : \varphi \in A_1(\Delta),\ \|\varphi\| = 1\} = \|\mu_0\|_\infty \tag{4.1}$$

and that the same relation is necessary and sufficient for μ_0 to be infinitesimally extremal, i.e., to satisfy $\|\mu_0\|_\infty = F_{\mathbf{T}}(\mathbf{0}, \phi'(\mathbf{0})\mu_0)$ (see [**EKK**], [**GL**], [**Ha**], [**Kr1**], [**RS**]).

In view of (4.1), there exists a maximizing sequence $\{\varphi_p\} \subset A_1(\Delta)$ for μ_0 so that

$$\|\mu_0\|_\infty = \lim_{p\to\infty} \Big| \iint_\Delta \mu_0(z)\varphi_p(z)\, dx\, dy \Big|. \tag{4.2}$$

Since $\|\omega(z) - \omega(rz)\|_{A_1(\Delta)} \to 0$ as $r \to 1$, one can assume that every φ_p is holomorphic in the closed disk $\overline{\Delta}$ and thus has there only a finite number of zeros. Setting

$$\mu_0^* = \mu_0/\|\mu_0\|_\infty,$$

equality (4.2) can be rewritten in the form

$$\lim_{p\to\infty} \Big| t \iint_\Delta \mu_0^*(z)\varphi_p(z)\, dx\, dy \Big| = |t|, \quad |t| \leqslant 1/k(f_0). \tag{4.3}$$

If there are infinitely many $\varphi_{p_1}, \varphi_{p_2}, \dots$ with zeros of even order in Δ, then the assertion of Theorem 1.1 for the Teichmüller disk $\Delta_{\mu_0^*}$ immediately follows from Theorem A. Indeed, by Parseval's equality, each such φ_{p_j} can be represented in the form

$$\varphi_{p_j}(z) = \frac{1}{\pi} \sum_{m+n=2}^{\infty} \sqrt{mn}\, x_m^{(p_j)} x_n^{(p_j)} z^{m+n-2}, \tag{4.4}$$

where the numbers $x_n^{(p_j)}$ are the coordinates of the points $\mathbf{x}^{(p_j)} = \big(x_n^{(p_j)}\big)$ of the Hilbert sphere

$$S(l^2) = \{\|\mathbf{x}\|_{l^2} = 1\}.$$

This allows us to define a sequence of holomorphic maps $\mathbf{T} \to \Delta$ by

$$h_{\mathbf{x}}^{(p_j)}(\psi) = \sum_{m,n=1}^{\infty} \sqrt{mn}\, \alpha_{mn}(\psi) x_m^{(p_j)} x_n^{(p_j)}, \tag{4.5}$$

and this sequence is maximizing for the Carathéodory distance $c_{\mathbf{T}}(\mathbf{0}, \psi_0)$, because by (4.1) we have

$$c_{\mathbf{T}}(\mathbf{0}, \psi_0) = \tau_{\mathbf{T}}(\mathbf{0}, \psi_0) \quad \text{for all } t \in \Delta.$$

Therefore, all invariant metrics on the holomorphic disk $\Delta_{\mu_0^*}$ coincide.

Step 2: Construction of defining plurisubharmonic functions for differentials with zeros of odd order.

Now consider the situation when all differentials $\varphi_1, \varphi_2, \dots$ have in the unit disk only zeros of odd order. In this case, we modify the previous arguments as follows. Let

$$a_1^{(p)}, a_2^{(p)}, \dots, a_{s(p)}^{(p)} \tag{4.6}$$

be the list of all zeros of a differential φ_p of odd order in the disk Δ. For convenience, the zeros are listed so that $|a_j^{(p)}| \leqslant |a_{j+1}^{(p)}|$, and $a_j^{(p)}, a_{j+1}^{(p)}$ are assumed to be located counterclockwise if $|a_j^{(p)}| = |a_{j+1}^{(p)}|$. We connect these points by distinct smooth arcs $\sigma_j^{(p)} = [a_j^{(p)}, a_{j+1}^{(p)}]$, which form nonzero angles at $a_j^{(p)}$, and add an arc $\sigma_{s(p)+1}^{(p)}$ joining the last zero $a_{s(p)}^{(p)}$ with the unit circle so that the union $\sigma^{(p)} = \bigcup_{j=1}^{s(p)+1} \sigma_j^{(p)}$ is a piecewise smooth slit whose complement $\Delta_p = \Delta \setminus \sigma^{(p)}$ is a simply connected domain.

Take a conformal map $z = g_p(\zeta)$ of the disk Δ onto this domain Δ_p and define the transform $\chi_p : \mu \mapsto \widetilde{\mu}$, setting

$$\widetilde{\mu}(\zeta) = \begin{cases} 0, & \text{if } |\zeta| > 1, \\ \mu|_{\Delta_p}(g_p(\zeta)) \overline{g_p'(\zeta)}/g_p'(\zeta), & \text{if } |\zeta| < 1. \end{cases} \tag{4.7}$$

This transform determines a biholomorphic self-isometry of the ball $\mathrm{Belt}(\Delta)_1$ and depends on the choice of the slit $\sigma^{(p)}$.

On the other hand, by applying the map g_p the original quadratic differential φ_p is transformed to the differential

$$g_p^*(\varphi_p) = \varphi_p(g_p(\zeta)) g_p'(\zeta)^2 =: \widetilde{\varphi}_p(\zeta) \tag{4.8}$$

which is holomorphic in the whole disk Δ and has in this disk only zeros of even order. Besides,

$$\langle \mu, \varphi_p \rangle_\Delta = \langle \widetilde{\mu}, \widetilde{\varphi}_p \rangle_\Delta. \tag{4.9}$$

Similar to (4.4), the function $\widetilde{\varphi}_p$ can be represented in the form

$$\widetilde{\varphi}_p(\zeta) = \frac{1}{\pi} \sum_{m+n=2}^{\infty} \sqrt{mn}\, x_m^{(p)} x_n^{(p)} \zeta^{m+n-2}, \tag{4.10}$$

with some $\mathbf{x}^{(p)} = (x_n^{(p)}) \in S(l^2)$.

We now define on the ball $\mathrm{Belt}(\Delta)_1$ a new equivalence relation, letting the Beltrami coefficients $\mu, \nu \in \mathrm{Belt}(\Delta)_1$ be *χ_p-equivalent* and write $\chi_p(\mu) \equiv \chi_p(\nu)$, if the corresponding quasiconformal homeomorphisms $w^{\chi_p(\mu)}$ and $w^{\chi_p(\nu)}$ *coincide on the unit circle* S^1 (and therefore on $\overline{\Delta^*}$). In other words, we require that the initial maps $w^\mu|\Delta$ and $w^\nu|\Delta$ coincide on $S^1 \cup \sigma^{(p)}$.

Factorization of $\chi_p(\mathrm{Belt}(\Delta)_1)$ by such a equivalence relation gives again the universal Teichmüller space $\mathbf{T}$, which we shall denote by $\mathbf{T}_p$. Its Bers embedding consists of the Schwarzian derivatives of conformal maps $w^{\chi_p(\mu)}|\Delta^*$.

Conversely, fix a point ζ_0 on the circle S^1 with $0 < \arg \zeta_0 < \pi$, and consider the conformal map g of the unit disk Δ onto Δ cut along the radius $[0, 1]$ so that $g(\pm\zeta_0) = 1$, $g(1) = 0$ and $g(-1) = -1$ (this map possesses the symmetry $g(\overline{\zeta}) = \overline{g(\zeta)}$). This defines a self-isomorphism

$$\mu \mapsto g * (\mu) = \mu \circ g^{-1}(z) g'(z)/\overline{g'(z)} \tag{4.11}$$

of the unit ball $L_\infty(\Delta)_1 = \{\nu \in L_\infty(\Delta) : \|\nu\| < 1\}$.

We extend the elements of $L_\infty(\Delta)_1$ (the Beltrami coefficients with the supports in Δ) by symmetry onto Δ^* and call two such coefficients μ_1, μ_2 to be *g-equivalent* if the corresponding quasiconformal automorphisms $w^{g*(\mu_1)}$ and $w^{g*(\mu_2)}$ coincide on S^1. This equivalence provides a Teichmüller space similar to $\mathbf{T}_p$.

It is evident that the arc $(-\zeta_0, z_0) \subset S^1$ can be replaced by any other subarc (ζ_0', ζ_0'') of S^1.

Lemma 4.1. *The isometry χ_p commutes with holomorphic projections ϕ_p : $\mathrm{Belt}(\Delta)_1 \to \mathbf{T}_p$ and $\phi : \mathrm{Belt}(\Delta)_1 \to \mathbf{T}$, i.e., the diagram*

$$\begin{array}{ccc} \mathrm{Belt}(\Delta)_1 & \xrightarrow{\phi_p} & \mathbf{T}_p \\ {\scriptstyle \chi_p^{-1}}\downarrow & & \downarrow{\scriptstyle \pi_p} \\ \mathrm{Belt}(\Delta)_1 & \xrightarrow{\phi} & \mathbf{T} \end{array} \tag{4.12}$$

is commutative, which determines a surjective holomorphic map $\pi_p : \mathbf{T}_p \to \mathbf{T}$.

Note that the map π_p does not preserve the Teichmüller distance, but the extremality of the Beltrami coefficients is preserved, which follows from (4.1) and (4.9).

Proof. One needs to make certain that if $\chi_p(\mu) \equiv \chi_p(\nu)$, then $\mu \sim \nu$ in Teichmüller's sense, but this evidently follows from the definition of χ_p-equivalence. $\square$

The fibers $\pi_p^{-1}(\psi)$ contain the elements $\psi^* \in \mathbf{T}_p$ which are located on arbitrarily large Teichmüller distance from $\psi^{\chi_p(\mu)}$. Such Schwarzians correspond to extensions of $w^\mu|S^1$ with large values $|\mu_w|$ in a neighborhood of $\sigma^{(p)}$. We shall need to restrict us by extensions with bounded $\|\chi_p(\mu)\|_\infty$.

To this end, we choose an extremal Beltrami coefficient $\mu_0(\psi) \in \mathrm{Belt}(\Delta)_1$ in the class $[f]$ of a map $f \in \bigcup_k \Sigma(k)$ with $S_f = \psi$ and take, for a fixed $K \in (0, \infty)$, the ball

$$B_p(\psi, K) = \{\varphi \in \mathbf{B} : d_{\mathbf{T}_p}(\varphi, \psi^{\chi_p(\mu_0)}) \leqslant K\} \subset \mathbf{T}_p.$$

Now consider the holomorphic maps

$$\widetilde{h}_{\mathbf{x}}(\psi^{\chi_p(\mu)}) = \sum_{m,n=1}^{\infty} \sqrt{mn}\, \alpha_{mn}(\chi_p(\mu)) x_m x_n : \mathbf{T}_p \to \Delta, \tag{4.13}$$

where $\psi^{\chi_p(\mu)}$ is the Schwarzian of $w^{\chi_p(\mu)}|\Delta^*$ and the defining parameter $\mathbf{x} = (x_n)$ ranges over the sphere $S(l^2)$, and define the upper envelopes

$$\widetilde{h}^*_{\mathbf{x},K,\sigma^{(p)}}(\psi) = \sup\{|\widetilde{h}_{\mathbf{x}}(\psi^{\chi_p(\mu)})| : \psi^{\chi_p(\mu)} \in \pi_p^{-1}(\psi) \cap B(\psi, K) \subset \mathbf{T}_p\} \tag{4.14}$$

and

$$\widetilde{h}_{\mathbf{x},K}(\psi) = \sup_{\sigma^{(p)}} \widetilde{h}^*_{\mathbf{x},K,\sigma^{(p)}}(\psi), \tag{4.15}$$

where the supremum is taken over all slits $\sigma^{(p)}$ joining the given zeros $a_j^{(p)}$. Both functions (4.14) and (4.15) range over $[0, 1)$.

The last function depends only on these zeros and descends to a function on the initial Teichmüller space $\mathbf{T}$ with the range $[0, 1)$. Let us denote the function pushed down by $h_{\mathbf{x},K}$. Since $h_{\mathbf{x},K_1}(\psi) \leqslant h_{\mathbf{x},K_2}(\psi)$ if $K_1 \leqslant K_2$, there exists a nonconstant limit function

$$h_{\mathbf{x}}(\psi) = \lim_{K \to 0} h_{\mathbf{x},K}(\psi).$$

It admits the following background property:

LEMMA 4.2. *The function $h_{\mathbf{x}}$ satisfies the mean value inequality*

$$h_{\mathbf{x}}(\psi_0) \leqslant \frac{1}{2\pi} \int_0^{2\pi} h_{\mathbf{x}}(\psi_0 + r\omega e^{i\theta})\, d\theta \tag{4.16}$$

for any $\omega \in \mathbf{B}$ and sufficiently small $r > 0$.

PROOF. Consider the planar set

$$\Omega(\psi_0) = \{\psi_0 + t\omega : t \in \mathbb{C}\} \cap \mathbf{T}.$$

It is open but does not need to be connected (see [**Kr3**]). We take its connected component $\Omega_0(\psi_0)$ containing ψ_0 and identify $\Omega_0(\psi_0)$ with the corresponding range domain of t in $\mathbb{C}$. It follows from Zhuravlev's theorem (see [**KK**, Part 1, Ch. V], [**Zh**]) that each connected component of $\Omega(\psi_0)$ is simply connected.

Schwarzians $\psi_t = \psi_0 + t\omega \in \Omega_0(\psi_0)$ determine a family of univalent functions

$$f_t(z) = z + b_0(t) + b_1(t)z^{-1} + \dots : \Delta^* \to \mathbb{C} \setminus \{0\}$$

with $S_{f_t} = \psi_t$ which admit quasiconformal extensions $\widehat{f}_t$ to the disk $\overline{\Delta}$ so that $\widehat{f}_t(0) = 0$; these functions are holomorphic in $t \in \Omega_0(\psi_0)$. The map

$$W(w, t) = f_t \circ f_0^{-1}(w) : f_0(\Delta^*) \times \Omega_0(\psi_0) \to \mathbb{C} \setminus \{0\}$$

determines a holomorphic motion of domain $f_0(\Delta^*)$ with the parameter space $\Omega_0(\psi_0)$ (i.e., it is injective in w for every fixed t, holomorphic in t for every fixed w, and $W(w,0)=w$ on $f_0(\Delta^*)$). Put $W(0,t)=0$ for all t.

By the properties of holomorphic motions (see, e.g., [**BR**], [**EKK**], [**EM**], [**MSS**], [**Sl**], [**ST**]), $W(w,t)$ extends from the set $f_0(\Delta^*)\cup\{0\}$ to a holomorphic motion $\widehat{W}(w,t):\widehat{\mathbb{C}}\times\Omega_0(\psi_0)$ of the whole sphere so that the fiber maps $W_t(w)=W(w,t)$ are quasiconformal homeomorphisms of $\widehat{\mathbb{C}}$ (conformal on $f_0(\Delta^*)$) whose Beltrami coefficients

$$\mu_W(w,t)=\partial_{\bar{w}}W(w,t)/\partial_w W(w,t)$$

depend holomorphically on parameter $t\in\Omega_0(\psi_0)$ as elements of $L_\infty(\mathbb{C})$.

Now choose $\nu\in\phi^{-1}(\psi_0)$ in $\mathrm{Belt}(\Delta)_1$ and consider the composite maps $W_t\circ w^\nu$. Their Beltrami coefficients

$$\mu_{W_t\circ w^\nu}=\frac{\mu_{W_t}\circ w^\nu+\nu\partial w^\nu/\overline{\partial w^\nu}}{1+\overline{\nu}(\mu_{W_t}\circ w^\nu)\partial w^\nu/\overline{\partial w^\nu}},\quad t\in\Omega_0^0,$$

range over a (simply connected) complex disk $\Delta_\nu\subset\mathrm{Belt}(\Delta)_1$. The transform (4.7), for a fixed collection of cuts $\sigma^{(p)}$, carries this disk onto the complex disk $\chi_p(\Delta_\nu)$, whiles the above construction of the space $\mathbf{T}_p$ yields that the restriction of projection ϕ_p onto the last disk is injective. Therefore, the composite $\phi_p\circ\chi_p\circ\phi^{-1}|\Omega_0(\psi_0)$ maps the distinguished domain $\Omega_0(\psi_0)$ biholomorphically (thus conformally) onto a simply connected holomorphic curve (Riemann surface)

$$\widetilde{\Omega}_0(\widetilde{\psi}_0)=\phi_p\circ\chi_p(\Delta_\nu)\subset\mathbf{T}_p,$$

where

$$\widetilde{\psi}_0=\phi_p\circ\chi_p(\nu)\in\pi_p(\psi_0).$$

Denote the indicated conformal map $\Omega_0(\psi_0)\to\widetilde{\Omega}_0(\widetilde{\psi}_0)$ by e_p.

The restrictions

$$\widetilde{h}_{\mathbf{x}}^{(p)}[\phi_p\circ\chi_p(\mu_{W_t\circ w^\nu})] \tag{4.17}$$

of the functions (4.13) to $\widetilde{\Omega}_0(\widetilde{\psi}_0)$ are holomorphic on this curve, thus the corresponding functions

$$h_{\mathbf{x}}^{(p)}\circ\phi_p\circ\chi_p(\mu_{W_t\circ w^\nu})\circ e_p \tag{4.18}$$

are holomorphic on the initial planar domain $\Omega_0(\psi_0)\subset\mathbf{T}$. The functions (4.17) and (4.18) depend on the choice of both $\nu\in\phi^{-1}(\psi_0)$ and $\sigma^{(p)}$.

On the other hand, let $\widetilde{\psi}_*$ be a point in the fiber $\pi_p(\psi_0)$ over ψ_0. Choose a Beltrami coefficient $\widetilde{\nu}=\phi_p^{-1}(\widetilde{\psi}_*)$ in the ball $\chi_p(\mathrm{Belt}(\Delta)_1)$. Its inverse image under χ_p, i.e., the Beltrami coefficient $\nu=\chi_p^{-1}(\widetilde{\nu})$, must lie in the fiber $\phi^{-1}(\psi_0)$ over ψ_0 in $\mathrm{Belt}(\Delta)_1$, because, by definition, the χ_p-equivalence is stronger than the initial equivalence relation defining the projection of $\mathrm{Belt}(\Delta)_1$ onto the base space $\mathbf{T}$.

This yields that when ν runs over the fiber $\phi^{-1}(\psi_0)\in\mathrm{Belt}(\Delta)_1$, we get for every point $\widetilde{\psi}_*\in\pi_p^{-1}(\psi_0)$ a similar holomorphic disk $\widetilde{\Omega}(\widetilde{\psi}_*)$ over $\Omega_0(\psi_0)$ centered at this point.

Therefore, defining on $\Omega_0(\psi_0)$, in accordance with (4.14) and (4.15), the envelope

$$h^0_{\mathbf{x}}(t) = \sup_{\sigma^{(p)}} \sup_{\nu \in \phi^{-1}(\psi_0)} h^{(p)}_{\mathbf{x}} \circ \phi_p \circ \chi_p(\mu_{W_t \circ w^\nu}) \circ e_p(t), \tag{4.19}$$

we get a function which coincides with the restriction to $\Omega_0(\psi_0)$ of the function $h_{\mathbf{x}}$ determined above on the whole space $\mathbf{T}$. The upper semi-continuous regularization $\limsup_{t' \to t} h^*_{\mathbf{x}}(t')$ of (4.19) is logarithmically subharmonic on $\Omega_0(\psi_0)$ and hence satisfies the mean value inequality on this planar domain. This yields the inequality (4.16) and completes the proof of Lemma 4.2. $\square$

Passing to the upper semi-continuous regularization

$$h^*_{\mathbf{x}}(\psi) = \limsup_{\varphi \to \psi} h_{\mathbf{x}}(\varphi) \tag{4.20}$$

of $h_{\mathbf{x}}$ on the whole space $\mathbf{T}$, one obtains a nonconstant logarithmically plurisubharmonic function on $\mathbf{T}$ whose properties are summarized in the following lemma.

LEMMA 4.3. *The function $h^*_{\mathbf{x}}$ is logarithmically plurisubharmonic on the space $\mathbf{T}$ and satisfies the relations*

$$\begin{aligned} &h^*_{\mathbf{x}}(\psi) \leqslant \tanh d_{\mathbf{T}}(\mathbf{0}, \psi) \leqslant k_0(f) < 1 \quad \textit{for all} \quad \psi = S_f \in \mathbf{T}; \\ &h^*_{\mathbf{x}}(\psi) = \|\psi\|_{\mathbf{B}} + O(1) \quad \textit{as} \quad \psi \to \mathbf{0}. \end{aligned} \tag{4.21}$$

PROOF. The first relation in (4.21) follows from (2.4) and from definition of $h^*_{\mathbf{x}}$, while the second equality immediately follows from the existence of the Ahlfors-Weill extension with harmonic Beltrami coefficient in Δ of conformal maps $f : \Delta^* \to \widehat{\mathbb{C}}$, provided $\|S_f\|_{\mathbf{B}} \leqslant 2$ (see [**AW**]). $\square$

We proceed with the proof of theorem and apply the Schwarz lemma to the lifts

$$\widetilde{H}_{\mathbf{x},p} = \widetilde{h}_{\mathbf{x}} \circ \phi_p : \mathrm{Belt}(\Delta)_1 \to \Delta, \quad p = 1, 2, \dots,$$

and obtain

$$|\widetilde{H}_{\mathbf{x},p}(t\chi_p(\mu))| = |\widetilde{h}_{\mathbf{x}} \circ \phi_p(t\chi_p(\mu))| \leqslant |t|, \tag{4.22}$$

provided $\|\mu\| = 1$ and $|t| < 1$. Taking into account that

$$\alpha_{mn}(\phi_p(\nu)) = -\frac{1}{\pi} \iint_{\Delta} \nu(z) z^{m+n-2} dx\, dy + O(\|\nu\|^2_\infty) \quad \text{as} \quad \|\nu\|_\infty \to 0,$$

because the functions $w^\nu \in \Sigma(k)$ with small $\|\nu\|_\infty$ are represented by means of the variational formula

$$w^\nu(\zeta) = \zeta - \frac{1}{\pi} \iint_{\Delta} \frac{\nu(z)}{z - \zeta}\, dx\, dy + O(\|\nu\|^2_\infty),$$

we conclude that the differential of the map $\widetilde{H}_{\mathbf{x},p}$ at the origin of $L_\infty(\mathbb{C})$ is given by

$$d\widetilde{H}_{\mathbf{x},p}(\mathbf{0})\nu = -\frac{1}{\pi}\iint\limits_{\Delta} \nu(z) \sum_{m+n=2}^{\infty} \sqrt{mn}\, x_m^{(p)} x_n^{(p)} z^{m+n-2} dx\, dy = -\langle \nu, \varphi_p \rangle_\Delta.$$

Comparison with Theorem A and (4.9) yields that, letting

$$\chi_p(\mu) = |\widetilde{\varphi}_p|/\widetilde{\varphi}_p =: \widetilde{\mu}_p,$$

we will have in (4.22) the case of equality

$$|\widetilde{h}_{\mathbf{x}} \circ \phi_p(t\widetilde{\mu}_p)| = |t\langle \widetilde{\mu}_p, \widetilde{\varphi}_p\rangle_\Delta| = |t| \quad \text{for all} \quad t \in \Delta. \tag{4.23}$$

The relations (4.14), (4.15), (4.20) and (4.23) together imply

$$\limsup_{t\to 0} \frac{h_{\mathbf{x}}^* \circ \phi(t\mu_0^*)}{|t|} \geqslant \limsup_{t\to 0} \frac{h_{\mathbf{x}}^* \circ \phi_p(t\widetilde{\mu}_p)}{|t|} \geqslant \lim_{t\to 0} \frac{|\widetilde{h}_{\mathbf{x}} \circ \phi_p(t\widetilde{\mu}_p)|}{|t|} = 1.$$

Hence, in view of (4.3),

$$\limsup_{t\to 0} \frac{h_{\mathbf{x}}^* \circ \phi(t\mu_0^*)}{|t|} = 1. \tag{4.24}$$

Now, let us consider the function

$$u_0(\psi) = \sup\{h_{\mathbf{x}}^*(\psi) : \mathbf{x} \in S(l^2)\} \tag{4.25}$$

for which, arguing similarly to Lemma 4.3, we get

LEMMA 4.4. *The upper semi-continuous regularization* $u_0^*(\psi) = \limsup_{\varphi\to\psi} u_0(\varphi)$ *of the function* (4.25) *is logarithmically plurisubharmonic on* $\mathbf{T}$ *and possesses the estimates similar to* (4.21).

Therefore the plurisubharmonicity of the Teichmüller distance $\tau_{\mathbf{T}}(\mathbf{0}, \psi)$ is an immediate consequence of the following straightforward extension of the Schwarz lemma (cf. [**Kl1**], [**Kr6**], [**Si**]).

LEMMA 4.5. *Let a function* $u(z) : \Delta \to [0, 1)$ *be logarithmically subharmonic in the disk* Δ *and such that the ratio* $u(z)/|z|^m$ *is bounded in a neighborhood of the origin for some* $m \geqslant 1$. *Then*

$$u(z) \leqslant |z|^m \quad \textit{for all} \quad z \in \Delta \tag{4.26}$$

and

$$\limsup_{|z|\to 0} \frac{u(z)}{|z|^m} \leqslant 1. \tag{4.27}$$

Equality in (4.26), *even for one* $z_0 \neq 0$ *or in* (4.27), *can only hold for* $u(z) = |z|^m$.

PROOF. The function

$$v(z) = \log \frac{u(z)}{|z|^m} = \log u_0(z) - m \log |z|$$

is subharmonic in the punctured disk $\Delta \setminus \{0\}$ and bounded from above near the origin; thus, due the removability of the one-point sets for the locally bounded above subharmonic functions, $v(z)$ extends to a subharmonic function in the whole disk Δ. We shall denote this extension also by $v(z)$. The function

$$v_1(z) = e^{v(z)} = u(z)/|z|^m$$

is also subharmonic on Δ; hence, applying to it the maximum principle in the disks $\overline{\Delta}_r = \{z : |z| \leqslant r\}$, $0 < r < 1$, we get

$$\sup_{\overline{\Delta}_r} v_1(z) = \sup_{|z|=r} v_1(z) = \sup_{|z|=r} \frac{u(z)}{|z|^m} \leqslant \frac{1}{r^m},$$

which implies (4.26).

In the case of equality in (4.26), even for one $z_0 \neq 0$ or in (4.27), we have that $v_1(z)$ attains at z_0 its maximum in Δ equal to 1, but this is possible only if $v_1(z) \equiv 1$. This completes the proof. □

Applying this lemma to subharmonic functions $u(t) = u_0^* \circ \phi(t\mu*)$, where $\mu^* = \mu/\|\mu\|$, $\mu \in \mathrm{Belt}(\Delta)_1$, one obtains $u_0^*(t\psi) \leqslant k(\mathbf{0}, t\psi)$ for all $\psi \in \mathbf{T}$.

On the other hand, it follows from (4.3), (4.21) and (4.24) that for $\mu = \mu_0$, where μ_0 is the initial extremal Beltrami coefficient for the point ψ_0, we have

$$\limsup_{t\to 0} \frac{u_0^* \circ \phi(t\mu_0^*)}{|t|} \geqslant \limsup_{t\to 0} \frac{h_{\mathbf{x}}^* \circ \phi(t\mu_0^*)}{|t|} = 1 = F_{\mathbf{T}}(\mathbf{0}, \phi'(\mu_0^*)),$$

which shows that we have the case of equality in (4.27). Hence,

$$u_0^*(t\psi_0) = k(\mathbf{0}, t\psi_0)$$

for all $t \in \mathbb{C}$ such that $t\psi_0 \in \mathbf{T}$, in particular, for $t = 1$.

Now for an arbitrary point $\psi \in \mathbf{T}$, one can apply the corresponding right translations of this space defined as follows.

Take a Beltrami coefficient $\nu \in \mathrm{Belt}(\Delta)_1$, and let its image in $\mathbf{T}$ be the Schwarzian $\psi = \phi_{\mathbf{T}}(\nu)$. Then $w^\nu(S^1)$ is a quasicircle with the interior domain $D_\nu = w^\nu(\Delta)$. Let w be a conformal map of the disk Δ onto Δ_ν, then we obtain (for fixed ν) a biholomorphic isomorphism

$$\mu \mapsto \sigma_\nu \mu = \frac{\mu - \nu}{1 - \bar{\nu}\mu} \frac{w^\nu \circ w^{-1}}{\overline{\overline{w^\nu \circ w^{-1}}}} \tag{4.28}$$

of the ball $\mathrm{Belt}(\Delta)_1$. This isomorphism is compatible with the canonical projection ϕ and thus descends to a holomorphic bijection $\widehat{\sigma}$ of $\mathbf{T}$ defined from the commutative diagram

$$\begin{array}{ccc}
\mathrm{Belt}(\Delta)_1 & \xrightarrow{\sigma_\nu} & \mathrm{Belt}(\Delta)_1 \\
{\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle\phi} \\
\mathbf{T} & \xrightarrow{\widehat{\sigma}_\nu} & \mathbf{T}
\end{array}$$

Thereby, the translation (4.28) determines the Teichmüller and Kobayashi isometries

$$\tau_{\mathbf{T}}(\phi(\nu), \phi(\mu)) = \tau_{\mathbf{T}}(\mathbf{0}, \phi(\sigma_\nu(\mu))), \quad d_{\mathbf{T}}(\phi(\nu), \phi(\mu)) = d_{\mathbf{T}}(\mathbf{0}, \phi(\sigma_\nu(\mu)))$$

and their infinitesimal analogues (for the corresponding Finsler structures)

$$\begin{aligned} F_{\mathbf{T}}(\phi(\nu), \phi'[\nu]\mu) &= \|\theta/(1-|\nu|^2)\|_\infty = F_{\mathbf{T}}(\mathbf{0}, (\phi\circ\sigma_\nu)'\mu), \\ \mathcal{K}_{\mathbf{T}}(\phi(\nu), \phi'[\nu]\mu) &= \mathcal{K}_{\mathbf{T}}(\mathbf{0}, (\phi\circ\sigma_\nu)'\mu); \end{aligned} \tag{4.29}$$

here $\theta = \mu - \nu$. In particular, the holomorphic disks $\Delta_\mu \in \mathbf{T}$ centered at $\mathbf{0}$ are moved conformally onto the corresponding holomorphic disks centered at φ; therefore, the holomorphic sectional curvatures of these metrics are preserved.

The function

$$\mathcal{A}(\phi(\nu), \phi'(\nu)\mu) = u_0^* \circ\ \widehat{\sigma}_\nu(\mu)$$

determines an infinitesimal plurisubharmonic metric on $\mathcal{T}(\mathbf{T})$ which is maximal among all such metrics majorized by $F_{\mathbf{T}}(\phi(\nu), \phi'(\nu)\mu)$. Hence, it must coincide with the Azukava metric of $\mathbf{T}$ generated by the pluricomplex Green function (1.2) (cf. [**Az**], [**Kl2**]), and therefore (cf. [**EM**]),

$$\mathcal{K}_{\mathbf{T}}(\phi(\nu), \phi'(\nu)\mu) = F_{\mathbf{T}}(\phi(\nu), \phi'(\nu)\mu). \tag{4.30}$$

Step 3: Curvature comparison for some Finsler metric.

As was already mentioned in Section 2.2, the holomorphic curvature of the Kobayashi metric of every complete hyperbolic complex Banach manifold M satisfies

$$\kappa_{\mathcal{K}_M}(\mathbf{x}, v) \geqslant -4.$$

Our goal is to show that the holomorphic maps constructed above allow us to establish for $M = \mathbf{T}$ the opposite inequality. Using the right translations (4.28), one reduces the proof to the case when $\mathbf{x} = \mathbf{0}$ and $v = \psi_0$ is a unit tangent vector to $\mathbf{T}$ at $\mathbf{0}$, i.e., such that $\|v\|_{T_0(\mathbf{T})} = 1$, where $T_0(\mathbf{T})$ means the tangent space to $\mathbf{T}$ at the base point.

Let $g : t \mapsto \psi(z,t)$ be a holomorphic map of a disk Δ_r $(r > 0)$ into $\mathbf{T}$ with $g(0) = \mathbf{0}$, $g'(0) = \psi_0$. This yields

$$\psi(z,t) = t\psi_0(z) + \omega(z,t), \quad \|\omega(z,t)\|_{\mathbf{B}} = O(t^2) \quad \text{as } t \to 0. \tag{4.31}$$

Assuming that r is sufficiently small (so that $\|\psi\|_{\mathbf{B}} < 2$), one can extend by [**AW**] the solutions w of the Schwarz equation $S_w(z) = \psi(z,t)$ in Δ^* to a quasiconformal homeomorphism w^{ν_ψ} of $\widehat{\mathbb{C}}$ with harmonic Beltrami coefficient

$$\nu_\psi(z,t) = -\frac{1}{2}(1-|z|^2)^2 \psi\Big(\frac{1}{\overline{z}}, t\Big)\frac{1}{\overline{z}^4},$$

in the disk Δ. Then $\nu_{\psi_0} - \mu^* \in A_1(\Delta)^\perp$ and by (4.31),

$$\sup\{|\langle \nu_{\psi(\cdot,t)} - t\mu_0^*, \varphi\rangle_\Delta| : \|\varphi\|_{A_1} = 1\} = O(t^2), \quad t \to 0. \tag{4.32}$$

Now consider the sequence of holomorphic maps

$$h_p(t) = \sum_{m,n=1}^{\infty} \sqrt{mn}\, \alpha_{mn}(\psi(t)) x_m^{(p)} x_n^{(p)} : \Delta_r \to \Delta \quad (p = 1, 2, \dots);$$

it is normal in the disk Δ_r. The derivatives of these maps at $t = 0$ are given by the formulas following (4.22). Applying the equality (4.32), one concludes that there exists $r > 0$ such that the sequence $\{h_p\}$ is convergent uniformly on the disk Δ_r to a conformal map

$$h_0(t) : \Delta_r \to \Delta.$$

The relations (4.3) and (4.23) imply that the derivative $h_0'(0)$ is the unit tangent vector to Δ at the origin.

Since the holomorphic maps decrease the hyperbolic metrics, we have

$$h_0^*\mathcal{A}(t) = \mathcal{A}(g(t), g'(t)) \geqslant \frac{|h_0'(t)|}{1 - |h_0(t)|^2}, \quad t \in \Delta_r.$$

The last relation shows that

$$\lambda_0(t) = \frac{|h_0'(t)|}{1 - |h_0(t)|^2}$$

is a *supporting metric* for $\lambda_{\mathcal{A}} = h_0^* A$ at $t = 0$, which means that $\lambda_0(0) = \lambda_{\mathcal{A}}(0)$ and $\lambda_0(t) \leqslant \lambda_{\mathcal{A}}(t)$ in a neighborhood of the origin. This metric has Gaussian curvature $\kappa(\lambda_0) = -4$ in noncritical points of h_0, thus in a neighborhood of the origin.

It follows that the ratio $\log \frac{\lambda_0}{\lambda_{\mathcal{A}}}$ has a local maximum at the point $t = 0$, and therefore, its generalized Laplacian in this point

$$\Delta \log \frac{\lambda_0}{\lambda_{\mathcal{A}}}(0) = \Delta \log \lambda_0(0) - \Delta \log \lambda_{\mathcal{A}}(0) \leqslant 0.$$

This implies

$$-\frac{\Delta \log \lambda_{\mathcal{A}}(0)}{\lambda_{\mathcal{A}}(0)^2} \leqslant -\frac{\Delta \log \lambda_0(0)}{\lambda_0^2(0)},$$

and the desired inequality $k(\lambda_{\mathcal{A}}) \leqslant -4$ follows.

Comparison of the obtained relations for the curvatures with the equality (4.30) completes the proof of Theorem 1.1.

5. Metric geometry of hyperbolic balls in T

By definition, the hyperbolic distance $d_{\Delta_\kappa}(0, t)$ on the disk $\Delta_\kappa = \{|t| < \kappa\}$ is given by

$$d_{\Delta_\kappa}(0, t) = d_\Delta\Big(0, \frac{t}{\kappa}\Big). \tag{5.1}$$

It is not known how to relate in the general case the Carathéodory and Kobayashi distances on the balls in a complex manifold X with the corresponding distances on X. As for the universal Teichmüller space, Theorem 1.1 and Corollary 1.3 ensure a relation similar to (5.1) for the Kobayashi metric on the hyperbolic (Teichmüller) balls

$$B_\kappa(\mathbf{T}) = \{\psi \in \mathbf{T} : d_{\mathbf{T}}(\psi, \mathbf{0}) < K\} \quad (0 < K < \infty),$$

where the values κ and K are related by $\kappa = \tanh K$.

THEOREM 5.1. *The Kobayashi metrics $d_{\mathbf{T}}$ and $d_{B_\kappa(\mathbf{T})}$ satisfy the relation*

$$d_{B_\kappa(\mathbf{T})}(\mathbf{0}, \psi) = \tanh^{-1}\Big(\frac{l(d_{\mathbf{T}}(\mathbf{0}, \psi))}{\kappa}\Big) = d_\Delta\Big(0, \frac{l(d_{\mathbf{T}}(\mathbf{0}, \psi))}{\kappa}\Big) \tag{5.2}$$

for the points $\psi \in B_\kappa(\mathbf{T})$, where $l(s) = \tanh s$.

The corresponding Green functions are related in a similar way; in particular, a representation of type (1.2) *for the balls* $B_\kappa(\mathbf{T})$ *assumes the form*

$$g_{B_\kappa(\mathbf{T})}(\mathbf{0},\psi) = \log \frac{k(\mathbf{0},\psi)}{\kappa} = \log \frac{e^{g_{\mathbf{T}}(\mathbf{0},\psi)}}{\kappa}. \tag{5.3}$$

Proof. Let $\psi_0 \in B_\kappa(\mathbf{T})$, and let $\mu_0 \in \mathrm{Belt}(\Delta)_1$ be an extremal Beltrami coefficient for ψ_0, i.e., $\tau_{\mathbf{T}}(\mathbf{0},\psi_0) = d_\Delta(0, \|\mu_0\|_\infty)$. Consider the Teichmüller isometries

$$h(t) = t\frac{\mu_0}{\|\mu_0\|_\infty} : \Delta \to \mathrm{Belt}(\Delta)_1 \quad \text{and} \quad \phi_{\mathbf{T}} \circ h : \Delta \to \mathbf{T}.$$

Then $\mu_0 = h(\|\mu_0\|)$ and $\phi_{\mathbf{T}} \circ h(\Delta_\kappa) \subset B_\kappa(\mathbf{T})$. Since the map $\phi_{\mathbf{T}} \circ h$ is holomorphic, we have

$$\begin{aligned} d_{B_\kappa(\mathbf{T})}(\mathbf{0},\psi_0) &= d_{B_\kappa(\mathbf{T})}(\phi_{\mathbf{T}} \circ h(0), \phi_{\mathbf{T}} \circ h(\|\mu_0\|)) \\ &\leqslant d_{\Delta_\kappa}(0, \|\mu_0\|) = d_\Delta\Big(0, \frac{\|\mu_0\|}{\kappa}\Big). \end{aligned} \tag{5.4}$$

To establish the opposite inequality, observe that the function

$$G_\kappa(\varphi,\psi) = \log \frac{k(\varphi,\psi)}{\kappa}$$

is separately plurisubharmonic in each of the variables φ, ψ on the whole space $\mathbf{T}$; the values of $G_\kappa(\mathbf{0},\psi)$ on $B_\kappa(\mathbf{T})$ belong to $[-\infty, 0)$, and

$$G_\kappa(\mathbf{0},\psi_0) = \log \frac{\|\mu_0\|}{\kappa} = \log \tanh d_{\Delta_\kappa}(0, \|\mu_0\|). \tag{5.5}$$

Since the Green function $g_{B_\kappa(\mathbf{T})}$ is maximal, (5.5) yields

$$g_{B_\kappa(\mathbf{T})}(\mathbf{0},\psi_0) \geqslant G_\kappa(\mathbf{0},\psi_0) \geqslant \log \tanh d_{\Delta_\kappa}(0, \|\mu_0\|). \tag{5.6}$$

The relations (5.4) and (5.6), together with (1.2), imply the desired equalities (5.2) and (5.3). ☐

6. Extension to Teichmüller space of the punctured disk

6.1. Many problems concerning holomorphic functions with quasiconformal extensions require to exploit more normalization conditions for these extensions.

Suitable model examples are provided by the class of univalent functions

$$f(z) = z + \sum_{n=3}^{\infty} a_n z^n$$

in the unit disk Δ, whose extensions $\widehat{f}$ to $\widehat{\mathbb{C}}$ satisfy $\widehat{f}(\infty) = \infty$, and the corresponding class of nonvanishing univalent functions $F(z) = z + b_1 z^{-1} + \cdots$ in Δ^* with the extensions $\widehat{F}$ to $\overline{\Delta}$ satisfying $\widehat{F}(0) = 0$. These classes are related by $F(z) = 1/f(1/z)$. Therefore, not all initial data become admissible to solve the problems.

Such maps are naturally connected with the Teichmüller space $\mathbf{T}(\Delta_*)$ of the punctured disk $\Delta_* = \Delta \setminus \{0\}$.

We shall show that Theorems 1.1 and 5.1 can be naturally extended to this space. To this end, we embed $\mathbf{T}(\Delta_*)$ holomorphically into $\mathbf{T}$.

6.2. The elements of $\mathbf{T}(\Delta_*)$ are the equivalence classes of the Beltrami coefficients $\mu \in \mathrm{Belt}(\Delta)_1$ so that the corresponding quasiconformal automorphisms W^μ of the unit disk coincide on $S^1 \cup \{0\}$ and are homotopic on the punctured disk Δ_*.

This space can be endowed with a canonical complex structure of a complex Banach manifold and embedded into $\mathbf{T}$ using uniformization.

Namely, Δ_* is conformally equivalent to Δ/Γ, where Γ is a cyclic parabolic Fuchsian group acting on Γ discontinously. The functions $\mu \in L_\infty(\Delta)$ are lifted to Δ as the Beltrami $((-1,1)$-measurable forms $\widetilde{\mu}\, d\overline{z}/dz$ in Δ with respect to Γ (i.e., via $(\widetilde{\mu} \circ \gamma)\overline{\gamma'}/\gamma' = \widetilde{\mu},\ \gamma \in \Gamma)$ forming the Banach space $L_\infty(\Delta, \Gamma)$. All these $\widetilde{\mu}$ are extended by zero to Δ^*; then the corresponding Schwarzians $S_{w^{\widetilde{\mu}}|\Delta^*}$ belong to $\mathbf{T}$. Moreover, $\mathbf{T}(\Delta_*)$ is isomorphic to the subspace $T(\Gamma) = \mathbf{T} \cap \mathbf{B}(\Gamma)$, where $\mathbf{B}(\Gamma)$ consists of elements $\psi \in \mathbf{B}$ satisfying $(\psi \circ \gamma)(\gamma')^2 = \psi$ in Δ^* for all $\gamma \in \Gamma$.

Due to the Bers isomorphism theorem, $\mathbf{T}(\Delta_*)$ is biholomorphically equivalent to the Bers fiber space

$$\mathrm{Fib}(\mathbf{T}) = \{\phi_{\mathbf{T}}(\mu), z) \in \mathbf{T} \times \mathbb{C} : \mu \in \mathrm{Belt}(\Delta)_1,\ z \in w^\mu(\Delta)\}$$

over $\mathbf{T}$ with holomorphic projection $\pi(\psi, z) = \psi$ (see [**Be**]).

6.3. Our approach is somewhat different and does not require uniformization.

We take now the Jordan curves σ_0 joining 0 with S^1 which can turn around the origin many times so that each turn determines a different homotopy class of curves on Δ_*. Fix such σ_0.

The converse construction by reducing the boundary values of quasiconformal maps $w^\mu|\overline{\Delta}$ along the subarcs $(\zeta_0', \zeta_0'') \subset S^1$ and transforming the Beltrami coefficients μ analogously to (4.11) exploited above must be modify as follows. We apply a conformal map g of Δ onto the slit disk $\Delta \setminus \sigma_0$, and call $\mu_1, \mu_2 \in \mathrm{Belt}(\Delta)_1$ to be *g-equivalent*, if $w^{g*(\mu_1)}$ and $w^{g*(\mu_2)}$ coincide on $S^1 \cup \{0\}$ and are homotopic on Δ^*. Then one obtains that the images $g * (\mu_1)$ and $g * (\mu_2)$ of any two $\mu_1, \mu_2 \in \mathrm{Belt}_1$ which are equivalent in the Teichmüller sense (i.e., $w^{\mu_1} = w^{\mu_2}$ on S^1) become g-equivalent.

Indeed, equality of the maps $w^{g*(\mu_1)}$ and $w^{g*(\mu_2)}$ on $S^1 \cup \{0\}$ allows us to construct the needed homotopy $w(z,t) : \Delta_* \times [0,1]$ for these maps in a standard way, when $w(z,t)$ is the point dividing the hyperbolic segment $[w^{g*(\mu_1)}(z), w^{g*(\mu_2)}(z)]$ on Δ_* in the ratio $t/(1-t)$.

Note that one can use also the hyperbolic metric of the whole disk Δ and obtain a homotopy of these maps which preserves the origin fixed. Then, by D. Epstein's theorem on the homotopy with a fixed base point, the restriction $w(z,t)$ to Δ_* yields a desired homotopy on the punctured disk (cf. [**Be**], [**Ep**]).

Now observe that the extremality condition for the elements $\mu \in \mathrm{Belt}(\Delta)_1$ to realize the Teichmüller metric of $\mathbf{T}(\Delta_*)$ is again of the form (4.1), but the corresponding quadratic differentials φ must be integrable and holomorphic on Δ_*. Thus they can have at most a simple pole at the origin.

Take again a maximizing sequence $\{\varphi_p\}$ for μ_0 as in (4.3) and denote the zeros of φ_p of odd order in Δ_* by (4.6). One only needs now to join these zeros by a piecewise smooth slit $\sigma^{(p)}$ located in $\Delta \setminus \sigma_0$. All other arguments in the proof of Theorems 1.1 and 5.1 remain valid.

7. Complex hyperconvexity of the Teichmüller spaces

Let X be an arbitrary hyperbolic Riemann surface. By the uniformization theorem, there is a torsion free Fuchsian group G acting discontinuously on the disks Δ and Δ^* so that the surface X is conformally homeomorphic to the factor-space $X = \Delta/\Gamma$. The limit set $\mathrm{L}(\Gamma)$ of Γ either coincides with the unit circle S^1 if Γ is of the first kind or $\mathrm{L}(\Gamma)$ is a nowhere dense subset of S^1 when Γ is of the second kind.

The Teichmüller space $\mathbf{T}(X) = \mathbf{T}(\Gamma)$ of the surface X (or of the group Γ) is the deformation space of the conformal structure on X. This space has a natural copy in the universal Teichmüller space; namely, the embedding of $\mathbf{T}(\Gamma)$ consists of those Schwarzians $\psi \in \mathbf{T}$ which are Γ-automorphic holomorphic forms of weight-4 in Δ^* (holomorphic quadratical differentials on Δ^*/Γ), i.e.,

$$(\psi \circ \gamma)(\gamma')^2 = \psi, \;\; \gamma \in \Gamma; \quad \psi(z) = O(|z|^{-4}) \text{ as } \; z \to \infty. \tag{7.1}$$

The Teichmüller spaces satisfy the relation

$$\mathbf{T}(\Gamma) = \mathbf{B}(\Gamma) \cap \mathbf{T},$$

where $\mathbf{B}(\Gamma)$ denotes the subspace of $\mathbf{B}$ formed by φ satisfying (7.1) (see, e.g., [**Le**]).

The Teichmüller metric $\tau_{\mathbf{T}(\Gamma)}$ on $\mathbf{T}(\Gamma)$ is defined similarly to (1.3). It is uniformly equivalent to the metric $\tau_{\mathbf{T}}|\mathbf{T}(\Gamma)$ induced on $\mathbf{T}(\Gamma)$ by the metric of the universal Teichmüller space: obviously,

$$\tau_{\mathbf{T}}(\varphi, \psi) < \tau_{\mathbf{T}(\Gamma)}(\varphi, \psi), \quad \varphi, \psi \in \mathbf{T}(\Gamma),$$

on the other hand, on every bounded set $E \subset \mathbf{T}(\Gamma)$,

$$\tau_{\mathbf{T}(\Gamma)}(\varphi, \psi) \leqslant 3(1 + \operatorname{diam} E) \; \tau_{\mathbf{T}}(\varphi, \psi),$$

where diam denotes the diameter of E in $\tau_{\mathbf{T}(\Gamma)}$ metric (see [**Le**, Theorem 4.7]).

As one of the important consequences of Theorem 1.1 we obtain the following result:

THEOREM 7.1. *Every Teichmüller space* $\mathbf{T}(\Gamma)$ *is complex hyperconvex, that is, there exists a negative continuous plurisubharmonic* $u(\psi)$ *on* $\mathbf{T}(\Gamma)$ *which tends to zero when* ψ *tends to infinity.*

The question of complex hyperconvexity of Teichmüller spaces was stated by M. Gromov. It is solved in the affirmative for the finite-dimensional Teichmüller spaces $\mathbf{T}(p, n) = \mathbf{T}(\Gamma)$ in [**Kr5**]. Theorem 1.1, together with the above relations between the spaces $\mathbf{T}$ and $\mathbf{T}(\Gamma)$, immediately provides this property for all $\mathbf{T}(\Gamma)$; as a needed plurisubharmonic function can serve, for example, $g_{\mathbf{T}}(\mathbf{0}, \psi)$.

Note that recently McMullen [**MM**] established the Kähler hyperbolicity of the finite-dimensional moduli spaces $\mathcal{M}(p, n) = \mathbf{T}(p, n)/\operatorname{Mod}(p, n)$, where $\operatorname{Mod}(p, n)$ denotes the Teichmüller modular group of the space $T(p, n)$. The complex hyperconvexity and the Kähler hyperbolicity are closely related properties.

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