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THE HOLOMORPHIC CONTRACTIBILITY OF TWO GENERALIZED TEICHMÜLLER SPACES

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ABSTRACT. We show by simple explicit constructions that the Teichmüller spaces $T_0(\Delta)$ and $T_0(\Delta \setminus \{0\})$ are holomorphically contractible. We also call attention to a useful criterion for a map between domains in complex Banach spaces to be holomorphic.

Introduction

We call a Riemann surface hyperbolic if its universal covering surface is conformally equivalent to the open unit disk Δ in the complex plane $\mathbb{C}$. We denote the Teichmüller space of the hyperbolic Riemann surface R by $T(R)$, and we denote by $T_0(R)$ the closed complex submanifold of $T(R)$ consisting of the Teichmüller equivalence classes of asymptotically conformal quasiconformal mappings with domain R. (A fuller description of $T_0(R)$ will be given in §1.2.)

A complex manifold X is said to be holomorphically contractible to the point x_0 in X if there is a continuous map $F\colon [0,1] \times X \to X$ such that for all x in X we have $F(0,x) = x$ and $F(1,x) = x_0$, and for all t in $[0,1]$ the map $x \mapsto F(t,x)$ from X to itself is holomorphic and fixes the point x_0. Such a map F is said to contract X holomorphically to x_0. For example, the map $(t,x) \mapsto (1-t)x$ contracts a complex Banach space X or its open unit ball holomorphically to 0. On the other hand, there are contractible bounded domains of holomorphy in $\mathbb{C}^2$ that are not holomorphically contractible (see Zaĭdenberg and Lin [**13**]).

Since the Teichmüller spaces $T(R)$ and $T_0(R)$ are contractible bounded domains, it is natural to ask whether they are holomorphically contractible. In fact, this question for the Teichmüller space $T(0,n)$ of the sphere with $n \geqslant 5$ punctures was one motivation for the examples in [**13**]. The question for arbitrary $T(R)$ was brought to our attention by Samuel Krushkal some time ago.

We have made no progress on that question, but we shall prove the following rather simple theorem. Although our methods shed no light on the general problem, the theorem has some interest, since both $T_0(\Delta)$ and $T_0(\Delta')$ can be interpreted as groups of symmetric homeomorphisms of the unit circle (see [**10**]).

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THEOREM. *Let Δ be the open unit disk, and let $\Delta' = \Delta \setminus \{0\}$. The Teichmüller spaces $T_0(\Delta)$ and $T_0(\Delta')$ are holomorphically contractible to their basepoints.*

We shall describe the Teichmüller spaces $T(R)$ and $T_0(R)$ in §1. Since they are domains in complex Banach spaces, we discuss holomorphic mappings between such domains in §2. Our discussion includes a somewhat unfamiliar characterization of holomorphic mappings, due to Dunford, which we state in §2 and prove in §4. Our theorem will be proved in §3.

1. $T(R)$ and $T_0(R)$ as domains in complex Banach spaces

1.1. The spaces $T(R)$. Let R be a hyperbolic Riemann surface. Choose a holomorphic universal covering $\varpi : \mathcal{H}^+ \to R$ of R by the upper half plane $\mathcal{H}^+$, and let Γ be the Fuchsian group of covering transformations for ϖ. By definition, the set of Beltrami differentials for Γ is the closed subspace $L^\infty(\mathcal{H}^+, \Gamma)$ of $L^\infty(\mathcal{H}^+)$ consisting of the μ in $L^\infty(\mathcal{H}^+)$ such that $(\mu \circ \gamma)\overline{\gamma'}/\gamma' = \mu$ for all γ in Γ.

The functions in the open unit ball $M(\Gamma)$ of $L^\infty(\mathcal{H}^+, \Gamma)$ are called the Beltrami coefficients for Γ. We extend any such Beltrami coefficient μ to an L^∞ function on $\mathbb{C}$ by setting it equal to zero in the complement of $\mathcal{H}^+$. Let f_μ be the unique quasiconformal map of $\mathbb{C}$ onto itself that fixes the points 0, and 1 and has the complex dilatation μ. As f_μ is holomorphic and injective in the lower half plane $\mathcal{H}^-$, we can form its Schwarzian derivative

$$\varphi_\mu(z) = S(f_\mu)(z) = \frac{f_\mu'''(z)}{f_\mu'(z)} - \frac{3}{2}\left(\frac{f_\mu''(z)}{f_\mu'(z)}\right)^2, \qquad z \in \mathcal{H}^-.$$

It is well known (see for example [**9**]) that φ_μ belongs to the Banach space $B(\Gamma)$ of holomorphic functions φ on $\mathcal{H}^-$ that have finite norm

$$\|\varphi\|_{B(\Gamma)} = \sup\{|z - \bar{z}|^2 |\varphi(z)| : z \in \mathcal{H}^-\}$$

and satisfy the Γ-invariance condition

$$(\varphi \circ \gamma)(\gamma')^2 = \varphi \qquad \text{for all } \gamma \text{ in } \Gamma. \tag{1.1}$$

We define the Teichmüller space $T(R)$ to be the image of $M(\Gamma)$ under the Bers map $\mu \mapsto \varphi_\mu$ from $M(\Gamma)$ to $B(\Gamma)$. It is well known (see for example [**9**] or [**12**]) that $T(R)$ is a bounded domain (connected open set) in $B(\Gamma)$. Since $f_\mu(z) = z$ for all z in $\mathbb{C}$ when $\mu = 0$, $T(R)$ contains the point 0 in $B(\Gamma)$.

REMARK. The space $T(R)$ can be defined intrinsically as a quotient space of the set of all quasiconformal maps whose domain is R. Its concrete realization as a domain in $B(\Gamma)$ was discovered by Bers (see [**1**] for the finite dimensional case and [**2**] for the general case).

1.2. The spaces $T_0(R)$. We continue to use the notation of §1.1. We say that μ in $L^\infty(\mathcal{H}^+, \Gamma)$ vanishes at infinity on R if for every $\epsilon > 0$ there is a compact set E in R such that $|\mu| < \epsilon$ almost everywhere in $\varpi^{-1}(R \setminus E)$. We denote by $M_0(\Gamma)$ the set of μ in $M(\Gamma)$ that vanish at infinity on R. We also define $B_0(\Gamma)$ to be the closed subspace of $B(\Gamma)$ consisting of the φ in $B(\Gamma)$ such that the Beltrami differential $\mu_\varphi(z) = \varphi(\bar{z})|z - \bar{z}|^2$, $z \in \mathcal{H}^+$, vanishes at infinity on R.

By definition, $T_0(R)$ is the image of $M_0(\Gamma)$ under the Bers map $\mu \mapsto \varphi_\mu$. By Theorems 2 and 4 in [**8**], $T_0(R)$ is contractible and equals the intersection of $T(R)$ and $B_0(\Gamma)$. (In particular, $T_0(R)$ is a bounded domain in $B_0(\Gamma)$.)

Remark. Let f be a quasiconformal map whose domain is R. We say f is asymptotically conformal if for every $\epsilon > 0$ there is a compact set E in R such that the restriction of f to $R \setminus E$ is $(1+\epsilon)$–quasiconformal. It is clear that f is asymptotically conformal if and only if the lift of its Beltrami coefficient to $\mathcal{H}^+$ belongs to $M_0(\Gamma)$. That fact makes possible an intrinsic definition of $T_0(R)$ as a quotient space of the set of asymptotically conformal quasiconformal maps whose domain is R. See [**7**], [**8**], or [**9**] for more details.

2. Holomorphic mappings in complex Banach spaces

Let X be a domain in the complex Banach space V, and let f be a map of X into the complex Banach space W. As usual, we say that f is holomorphic if it is Fréchet differentiable and its Fréchet derivative $f'(x)$ is a $\mathbb{C}$–linear map for each x in X. These conditions are implied by the apparently weaker conditions that f is locally bounded and its directional derivative

$$df(x,v) = \lim_{t\to 0} \frac{f(x+tv)-f(x)}{t} \qquad (t \in \mathbb{C})$$

exists for every x in X and v in V. They imply the apparently stronger condition that f has a Taylor series expansion at each point of X. (See for example Proposition 2 in §1 of Douady [**5**].)

In practice, the criterion above need be applied only to maps from X to $\mathbb{C}$, since $f\colon X \to W$ is holomorphic if $\ell \circ f$ is holomorphic for sufficiently many linear functionals ℓ on W. A very convenient statement along these lines is given in Dunford [**6**]. We need a definition from [**6**].

Definition. Let W^* be the (complex) dual space of the complex Banach space W. The subset A of W^* is *determining* if there is a positive number C such that

$$\|w\| \leqslant C \sup\left\{\frac{|\ell(w)|}{\|\ell\|} : \ell \in A \text{ and } \|\ell\| > 0\right\} \quad \text{for all } w \text{ in } W. \tag{2.1}$$

Proposition 1 (Dunford). *Let V and W be complex Banach spaces, let X be a domain in V, and let f be a map of X into W. Let A_f be the set of ℓ in W^* such that $\ell \circ f$ is holomorphic in X. Then,*

(a) *if A_f contains a closed determining subspace of W^*, then f is holomorphic,*
(b) *if f is locally bounded in X, then A_f is a closed subspace of W^*.*

Corollary 1. *If $\ell \circ f$ is holomorphic for all ℓ in W^*, then f is holomorphic. If f is locally bounded and $\ell \circ f$ is holomorphic for all ℓ in a determining subset of W^*, then f is holomorphic.*

Corollary 1 is an obvious consequence of Proposition 1. Part (a) of the proposition is Theorem 76 in [**6**]. One of its striking features is that its hypothesis does not mention local boundedness. Dunford's statement of Theorem 76 in [**6**] requires X to be a domain in $\mathbb{C}$, but his proof works equally well for domains in complex Banach spaces. We paraphrase that proof in §4.

REMARK. More general and even less stringent sufficient conditions for a map f to be holomorphic can be found in Chapter 3 of Dineen [**4**].

In addition, Exercise 3.60 in that chapter implies that if f maps the domain X into the dual space Y^* of a complex Banach space Y, and if $x \mapsto f(x)y$ is holomorphic for all y in Y, then f is holomorphic. Part (a) of Proposition 1 follows from that special case, for if A is a closed determining subspace of W^*, then Lemma 3 in §4 allows us to interpret $f\colon X \to W$ as a map from X to A^*.

3. Proof of the theorem

3.1. A lemma. Our holomorphic contractions of $T_0(\Delta)$ and $T_0(\Delta')$ rely on the following observation. We use the notation of §1.

LEMMA 1. *Let g be a Möbius transformation that maps $\mathcal{H}^-$ into itself, and let $g^*(\varphi) = (\varphi \circ g)(g')^2$ for all holomorphic functions φ on $\mathcal{H}^-$. If $g \circ \gamma \circ g^{-1}$ belongs to Γ for all γ in Γ, then g^* is a bounded $\mathbb{C}$–linear map of $B(\Gamma)$ into itself, and it maps $T(R)$ into itself.*

PROOF. This is a standard fact when $g(\mathcal{H}^-) = \mathcal{H}^-$, and the usual proof applies almost verbatim when $g(\mathcal{H}^-)$ is a proper subset of $\mathcal{H}^-$. We sketch it for the reader's convenience.

Let φ in $B(\Gamma)$ and γ in Γ be given. By hypothesis, there is $\widehat{\gamma}$ in Γ such that $g \circ \gamma = \widehat{\gamma} \circ g$. Set $\widehat{\varphi} = (\varphi \circ \widehat{\gamma})(\widehat{\gamma}')^2$. Then $(g^*(\varphi) \circ \gamma)(\gamma')^2 = g^*(\widehat{\varphi})$ by direct calculation, so $g^*(\varphi)$ satisfies the invariance condition (1.1) whenever φ does. If in addition φ belongs to $B(\Gamma)$, then

$$|g^*(\varphi)(z)|\,|z-\bar{z}|^2 = |\varphi(g(z))|(|g'(z)|\,|z-\bar{z}|)^2 \leqslant |\varphi(g(z))|\,|g(z)-\overline{g(z)}|^2 \leqslant \|\varphi\|_{B(\Gamma)}$$

for all z in $\mathcal{H}^-$, so g^* is a $\mathbb{C}$–linear map of $B(\Gamma)$ into itself, and its operator norm is at most one. (Observe that we have applied the infinitesimal Schwarz–Pick lemma to the map $g\colon \mathcal{H}^- \to \mathcal{H}^-$.)

Now suppose $\varphi \in T(R)$. That means $\varphi = S(f_\mu)$ for some μ in $M(\Gamma)$. Since $g(\mathcal{H}^-) \subset \mathcal{H}^-$, $f_\mu \circ g$ is holomorphic and injective in $\mathcal{H}^-$. Let ν be the Beltrami coefficient of $f_\mu \circ g$. The chain rule gives the formula $\nu = (\mu \circ g)\overline{g'}/g'$, from which it follows readily that ν belongs to $M(\Gamma)$. (Recall that we are identifying $L^\infty(\mathcal{H}^+)$ with the space of functions in $L^\infty(\mathbb{C})$ that vanish identically on the complement of $\mathcal{H}^+$.) Using standard properties of the Schwarzian derivative, we obtain

$$g^*(\varphi) = g^*(S(f_\mu)) = S(f_\mu \circ g) = S(f_\nu) \in T(R). \qquad \square$$

REMARK. If Γ contains a hyperbolic transformation γ and g is a Möbius transformation such that $g \circ \gamma \circ g^{-1} \in \Gamma$ and $g(\mathcal{H}^-) \subset \mathcal{H}^-$, then $g(\mathcal{H}^-) = \mathcal{H}^-$. When we apply Lemma 1, Γ will contain no hyperbolic transformations, and $g(\mathcal{H}^-)$ will generally be a proper subset of $\mathcal{H}^-$.

3.2. The space $T_0(\Delta)$. When $R = \Delta$, the covering map $\varpi\colon \mathcal{H}^+ \to R$ is a Möbius transformation, and the group Γ is trivial. In particular, $B(\Gamma)$ is simply the Banach space $B(\mathcal{H}^-)$ of holomorphic functions φ on $\mathcal{H}^-$ with finite norm

$$\|\varphi\|_{B(\mathcal{H}^-)} = \sup\{|z-\bar{z}|^2|\varphi(z)| : z \in \mathcal{H}^-\},$$

$B_0(\Gamma)$ is the set $B_0(\mathcal{H}^-)$ of φ in $B(\mathcal{H}^-)$ such that the function $|z-\bar{z}|^2|\varphi(z)|$ vanishes at infinity on $\mathcal{H}^-$, $T(\Delta)$ is a bounded domain in $B(\mathcal{H}^-)$, and $T_0(\Delta)$ is the intersection of $T(\Delta)$ and $B_0(\mathcal{H}^-)$.

Our contraction of $T_0(\Delta)$ is easier to describe if we replace the spaces $B(\mathcal{H}^-)$ and $B_0(\mathcal{H}^-)$ by their counterparts on Δ, so we map Δ onto $\mathcal{H}^-$ by the Möbius transformation $A(z) = i(z-1)(z+1)^{-1}$, z in Δ. The formula $A^*(\varphi) = (\varphi \circ A)(A')^2$, φ in $B(\mathcal{H}^-)$, defines an isometric isomorphism A^* of $B(\mathcal{H}^-)$ onto the space $B(\Delta)$ of holomorphic functions ψ on Δ with finite norm

$$\|\psi\|_{B(\Delta)} = \sup\{(1-|z|^2)^2|\psi(z)| : z \in \Delta\}.$$

Let $B_0(\Delta)$ be the set of ψ in $B(\Delta)$ such that $\lim_{|z|\to 1}(1-|z|^2)^2|\psi(z)| = 0$. Then $A^*(B_0(\mathcal{H}^-)) = B_0(\Delta)$. We shall identify $T(\Delta)$ and $T_0(\Delta)$ with their images under A^*, so $T(\Delta)$ becomes a bounded domain in $B(\Delta)$, and $T_0(\Delta) = T(\Delta) \cap B_0(\Delta)$.

Lemma 2. *Given τ in the closed unit disk $\overline{\Delta}$ and ψ in $B_0(\Delta)$, let g_τ be the map $z \mapsto \tau z$ of Δ into itself, and let $g_\tau^*(\psi)$ be the holomorphic function $(\psi \circ g_\tau)(g_\tau')^2$ on Δ. The formula $G(\tau,\psi) = g_\tau^*(\psi)$ defines a continuous map G from $\overline{\Delta} \times B_0(\Delta)$ to $B_0(\Delta)$, and G maps $\overline{\Delta} \times T_0(\Delta)$ into $T_0(\Delta)$. The restriction of G to $\Delta \times B_0(\Delta)$ is holomorphic.*

Proof. For any given (τ,ψ) in $\overline{\Delta} \times B_0(\Delta)$ and z in Δ, the explicit formula

$$G(\tau,\psi)(z) = g_\tau^*(\psi)(z) = \tau^2\psi(\tau z) \tag{3.1}$$

implies that

$$(1-|z|^2)^2|G(\tau,\psi)(z)| = (1-|z|^2)^2|\psi(\tau z)\tau^2| \leqslant (1-|\tau z|^2)^2|\psi(\tau z)\tau^2| \leqslant |\tau|^2\|\psi\|.$$

Therefore $(1-|z|^2)^2||G(\tau,\psi)(z)| \to 0$ as $|z| \to 1$, $\|G(\tau,\psi)\|_{B(\Delta)} \leqslant |\tau|^2\|\psi\|_{B(\Delta)}$, and G is a locally bounded map of $\overline{\Delta} \times B_0(\Delta)$ into $B_0(\Delta)$.

Clearly, $G(0,\psi) = 0 \in T(\Delta)$ for all ψ in $B_0(\Delta)$. If $0 < |\tau| \leqslant 1$, we apply Lemma 1 with $g = A \circ g_\tau \circ A^{-1}\colon \mathcal{H}^- \to \mathcal{H}^-$ to conclude that $G(\tau,\psi) \in T(\Delta)$ for all ψ in $T(\Delta)$. Therefore, G maps $\overline{\Delta} \times B(\Delta)$ into $T(\Delta) \cap B_0(\Delta) = T_0(\Delta)$.

Now we examine the continuity of G in $\overline{\Delta} \times B_0(\Delta)$. Since

$$\begin{aligned}\|G(\tau,\psi_1) - G(\tau,\psi_2)\|_{B(\Delta)} &= \|G(\tau,\psi_1-\psi_2)\|_{B(\Delta)}\\ &\leqslant |\tau|^2\|\psi_1-\psi_2\|_{B(\Delta)} \leqslant \|\psi_1-\psi_2\|_{B(\Delta)}\end{aligned}$$

for all τ in $\overline{\Delta}$ and ψ_1 and ψ_2 in $B_0(\Delta)$, it suffices to show that $G(\tau,\psi)$ is a continuous function of τ for every fixed ψ in $B_0(\Delta)$. Let ψ be given, and let the sequence (τ_n) in $\overline{\Delta}$ converge to τ. We must show that the functions $f_n(z) = (1-|z|^2)^2G(\tau_n,\psi)(z)$ converge to $f(z) = (1-|z|^2)^2G(\tau,\psi)$ uniformly in Δ. For that purpose, we extend the functions f_n and f to continuous functions on $\overline{\Delta}$ by setting them equal to zero when $|z| = 1$. As is well known (see Theorem 5 in Chapter 7 of [**11**] or the discussion of continuous convergence in §§174–180 of [**3**]), it suffices to show that

$$\lim_{n\to\infty} f_n(z_n) = f\Big(\lim_{n\to\infty} z_n\Big) \tag{3.2}$$

for every convergent sequence (z_n) in $\overline{\Delta}$. Since $\psi \in B_0(\Delta)$, (3.2) follows readily from (3.1), so G is continuous in $\overline{\Delta} \times B_0(\Delta)$.

It remains to show that G is holomorphic in $\Delta \times B_0(\Delta)$. Since G is locally bounded and the linear functionals $\psi \mapsto (1-|z|^2)^2\psi(z)$, z in Δ, are a determining subset of $B_0(\Delta)^*$, Corollary 1 says that we need only consider the functions

$$h(\tau,\psi) = (1-|z|^2)^2 g_\tau^*(\psi)(z)$$

for arbitrary fixed z in Δ. Each of these functions h is locally bounded in $\Delta\times B_0(\Delta)$, so we need only show that the directional derivative

$$dh((\tau,\psi),(\sigma,\phi)) = \lim_{t\to 0}\frac{h(\tau+t\sigma,\psi+t\phi)-h(\tau,\psi)}{t}$$

exists for arbitrary (τ,ψ) in $\Delta\times B_0(\Delta)$, (σ,ϕ) in $\mathbb{C}\times B_0(\Delta)$, and z in Δ. A routine calculation using (3.1) shows that the required limit exists. $\square$

COROLLARY 2. *The map $F(t,\psi) = G((1-t),\psi)$ from $[0,1]\times T_0(\Delta)$ to $T_0(\Delta)$ contracts $T_0(\Delta)$ holomorphically to* 0.

REMARK. We thank Fred Gardiner for pointing out to us that G is holomorphic in $\Delta\times T_0(\Delta)$. Although this fact plays no essential role in showing that $T_0(\Delta)$ is holomorphically contractible, we could not resist including its proof. The same argument shows that (3.1) defines a holomorphic map of $\Delta\times T(\Delta)$ into $T_0(\Delta)$, but that observation does not imply that $T(\Delta)$ is holomorphically contractible. Arbitrary points of $T(\Delta)$ cannot be approximated by points in the proper closed subspace $T_0(\Delta)$.

3.3. The space $T_0(\Delta')$. When $R = \Delta'$, we choose $\varpi(z) = \exp(2\pi i z)$ for the covering map $\varpi\colon \mathcal{H}^+ \to R$. In this case, Γ is generated by the map $z \mapsto z+1$, and $B(\Gamma)$ is the set of functions φ in $B(\mathcal{H}^-)$ such that $\varphi(z+1) = \varphi(z)$ for all z in $\mathcal{H}^-$.

Let $f\colon \mathcal{H}^- \to \Delta'$ be the map $f(z) = \varpi(-z)(=\exp(-2\pi i z))$, z in $\mathcal{H}^-$. Let $B(\Delta')$ be the Banach space of holomorphic functions φ on Δ' with finite norm

$$\|\varphi\|_{B(\Delta')} = \sup\{(2|w|\log|w|)^2|\varphi(w)| : w\in\Delta'\}.$$

Every function in $B(\Delta')$ is meromorphic in Δ with at worst a simple pole at 0, so $\lim_{w\to 0}(2|w|\log|w|)^2|\varphi(w)| = 0$ for all φ in $B(\Delta')$.

The map $f^*(\varphi) = (\varphi\circ f)(f')^2$, φ in $B(\Delta')$, is an isometric isomorphism of $B(\Delta')$ onto $B(\Gamma)$, and $(f^*)^{-1}(B_0(\Gamma))$ is the space $B_0(\Delta')$ of φ in $B(\Delta')$ such that $\lim_{|w|\to 1}(2|w|\log|w|)^2|\varphi(w)| = 0$. We shall identify $T(\Delta')$ and $T_0(\Delta')$ with their images in $B(\Delta')$ under $(f^*)^{-1}$, so that $T(\Delta')$ becomes a bounded domain in $B(\Delta')$, and $T_0(\Delta') = T(\Delta)\cap B_0(\Delta')$.

The Möbius transformations $A_\zeta(z) = z+\zeta$, ζ in $\mathcal{H}^-\cup\mathbb{R}$, map $\mathcal{H}^-$ into itself and commute with Γ. By Lemma 1, the maps $(f^*)^{-1}\circ A_\zeta^*\circ f^*$ carry $B(\Delta')$ and $T(\Delta')$ into themselves. By direct calculation, these maps are precisely the maps

$\varphi \mapsto g_\tau^*(\varphi) = (\varphi \circ g_\tau)(g_\tau')^2$, where $\tau = \exp(-2\pi i\zeta)$ and $g_\tau(w) = \tau w$, as in Lemma 2. We can therefore proceed much as we did in §3.2.

Set $G(0,\varphi)(w) = 0$ and $G(\tau,\varphi)(w) = g_\tau^*(\varphi)(w) = \tau^2\varphi(\tau w)$ for φ in $B_0(\Delta')$, w in Δ', and $0 < |\tau| \leqslant 1$. It is easy to verify that G maps $\overline{\Delta} \times B_0(\Delta')$ into $B_0(\Delta')$ and $\overline{\Delta} \times T_0(\Delta')$ into $T_0(\Delta')$. In addition, if $w \in \Delta'$ and $0 < \tau \leqslant 1$, we can apply the maximum principle to $w\varphi(w)$ (which has a removable singularity at 0) and obtain the inequality

$$|\tau w\varphi(\tau w)| \leqslant \max\{|\zeta\varphi(\zeta)| : |\zeta| = |w|\} \leqslant \frac{\|\varphi\|_{B(\Delta')}}{4|w|(\log|w|)^2}.$$

It follows readily that $\|G(\tau,\varphi)\|_{B(\Delta')} \leqslant |\tau|\|\varphi\|_{B(\Delta')}$ for all τ in $\overline{\Delta}$.

To see that G is continuous, interpret the functions $(2|w|\log|w|)^2\varphi(w)$, φ in $B_0(\Delta')$, as continuous functions on $\overline{\Delta}$, and argue as in §3.2.

The map $\varphi \mapsto G(\tau,\varphi)$ of $B_0(\Delta')$ into itself is a bounded $\mathbb{C}$–linear transformation, hence holomorphic, for each τ in $\overline{\Delta}$. Therefore the map $F(t,\varphi) = G(1-t,\varphi)$ from $[0,1] \times T_0(\Delta')$ to $T_0(\Delta')$ contracts $T_0(\Delta')$ holomorphically to 0.

4. Proof of Proposition 1

Let W be a complex Banach space, and let A be a closed determining subspace of W^*. We start by reformulating the inequality (2.1).

LEMMA 3. *For each w in W, define λ_w in A^* by setting $\lambda_w(\ell) = \ell(w)$ for all ℓ in A. If C is any positive number that satisfies condition* (2.1)*, then*

$$\|\lambda_w\| \leqslant \|w\| \leqslant C\|\lambda_w\| \qquad \text{for all } w \text{ in } W. \tag{4.1}$$

PROOF OF THE LEMMA. The inequality $\|\lambda_w\| \leqslant \|w\|$ is obvious from the definition of λ_w. The inequality $\|w\| \leqslant C\|\lambda_w\|$ simply restates condition (2.1), since the right hand side of (2.1) is C times the norm of λ_w on A. □

PROOF OF THE PROPOSITION. Let $f\colon X \to W$ be given, and let $\ell \circ f$ be holomorphic in X for all ℓ in the closed determining subspace A of W^*. To prove assertion (a) of the proposition, we must show that f is holomorphic in X. We follow Dunford's proof of Theorem 76 in [**6**].

First we shall prove that f is locally bounded in X. Let the sequence (x_n) in X converge to some x_0 in X. We must show that the sequence of numbers $\|f(x_n)\|$ is bounded. Since $\ell(f(x_n)) \to \ell(f(x_0))$ as $n \to \infty$ for each ℓ in A, the sequence of numbers $\lambda_{f(x_n)}(\ell) = \ell(f(x_n))$ is bounded for each ℓ in A. By the uniform boundedness principle, there is a number M such that $\|\lambda_{f(x_n)}\| \leqslant M$ for all n. Since A is a determining set, there is a number C that satisfies (2.1). By Lemma 3,

$$\|f(x_n)\| \leqslant C\|\lambda_{f(x_n)}\| \leqslant CM$$

for every n, so f is locally bounded.

Now we shall prove that the directional derivative $df(x,v)$ exists for all x in X and v in V. Given x and v, we choose positive numbers r and K so that $x + tv$ is in

X and $\|f(x+tv)\| \leqslant K$ for all t in $\mathbb{C}$ with $|t| \leqslant 2r$. By Cauchy's integral formula,

$$
\begin{aligned}
|\ell(f(x+tv)-f(x+sv))| &= \left|\frac{1}{2\pi i}\int_{|\zeta|=2r}\left[\frac{1}{\zeta-t}-\frac{1}{\zeta-s}\right]\ell(f(x+\zeta v))\,d\zeta\right| \\
&\leqslant \frac{2K}{r}\|\ell\|\,|t-s|
\end{aligned}
\tag{4.2}
$$

for all s and t in the closed disk $D = \{z \in \mathbb{C} : |z| \leqslant r\}$ and ℓ in A. Since A is a determining set, (4.2) implies that the map $t \mapsto f(x+tv)$ is Lipschitz continuous in D. We can therefore define a holomorphic map g from the interior of D to W by the formula

$$g(t) = \frac{1}{2\pi i}\int_{|\zeta|=r}\frac{f(x+\zeta v)}{\zeta - t}d\zeta, \qquad |t| < r.$$

Since $\ell(g(t)) = \ell(f(x+tv))$ for all ℓ in the determining set A when $|t| < r$, we must have $g(t) = f(x+tv)$ for such t. Therefore $df(x,v)$ exists and equals $g'(0)$, and f (being locally bounded) is holomorphic in X. That proves (a).

Since A_f is always a (not necessarily closed) subspace of W^*, the point of assertion (b) is that A_f is a closed set if f is locally bounded. For the proof, let ℓ belong to W^*, let (ℓ_n) be a sequence in W^* that converges to ℓ, and let $f\colon X \to W$ be a locally bounded function such that $g_n = \ell_n \circ f$ is holomorphic in X for all n. We must show that $g = \ell \circ f$ is holomorphic.

That is easy. Since f is locally bounded, so is g. Since in addition $\|\ell_n - \ell\| \to 0$ as $n \to \infty$, $g_n \to g$ locally uniformly in X. It follows readily that $dg(x,v)$ exists for any given x in X and v in V. In fact, $dg(x,v)$ is the derivative at $t=0$ of the function $t \mapsto g(x+tv)$, which is holomorphic in a neighborhood of zero since it is the uniform limit of the holomorphic functions $t \mapsto g_n(x+tv)$ in the disk $\{t \in \mathbb{C} : |t| < \epsilon\}$ if $\epsilon > 0$ is sufficiently small. □

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