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VARIOUS CONSTANTS ASSOCIATED WITH QUASIDISKS AND QUASISYMMETRIC HOMEOMORPHISMS

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Abstract. In this expository paper, we will discuss some constants associated with quasidisks and quasisymmetric homeomorphisms.

1. Quasidisks and quasisymmetric homeomorphisms

This is a survey article on various constants associated with quasidisks and quasisymmetric homeomorphisms. Before discussing these constants, we recall some basic definitions and notations.

Let $D = \{|z| < 1\}$ denote the unit disk in the extended complex plane $\overline{C}$, and $D^* = \overline{C} - \overline{D}$. Let Ω be a Jordan domain in the extended complex plane $\overline{C}$, and $\Omega^* = \overline{C} - \overline{\Omega}$. Let f_1 and f_2 map Ω and Ω^* conformally onto D and D^*, respectively. Extend f_1 and f_2 to the boundary $\partial\Omega = \partial\Omega^*$ and define the homeomorphism $h_\Omega : \partial D \to \partial D$ by $h_\Omega = f_2 \circ f_1^{-1}|\partial D$. Then h_Ω is called the sewing (welding) mapping of the domains Ω and Ω^*.

A Jordan domain Ω is called a quasidisk if it is the image of the unit disk D under a quasiconformal self-mapping of the extended complex plane $\overline{C}$. On the other hand, a homeomorphism $h : \partial D \to \partial D$ is called a quasisymmetric homeomorphism if h has a quasiconformal extension into the unit disk D (see [BA]). Then Ω is a quasidisk if and only if h_Ω is a quasisymmetric homeomorphism (see [Le]). Conversely, a quasisymmetric homeomorphism h also determines a pair of complementary quasidisks, which we denote by Ω_h and Ω_h^*, respectively.

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We say a Jordan domain Ω is an asymptotically conformal extension domain if the Riemann mapping from D to Ω has a quasiconformal extension to the whole plane whose complex dilatation μ satisfies $|\mu(z)| \to 0$ as $|z| \to 1+$. Ω is a disk if it is the image of D under a Möbius transformation. As usual, we say Ω is smooth if its boundary curve $\partial\Omega$ has a continuously varying tangent. Clearly, a smooth domain is always an asymptotically conformal extension domain, but the converse is not true. A quasisymmetric homeomorphism h is said to be symmetric if it has a quasiconformal extension into the unit disk D whose complex dilatation μ satisfies $|\mu(z)| \to 0$ as $|z| \to 1-$. Then it is easy to see that Ω is an asymptotically conformal extension domain iff h_Ω is a symmetric homeomorphism (see [GS]).

2. Constants attached to quasidisks

2.1. Fredholm eigenvalue and Schober functional. Let Ω be a Jordan domain in the extended complex plane $\overline{C}$, and Γ its boundary. Then the least positive Fredholm eigenvalue λ_Γ is defined by the equality (see Schiffer [S2])

$$\frac{1}{\lambda_\Gamma} = \sup_{v\in\mathcal{H}} \frac{|\mathcal{D}_\Omega[v] - \mathcal{D}_{\Omega^*}[v]|}{\mathcal{D}_\Omega[v] + \mathcal{D}_{\Omega^*}[v]} = \frac{\max\{s(\Omega,\Omega^*), s(\Omega^*,\Omega)\} - 1}{\max\{s(\Omega,\Omega^*), s(\Omega^*,\Omega)\} + 1}. \tag{2.1}$$

Here and in what follows,

$$\mathcal{D}_G[v] = \iint_G |\nabla v|^2 = 2\iint_G (|v_z|^2 + |v_{\bar z}|^2)\,dx\,dy \tag{2.2}$$

is the Dirichlet integral, and $\mathcal{H}$ is the set of all real-valued functions v, which are continuous in $\overline{\mathbb{C}}$, harmonic in $\Omega \cup \Omega^*$ and $\mathcal{D}_\Omega[v] + \mathcal{D}_{\Omega^*}[v] < +\infty$. Also

$$(2.3)\qquad s(\Omega,\Omega^*) = \sup_{v\in\mathcal{H}} \frac{\mathcal{D}_\Omega[v]}{\mathcal{D}_{\Omega^*}[v]} \qquad \text{and} \qquad (2.4)\qquad s(\Omega^*,\Omega) = \sup_{v\in\mathcal{H}} \frac{\mathcal{D}_{\Omega^*}[v]}{\mathcal{D}_\Omega[v]}$$

are known as the Schober's domain functionals (see Schober [Sc1]). It is obvious that $\lambda_\Gamma \geqslant 1$, and it is also known that $\lambda_\Gamma > 1$ iff Ω is a quasidisk.

It is known that the least positive Fredholm eigenvalue λ_Γ plays an important role in determining the rate of convergence of the classical Neumann-Poincaré series (see [BS], [S1]), and many interesting properties of λ_Γ have been obtained in the literature (see [Ah1], [Ku], [S1-2], [Sc1-2], [Sp]). As pointed out by Schober [Sc1, p. 379], when Γ is three times continuously differentiable, both supremums in (2.3) and (2.4) can be attained, and in this situtation, $s(\Omega,\Omega^*) = s(\Omega^*,\Omega)$, so Schober's domain functionals s are actually curve functionals. However, in the general case, the situtation is somewhat different. In fact, the following results were shown by the author [Sh4].

THEOREM 2.1 [Sh4]. *For any quasidisk Ω, $s(\Omega,\Omega^*) = s(\Omega^*,\Omega)$.*

PROPOSITION 2.1 [Sh4]. *There exists a class of pairs of quasidisks Ω and Ω^* such that $s(\Omega,\Omega^*) = s(\Omega^*,\Omega)$ can not be attained at any $v \in \mathcal{H}$.*

2.2. Quasiconformal reflection constant. It is known that a Jordan domain Ω in the extended complex plane $\overline{C}$ is a quasidisk iff there exists a quasiconformal reflection in $\partial\Omega$, i.e. a mapping f which is a quasiconformal mapping of $\overline{C}$ and interchanges Ω and Ω^* and keeps every point of $\partial\Omega$ fixed (see [Le]). Thus, a quasidisk Ω determines the quasiconformal reflection constant $R(\Omega)$, defined as

$$R(\Omega) = \inf\{K[f] : \text{ for all quasiconformal reflections } f \text{ in } \partial\Omega\}. \tag{2.5}$$

Clearly, $R(\Omega) = R(\Omega^*)$. There always exists a so-called extremal quasiconformal reflection which attains the infimum in (2.5). By the quasi-invariance property of Dirichlet integral under quasiconformal mappings, we conclude that $s(\Omega, \Omega^*) = s(\Omega^*, \Omega) \leqslant R(\Omega)$ and consequently that (see [Ah1])

$$\frac{1}{\lambda_\Gamma} \leqslant \frac{R(\Omega) - 1}{R(\Omega) + 1}. \tag{2.6}$$

A necessary and sufficient condition for the equality in (2.6) will be given later (see Corollary 4.1).

2.3. Quasiextremal distance constant. Before defining the quasiextremal distance constant, we recall some basic relation between moduli and harmonic functions. For reference, we refer the reader to Gerhing [Ge] or Ahlfors [Ah2, Chapter 4]. Given any pair of disjoint nondegenerate continua A and B in $\overline{C}$, let $\operatorname{mod}(A, B; C)$ denote the modulus of the family $\Gamma(A, B; C)$ of curves that join A and B in C. Then there exists a unique real-valued function $u_{A,B}$, which is continuous in $\overline{C}$, harmonic in $\overline{C} - (A \cup B)$, with constant values 0 and 1 in A and B, respectively, such that $\operatorname{mod}(A, B; C) = \mathcal{D}_C[u_{A,B}]$. Note that the the modulus $\operatorname{mod}(A, B; C)$ of the family $\Gamma(A, B; C)$ is also closely related to the conformal module of the ring domain $\overline{C} - (A \cup B)$. Recall that for a ring domain R, its conformal module is defined as $M(R) = \log(r_2/r_1)$, if R is mapped conformally onto $\{r_1 < |z| < r_2\}$. Then the conformal module of the ring domain $\overline{C} - (A \cup B)$ is $2\pi / \operatorname{mod}(A, B; C)$. Furthermore, if ϕ maps $\overline{C} - (A \cup B)$ conformally onto $\{1 < |z| < \exp(2\pi / \operatorname{mod}(A, B; C))\}$, then $u_{A,B} = \chi_B + (\log|\phi|)/(2\pi / \operatorname{mod}(A, B; C))\chi_{C-(A\cup B)}$, where χ is the characteristic function of a set. When $A, B \subset \partial D$,

$$u_{A,B}|D = \frac{1}{2}(\phi_{A,B} + \overline{\phi}_{A,B}),$$

where $\phi_{A,B}$ is a conformal mapping of D onto

$$R_{A,B} = \{w = u + iv : 0 < u < 1, 0 < v < \operatorname{mod}(A, B; D)\}.$$

Let Ω be a domain in the extended complex plane $\overline{C}$. Given a pair of disjoint nondegenerate continua A and B in $\overline{\Omega}$, let $\operatorname{mod}(A, B; \Omega)$ denote the modulus of the family $\Gamma(A, B; \Omega)$ of curves that join A and B in Ω. The following so-called quasiextremal distance constant (or QED constant) was introduced in [Y1]:

$$M(\Omega) = \sup\left\{\frac{\operatorname{mod}(A, B; C)}{\operatorname{mod}(A, B; \Omega)} : \text{for all pairs } A \text{ and } B \text{ in } \overline{\Omega}\right\}. \tag{2.7}$$

The domain Ω is a QED domain if its QED constant $M(\Omega)$ is finite. QED domains were introduced by Gehring and Martio [GM] as a useful class of domains in the study of quasiconformal mappings.

It was proved in [GM] that a Jordan domain Ω is a QED domain if and only if Ω is a quasidisk. Now the two domain constants $M(\Omega)$ and $R(\Omega)$ are closely related to one another. For example, by the quasi-invariance property of modulus under quasiconformal mappings, it holds that (see [Y1])

$$M(\Omega) \leqslant R(\Omega) + 1. \tag{2.8}$$

It was conjectured by Garnett and Yang [GY] that the equality in (2.8) holds for all quasidisks. But this was disproved by Yang [Y3] for ellipses. In [Y3] the following boundary quasiextremal distance constant (or BQED constant) $M_b(\Omega)$ was also introduced:

$$M_b(\Omega) = \sup\left\{\frac{\operatorname{mod}(A, B; C)}{\operatorname{mod}(A, B; \Omega)} : \text{for all pairs } A \text{ and } B \text{ in } \partial\Omega\right\}. \tag{2.9}$$

Clearly, $M_b(\Omega) \leqslant M(\Omega)$. However, the question whether $M_b(\Omega) = M(\Omega)$ still remains open.

There have been much work related to the Garnet–Yang conjecture. Yang [Y4] proved that $M(\Omega) < R(\Omega) + 1$ for all smooth domains other than disks and asked whether $M(\Omega) < R(\Omega)+1$ for all asypmtotically conformal extension domains other than disks. On the other hand, Wu and Yang [WY] proved that $M_b(\Omega) < R(\Omega)+1$ for all asypmtotically conformal extension domains other than disks. Recently, the author [Sh5] proved the following result, which contains the above-mentioned results obtained by Wu and Yang and gives an affirmative answer to the question of Yang as well. Consequently, the Garnett–Yang conjecture is not true for all asypmtotically conformal extension domains other than disks.

Theorem 2.2 [Sh5]. *Let Ω be an asymptotically conformal extension domain. Then there exists a pair of disjoint nondegenerate continua A and B in $\overline{\Omega}$ such that $M(\Omega) = \frac{\operatorname{mod}(A,B;C)}{\operatorname{mod}(A,B;\Omega)}$. Furthermore, $M(\Omega) < R(\Omega) + 1$ unless Ω is a disk.*

3. Constants attached to quasisymmetric homeomorphisms

3.1 Extremal maximal dilatation and boundary dilatation. Given a quasisymmetric homeomorphism h of the unit circle onto itself, we denote by $Q(h)$ the class of all quasiconformal mappings of the unit disk D with boundary values h. The homeomorphism h then determines the extremal maximal dilatation $K^*(h)$, defined as

$$K^*(h) = \inf_{f \in Q(h)} K[f]. \tag{3.1}$$

Clearly, $K^*(h^{-1}) = K^*(h)$. $f \in Q(h)$ is called extremal if $K[f] = K^*(h)$ (see [St3]). It is well known that there always exists at least one extremal mapping in the class $Q(h)$. Furthermore, the extremality of a quasiconformal mapping is completely characterized by the following theorem due to Hamilton–Krushkal–Reich–Strebel (see [Ga], [GL], [St3]).

HAMILTON–KRUSHKAL–REICH–STREBEL THEOREM. *A quasiconformal mapping f is extremal iff its complex dilatation μ satisfies the Hamilton–Krushkal condition:*

$$\sup_{\phi\in\mathcal{A}} \frac{\mathrm{Re}\iint_D \mu(z)\phi(z)\,dx\,dy}{\iint_D |\phi(z)|\,dx\,dy} = \|\mu\|_\infty,$$

where $\mathcal{A}$ is the Banach space of all holomorphic and integrable functions ϕ in the unit disk.

A quasisymmetric homeomorphism h also determines the boundary dilatation $H(h)$ (see [St3]), defined as

(3.2)

$$H(h) = \inf\{K[f|D-E] : \text{ for all } f\in Q(h) \text{ and all compact subsets } E\subset D\}.$$

Then, $H(h) = H(h^{-1})$, $H(h) \leqslant K^*(h)$. The set of all normalized (fixing three boundary points on ∂D) quasisymmetric homeomorphisms of the unit circle onto itself is known as the universal Teichmüller space T of Bers (see [Le], [Na]). Following Earle–Li [EL], a point h is called a Strebel point if $H(h) < K^*(h)$. Then, by a result of Lakić [La], the set of Strebel points is open and dense in the universal Teichmüller space T. It is also known that h is symmetric iff $H(h) = 1$ (see [GS]), and all the normalized symmetric homeomorphisms comprise a closed set T_0 of the universal Teichmüller space T.

3.2 Maximal dilatation. The maximal dilatation $K(h)$ of h is defined as

$$K(h) = \sup\Big\{\frac{\mathrm{mod}(h(A), h(B); D)}{\mathrm{mod}(A,B;D)} : \text{for all pairs } A \text{ and } B \text{ in } \partial D\Big\}. \tag{3.3}$$

Clearly, $K(h^{-1}) = K(h)$. By the quasi-invariance property of modulus under quasiconformal mappings, it follows that $K(h) \leqslant K^*(h)$. It was an open question for a long time to determine whether or not $K(h) = K^*(h)$ always holds before Anderson and Hinkkanen disproved this by giving concrete examples of a family of affine mappings of some parallelograms (see [AH]). Later, a necessary condition for $K(h) = K^*(h)$ was obtained independently by Wu [Wu] and Yang [Y2]. We say h is induced by affine mappings if it is the restriction to ∂D of a map of the form $\phi_2\circ f_K\circ\phi_1^{-1}$, where $f_K(x+iy) = x+iKy$, while ϕ_1 and ϕ_2 are conformal mappings from a rectangle $\{x+iy : 0<x<a, 0<y<b\}$ and its image $\{u+iv : 0<u<a, 0<v<Kb\}$ under f_K onto D, respectively. Then the necessary condition for $K(h) = K^*(h)$ obtained by Wu [Wu] and Yang [Y2] can be stated as

THEOREM 3.1 [Wu], [Y2]. *Let $h : \partial D \to \partial D$ be a quasisymmetric homeomorphism. If $K(h) = K^*(h)$, then either h is induced by an affine mapping or $H(h) = K^*(h)$.*

In their papers [Wu] and [Y2], Wu and Yang also asked whether the converse of Theorem 3.1 was true. Recently, the author [Sh2] proved that there exists a family of quasisymmetric homeomorphisms h such that $K(h) < K^*(h) = H(h)$, which gives a negative answer to the question. On the other hand, some classes of quasisymmetric homeomorphisms h for which $K(h) = K^*(h) = H(h)$ were discussed in [CZL] and [LWQ].

EXAMPLE 1 [Sh2]. For convenience we use the upper half plane $\mathbb{H} = \{z : Imz > 0\}$ instead of the unit disk D. For any $K > 1$, we consider the quasisymmetric homeomorphism h of Strebel (see [St1], [St2]), namely, $h = h_K : \partial\mathbb{H} \to \partial\mathbb{H}$ is as $h(x) = x$ for $x \leqslant 0$ and $h(x) = Kx$ for $x > 0$. It is easily computed from [St1] that

$$H(h) = K^*(h) = 1 + \frac{1}{2\pi^2}\log^2 K + \frac{1}{\pi}\log K\sqrt{1 + \frac{1}{4\pi^2}\log^2 K}.$$

In fact, let

$$f(z) = K^{1-\frac{1}{\pi}\arg z} z.$$

Then f is an extremal mapping in $Q(h)$. It was calculated in [Sh2] that

$$K(h) = \sup\{\Lambda(K\rho)/\Lambda(\rho) : \rho > 0\},$$

where $\Lambda(\rho)$ is the conformal module of the quadrilateral Q with domain $\mathbb{H}$ and vertices $\infty, -1, 0$ and ρ. It was also proved there that, when K is large,

$$K(h) < 1 + \frac{1}{\pi}\log K + \frac{1}{4\pi^2}\log^2 K.$$

Another approach due to Reich [Re] and developed in [CC] gave a necessary and sufficient condition for $K(h) = K^*(h)$.

THEOREM 3.2 [RE], [CC]. *Let $h : \partial D \to \partial D$ be a quasisymmetric homeomorphism. Then $K(h) = K^*(h)$ if and only if $Q(h)$ contains an extremal mapping whose complex dilatation μ satisfies*

$$\sup_{A,B\subset\partial D} \frac{\operatorname{Re}\iint_D \mu(z){\phi'}^2_{A,B}(z)\,dx\,dy}{\iint_D |{\phi'}^2_{A,B}(z)|\,dx\,dy} = \|\mu\|_\infty. \tag{3.4}$$

3.3 Two other constants. Now let $\mathcal{D}$ denote the set of all real-valued functions u harmonic in D with finite Dirichlet integral. We also denote by $\mathcal{A}D$ the set of all functions ϕ holomorphic in D with finite Dirichlet integral. For any $u \in \mathcal{D}$, let $P(u \circ h)$ denote the Poission integral of $u \circ h$. We define

$$K_1(h) = \sup_{A,B\subset\partial D} \frac{\mathcal{D}_D[P(u_{A,B}\circ h)]}{\mathcal{D}_D[u_{A,B}]}. \tag{3.5}$$

Since

$$\operatorname{mod}(h^{-1}(A), h^{-1}(B); D) = \mathcal{D}_D[u_{h^{-1}(A),h^{-1}(B)}] \leqslant \mathcal{D}_D[P(u_{A,B}\circ h)]$$

for all pairs $A, B \subset \partial D$, it follows that $K(h) = K(h^{-1}) \leqslant K_1(h)^*$. We also define

$$K_2(h) = \sup_{u\in\mathcal{D}} \frac{\mathcal{D}_D[P(u\circ h)]}{\mathcal{D}_D[u]}. \tag{3.6}$$

Note that the invariant $K_2(h)$ (more precisely, $\max(K_2(h), K_2(h^{-1}))$) was already introduced by Beurling and Ahlfors in their famous paper [BA] and has been much investigated recently (see [KP], [NS], [P1–P5], [Sh1–Sh5]). Particularly, the following result was pointed out by the author [Sh4].

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*It is not known whether or not $K_1(h) = K_1(h^{-1})$ for any quasisymmetric homeomorphism h.

THEOREM 3.3 [Sh4]. *For any quasisymmetric homeomorphism h, it holds that* $K_2(h) = K_2(h^{-1})$.

It is obvious that $K_1(h) \leqslant K_2(h)$. On the other hand, by the quasi-invariance property of the Dirichlet integral under quasiconformal mappings, it holds that $K_2(h) \leqslant K^*(h)$. Consequently, for any quasisymmetric homeomorphism h, it holds that

$$K(h) \leqslant K_1(h) \leqslant K_2(h) \leqslant K^*(h). \tag{3.7}$$

To determine when $K_2(h) = K^*(h)$, the author [Sh1] proved

THEOREM 3.4 [Sh1]. *Let $h : \partial D \to \partial D$ be a quasisymmetric homeomorphism. Then $K_2(h) = K^*(h)$ if and only if $Q(h)$ contains an extremal quasiconformal mapping whose Beltrami differential μ satisfies*

$$\sup_{\phi \in \mathcal{A}D} \frac{\operatorname{Re} \iint_D \mu(z) {\phi'}^2(z)\, dx\, dy}{\iint_D |{\phi'}^2(z)|\, dx\, dy} = \|\mu\|_\infty. \tag{3.8}$$

Remark. Here it is a convenient place to point out a result of Shiga and Tanigawa. After the paper [Sh2] was published, the paper [ST] by Shiga and Tanigawa was called to the author's attention. In their paper, among other things, Shiga and Tanigawa proved that there exists a quasisymmetric homeomorphism h such that $H(h) = K^*(h)$, and that the relation (3.8) and consequently the relation (3.4) doesn't hold for any extremal quasiconformal mapping in $Q(h)$, which implies that $K(h) < K^*(h)$ by Theorem 3.2. Therefore, this has already given an example h for which $H(h) = K^*(h)$ but $K(h) < K^*(h)$. However, this example was abstractly constructed and somewhat complicated (but has some further properties). Note that for this construction, $K_2(h) < K^*(h)$ by Theorem 3.4, while for Strebel's quasisymmetric homeomorphism h in Example 1, it holds that $K_2(h) = K^*(h)$ (see [Sh2]).

To determine when $K_1(h) = K^*(h)$, the author [Sh5] proved

THEOREM 3.5 [Sh5]. *Let $h : \partial D \to \partial D$ be a quasisymmetric homeomorphism. Then*

(1) $K_1(h) = K^*(h)$ *if and only if $Q(h)$ contains an extremal mapping whose complex dilatation μ satisfies the relation* (3.4)*;*

(2) *If, in addition, $K_1(h)$ is attained by a pair of disjoint nondegenerate continua in ∂D, then $K_1(h) = K^*(h)$ if and only if h is induced by an affine mapping.*

4. Further results

Let Ω be a quasidisk in the extended complex plane $\overline{C}$. Recall that $h_\Omega = f_2 \circ f_1^{-1}|\partial D$ is the sewing mapping of the domains Ω and Ω^*. In this section, we will give some further relations among the constants attached to the quasidisk Ω and the constants to the quasisymmetric homeomorphism h_Ω.

First we note the following proposition, the second part of which was proved in [P4] and [Sh3], while the third part of which was proved in [Sh5].

PROPOSITION 4.1. *Let Ω be a quasidisk in the extended complex plane $\overline{C}$. Then we have*

(1) $R(\Omega) = K^*(h_\Omega)$;
(2) $s(\Omega^*, \Omega) = K_2(h_\Omega)$, $s(\Omega, \Omega^*) = K_2(h_\Omega^{-1})$;
(3) $K(h_\Omega) + 1 \leqslant M_b(\Omega) \leqslant K_1(h_\Omega^{-1}) + 1$.

Noting that (see Theorems 2.1 and 3.3)

$$\frac{1}{\lambda_\Gamma} = \frac{s(\Omega, \Omega^*) - 1}{s(\Omega, \Omega^*) + 1} = \frac{s(\Omega^*, \Omega) - 1}{s(\Omega^*, \Omega) + 1} = \frac{K_2(h_\Omega) - 1}{K_2(h_\Omega) + 1} = \frac{K_2(h_\Omega^{-1}) - 1}{K_2(h_\Omega^{-1}) + 1},$$

we find out that the equality in (2.6) holds, that is, $\frac{1}{\lambda_\Gamma} = \frac{R(\Omega)-1}{R(\Omega)+1}$, iff $K_2(h_\Omega) = K^*(h_\Omega)$, so we can conclude from Theorem 3.4 the following result. Note that by the Schiffer–Kühnau result [Ku], [S2], λ_Γ^{-1} is just the Grunsky functional (see [Kr2]), so it is also implied by Krushkal's work [Kr2].

COROLLARY 4.1. $\frac{1}{\lambda_\Gamma} = \frac{R(\Omega)-1}{R(\Omega)+1}$ *iff $Q(h_\Omega)$ contains an extremal quasiconformal mapping whose Beltrami differential μ satisfies* (3.8).

The following theorem gives a characterization of the domains Ω for which $M_b(\Omega) = R(\Omega) + 1$.

THEOREM 4.1 [Sh5]. *Let Ω be a quasidisk in the extended complex plane. Then*

(1) *$M_b(\Omega) = R(\Omega) + 1$ if and only if $Q(h_\Omega)$ contains an extremal mapping whose complex dilatation μ satisfies the relation* (3.4).

(2) *If, in addition, $M_b(\Omega)$ is attained by a pair of disjoint nondegenerate continua in $\partial\Omega$, then $M_b(\Omega) = R(\Omega) + 1$ if and only if h_Ω is induced by an affine mapping.*

An immediate consequence of Theorems 3.2, 3.5 and 4.1 is the following

COROLLARY 4.2. *Let Ω be a quasidisk in the extended complex plane. Then the following conditions are all equivalent:*

(1) $M_b(\Omega) = R(\Omega) + 1$; (2) $K(h_\Omega) = K^*(h_\Omega)$; (3) $K_1(h_\Omega) = K^*(h_\Omega)$;

(4) *$Q(h_\Omega)$ contains an extremal mapping whose complex dilatation μ satisfies the relation* (3.4).

Now we want to give some relation between $M_b(\Omega)$ and $K(h_\Omega)$. When h_Ω is induced by an affine mapping, it holds that $M_b(\Omega) = K(h_\Omega) + 1$. The following theorem says that this is the only case when $M_b(\Omega) = K(h_\Omega) + 1$ if $K(h_\Omega)$ is in addition attained by a pair of disjoint nondegenerate continua.

THEOREM 4.2 [Sh5]. *Let Ω be a quasidisk in the extended complex plane. If A and B is a pair of disjoint nondegenerate continua in ∂D such that*

$$K(h_\Omega) = \frac{\operatorname{mod}(h_\Omega(A), h_\Omega(B); D)}{\operatorname{mod}(A, B; D)},$$

then $M_b(\Omega) = K(h_\Omega) + 1$ if and only if h_Ω is induced by an affine mapping.

As an application of previous results, we have a rather satisfactory discription of the various constants when Ω is an asymptotically conformal extension domain. Note that in this case, h_Ω can not be induced by affine mappings unless Ω is a disk.

THEOREM 4.3 [Sh5]. *Let Ω be an asymptotically conformal extension domain. Then all the supremums in* (2.9), (3.3), (3.5) *and* (3.6) *can be attained. Furthermore, $K(h_\Omega) < M_b(\Omega) - 1 \leqslant K_1(h_\Omega) < K^*(h_\Omega) = R(\Omega)$ unless Ω is a disk.*

Remark. As stated in section 2, Wu and Yang [WY, Theorem 2.3] proved that $M_b(\Omega) < R(\Omega) + 1$ for all asymptotically conformal extension domains other than disks. On the other hand, Wu [Wu, Theorem 4] and Yang [Y2, Corollary 2.6] proved independently that $K(h_\Omega) < K^*(h_\Omega)$ for all asymptotically conformal extension domains other than disks. Theorem 4.3 implies the stronger result that $K(h_\Omega) < M_b(\Omega) - 1 < K^*(h_\Omega)$ for such domains.

We also have some further results concerning the Garnett–Yang conjecture.

THEOREM 4.4 [Sh5]. *Let Ω be a quasidisk in the extended complex plane. If $M(\Omega)$ is attained by a pair of disjoint nondegenerate continua A and B in $\overline{\Omega}$, then $M(\Omega) = R(\Omega) + 1$ if and only if h_Ω is induced by an affine mapping.*

THEOREM 4.5 [Sh5]. *Let Ω be a quasidisk in the extended complex plane. Then either there exists a pair of disjoint nondegenerate continua A an B in $\overline{\Omega}$ such that $M(\Omega) = \frac{\operatorname{mod}(A,B;C)}{\operatorname{mod}(A,B;\Omega)}$ or $M(\Omega) \leqslant 2H(h_\Omega)$.*

Remark. An interesting question is to determine whether the bound $2H(h_\Omega)$ in Theorem 4.5 can be replaced by $1 + H(h_\Omega)$. If the answer to the question were affirmative, then there would be a large class of domains Ω for which $M(\Omega) < R(\Omega) + 1$, namely, the domains Ω whose associated sewing mappings h_Ω are Strebel points and are not induced by affine mappings.

We end this section with some questions relating to the quasiextremal distance constant and the quasiconformal reflection constant. For a pair of complementary quasidisks, we do not know whether the relations hold: $M_b(\Omega) = M(\Omega)$, $M_b(\Omega) = M_b(\Omega^*)$, $M(\Omega) = M(\Omega^*)$. Note that the last relation was also conjectured by Garnett–Yang [GY]. Even for asymptotically conformal extension domains, these questions still remain open. Theorems 2.2 and 4.3 may shed some new light on these questions for asymptotically conformal extension domains. We also have no characterizations of the domains Ω for which the relation $M(\Omega) = R(\Omega) + 1$ holds.

5. Metrics on the Universal Teichmüller space

Using some constants attached to quasisymmetric homeomorphisms, one can define some invariant metrics on the universal Teichmüller space. Given two points $[h_1]$ and $[h_2]$ in the universal Teichmüller space T, we define

$$d^*([h_1],[h_2]) = \frac{1}{2}\log K^*(h_2 \circ h_1^{-1}), \tag{5.1}$$

$$d([h_1],[h_2]) = \frac{1}{2}\log K(h_2 \circ h_1^{-1}), \tag{5.2}$$

$$d_2([h_1],[h_2]) = \frac{1}{2}\log K_2(h_2 \circ h_1^{-1}). \tag{5.3}$$

Note that d^* is the well-known Teichmüller metric (see [Ga], [GL]), d is the metric defined by Lehto [Le], while d_2 is precisely the metric defined previously by the author [Sh3]. It should be pointed out that the metric d has been much discussed recently by Partyka, Sakan and Zajac (see [PS1], [PS2], [SZ], [Z]). Although we do not know whether these metrics are metrically equivalent, they are actually uniformly topologically equivalent. In fact, they are metrically equivalent in the infinitesimal sense.

Let h_t be a family of quasisymmetric homeomorphisms which are the boundary values of the quasiconformal mappings f_t with complex dilatation $\mu_t = t\mu + o(t)$ for small $t > 0$. It is known (see [Ga], [GL]) that

$$d^*([0],[h_t]) = t\|\mu\|^* + o(t) = t \sup_{\phi\in\mathcal{A}} \frac{\operatorname{Re}\iint_D \mu(z)\phi(z)\,dx\,dy}{\iint_D |\phi(z)|\,dx\,dy} + o(t). \tag{5.4}$$

It was calulated by the author [Sh3] that

$$d_2([0],[h_t]) = t\|\mu\|_2 + o(t) = t \sup_{\phi\in\mathcal{A}D} \frac{\operatorname{Re}\iint_D \mu(z)\phi'^2(z)\,dx\,dy}{\int_D |\phi'^2(z)|\,dx\,dy} + o(t). \tag{5.5}$$

A direct computation will also show that

$$d([0],[h_t]) = t\|\mu\| + o(t) = t \sup_{A,B\subset\partial D} \frac{\operatorname{Re}\iint_D \mu(z)\phi'^2_{A,B}(z)\,dx\,dy}{\iint_D |\phi'^2_{A,B}(z)|\,dx\,dy} + o(t). \tag{5.6}$$

$\|\mu\|^*$ is known as the Teichmüller norm of μ. $\|\mu\|_2$ can be regarded as another norm of μ, which was proved to be equivalent to $\|\mu\|^*$ (see [Sh3]). Now $\|\mu\|$ is precisely the cross-ratio norm of μ in the sense of Earle–Gardiner–Lakić [EGL]. They also proved that $\|\mu\|$ is equivalent to $\|\mu\|^*$. In fact, for any pair of disjoint arcs A and B in ∂D joining z_1, z_2 and z_3, z_4, respectively, by the well-known Christoffel–Schwarz formula, we have

$$\frac{\phi'^2_{A,B}}{\iint_D |\phi'^2_{A,B}|} = \frac{-\frac{(z_1-z_2)(z_3-z_4)}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}}{\iint_D \frac{|(z_1-z_2)(z_3-z_4)|}{|(z-z_1)(z-z_2)(z-z_3)(z-z_4)|}}. \tag{5.7}$$

Consequently, if we extend μ to the whole plane by reflection, we have

$$\begin{aligned}
\|\mu\| &= \sup_{A,B\subset\partial D} \frac{\operatorname{Re}\iint_D \mu(z)\phi'^2_{A,B}(z)\,dx\,dy}{\iint_D |\phi'^2_{A,B}(z)|\,dx\,dy} \\
&= \sup_{z_1,z_2,z_3,z_4} \frac{\operatorname{Re}\iint_D \frac{\mu(z)(z_1-z_2)(z_3-z_4)}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}\,dx\,dy}{\iint_D \frac{|(z_1-z_2)(z_3-z_4)|}{|(z-z_1)(z-z_2)(z-z_3)(z-z_4)|}} \\
&= \sup_{z_1,z_2,z_3,z_4} \frac{\left|\iint_C \frac{\mu(z)}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}\,dx\,dy\right|}{\iint_C \frac{1}{|(z-z_1)(z-z_2)(z-z_3)(z-z_4)|}}.
\end{aligned} \tag{5.8}$$

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