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REMARKS ON THE CONDITIONS FOR UNIQUE EXTREMALITY OF QUASICONFORMAL MAPPINGS

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Abstract. In work dating to 1981, two different sufficient conditions for unique extremality of quasiconformal mappings, referred to here as Theorem A and Theorem B were found. In the case of Theorem A, the condition is now known to also be necessary. In spite of this, Theorem B has been found to be the more useful of the two conditions in delicate situations. However, the question of whether the conditions of Theorem B are necessary is still open. We discuss some matters relevant to this problem. As a new illustration of Theorem B, we show that it can be used to give a self-contained proof of a result of Strebel's.

0. Introduction

In [3], two separate sufficient conditions for unique extremality of quasiconformal mappings of Teichmüller type, referred to as Theorem A and Theorem B, were proved. A review of the statements and an outline of their proofs are given in Section 1, following. It turned out [1] that the conditions for Theorem A were not only sufficient but also necessary for unique extremality. For Theorem B the question of necessity is open. In Section 3 we explore some of the difficulties of this question and state a positive partial result, Theorem 3.1. In practice, Theorem B has turned out to be more useful than Theorem A for proving unique extremality in delicate cases. A rather spectacular illustration of this is found in a recent paper [2] of V. Marković. A less difficult example is given in Section 2, below.

1. Review of the conditions and their proofs

Assume that $f(z)$ and $g(z)$ are quasiconformal mappings of a region Ω of the plane, agreeing on $\partial\Omega$.

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Let μ_f, μ_g, $\mu_{f^{-1}}$, $\mu_{g^{-1}}$, respectively denote the complex dilatations of f, g, f^{-1}, g^{-1}, and let

$$\kappa(z) = \mu_f(z),\ \alpha(z) = (\mu_{f^{-1}} \circ f)(z),\ \beta(z) = (\mu_{g^{-1}} \circ f)(z). \tag{1.1}$$

Furthermore, let $L^1_a(\Omega)$ denote the class of functions holomorphic in Ω and belonging to $L^1(\Omega)$. Under the foregoing conditions, it is known[1] that the following inequality holds for all $\varphi \in L^1_a(\Omega)$:

$$\begin{aligned}(1.2) \qquad & \iint_\Omega \frac{(|\alpha|^2 - |\beta|^2) + (1-|\kappa|)\left(|\alpha| - \mathrm{Re}\,\frac{\overline{\beta}\alpha}{|\alpha|}\right)}{(1+|\kappa|)(1-|\beta|^2)}\,|\varphi|\,dx\,dy \\ & \leqslant \mathrm{Re} \iint_\Omega \frac{\overline{\alpha}}{|\alpha|}\left(|\varphi| - \frac{\kappa}{|\kappa|}\,\varphi\right)\frac{(1-\overline{\beta}\alpha)(\alpha-\beta)}{(1-|\kappa|^2)(1-|\beta|^2)}\,dx\,dy\end{aligned}$$

Our object is to discuss some known conditions for the unique extremality of f among the class of quasiconformal mappings of Ω with the same boundary values as f and belonging to the same homotopy class as f, for the case when

$$\kappa(z) = \mu_f(z) = k\,\frac{\overline{\varphi_o(z)}}{|\varphi_o(z)|}\,, \tag{1.3}$$

where $\varphi_o(z)$ is holomorphic in Ω but not necessarily an element of $L^1_a(\Omega)$. If one analyzes the development in [3] it is seen to implicitly involve two lemmas, stated as Lemma A and Lemma B below. These lemmas do not occur explicitly in [3], but their proofs are actually there.

Before proceeding, *disregarding the definitions* (1.1) *and* (1.3), we shall say that the constant k and the *arbitrary* functions, $\kappa \in L^\infty(\Omega)$, $\alpha \in L^\infty(\Omega)$, $\beta \in L^\infty(\Omega)$, $\varphi \in L^1(\Omega)$, *satisfy the Main Inequality*, written

$$\{k, \kappa, \alpha, \beta, \varphi\} \in \mathcal{M} \tag{1.4}$$

for short, to mean solely that $0 \leqslant k < 1$, and that

$$|\kappa(z)| = |\alpha(z)| \leqslant k,\ |\beta(z)| \leqslant k \ \text{ a.e. in } \Omega, \tag{1.5}$$

and that the inequality (1.2) is satisfied.[2]

Lemma 1.1. *Suppose* $\{k, \kappa, \alpha, \beta, \varphi\} \in \mathcal{M}$*, where* $|\kappa(z)| \equiv k$ *a.e. in* Ω*. Then*

(1.6)
$$\iint_\Omega |\alpha-\beta|^2|\varphi|\,dx\,dy \leqslant \sqrt{C_o(k)} \iint_\Omega |\alpha-\beta|\,|\varphi|^{1/2}\Big(k|\varphi| - \mathrm{Re}(\kappa\varphi)\Big)^{1/2}\,dx\,dy,$$
$$C_o(k) = \frac{8k(1+k^2)^2}{(1+k)^2(1-k)^6}.$$

[1]Inequality (1.2) is a version of an inequality due to K. Strebel and the author, sometimes referred to as the "Main Inequality". See [3] and the references there.

[2]In particular, note that whether or not φ is holomorphic plays no part in (1.4).

Proof. The proof consists merely in algebraic manipulation of (1.2). See [3, pp. 292–294]. □

For κ, φ as in (1.5), let

$$\delta\{\varphi\} = \operatorname{Re} \iint_{\Omega} \big(k|\varphi(z)| - \kappa(z)\varphi(z)\big)\, dx\, dy.$$

LEMMA A. *Suppose* $|\kappa(z)| \equiv k$ *a.e. in* Ω, *and* $\{k, \kappa, \alpha, \beta, \varphi_n\} \in \mathcal{M}$, $n = 1, 2, \ldots$, *where*

$$\lim_{n\to\infty} \varphi_n(z) \doteq \varphi_o(z) \neq 0 \text{ a.e. in } \Omega, \tag{1.7}$$

and

$$\lim_{n\to\infty} \delta\{\varphi_n\} = 0. \tag{1.8}$$

Then $\alpha = \beta$.

Proof. Using Schwarz's Inequality in (1.6) gives

$$\iint_{\Omega} |\alpha - \beta|^2 |\varphi_n|\, dx\, dy \leqslant C_o(k)\delta\{\varphi_n\}, \quad n = 1, 2, \ldots.$$

By (1.7) and Fatou's Lemma,

$$\iint_{\Omega} |\alpha - \beta|^2 |\varphi_o|\, dx\, dy = 0,$$

and the conclusion follows. □

LEMMA B. *Suppose* $|\kappa(z)| \equiv k$ *a.e. in* Ω, *and* $\{k, \kappa, \alpha, \beta, \varphi_n\} \in \mathcal{M}$, $n = 1, 2, \ldots$, *where* (1.7), (1.9), *and* (1.10) *hold.*

$$\big\{\delta\{\varphi_n\}\big\} \text{ is a bounded sequence} \tag{1.9}$$

$$\liminf_{t\to 0} \iint_{\Omega(n,t)} |\varphi_n(z)|\, dx\, dy = 0 \text{ uniformly with respect to } n, \tag{1.10}$$

where $\Omega(n,t) = \{z \in \Omega : |\varphi_n(z)| > (1/t)|\varphi_o(z)|\}$. *Then* $\alpha = \beta$.

Proof. The proof also starts with (1.6), but involves a more refined procedure than the one used for Lemma A. See [3, pp. 294–295]. □

The basic results of [3] follow immediately from (1.1) and (1.2) in conjunction with Lemmas A and B; namely,

THEOREM A. *Suppose* f *is a quasiconformal mapping* Ω *with* $\mu_f(z)$ *of the form* (1.3), *and such that* (1.7) *and* (1.8) *hold for a sequence* $\varphi_n \in L^1_a(\Omega)$, $n = 1, 2, \ldots$. *Then* f *is uniquely extremal.*

THEOREM B. *Suppose* f *is a quasiconformal mapping of* Ω *with* $\mu_f(z)$ *of the form* (1.3), *and such that* (1.7), (1.9) *and* (1.10) *hold for a sequence* $\varphi_n \in L^1_a(\Omega)$, $n = 1, 2, \ldots$. *Then* f *is uniquely extremal.*

2. New proof of a theorem of Strebel

The mapping $f(z) = r^s e^{i\theta}$, $(z = re^{i\theta})$, where s is a positive constant, occurs in numerous contexts in the theory of plane quasiconformal mappings. This mapping has complex dilatation

$$\mu_f(z) = \frac{f_{\bar z}}{f_z} = \frac{s-1}{s+1}\,\nu(z),\ \nu(z) = \frac{z}{\bar z}, \quad (z \neq 0). \tag{2.1}$$

From (2.1) one sees that f is a K-quasiconformal mapping of the complex plane $\mathcal{C}$ onto itself, with $K = \max(s, 1/s)$.

Strebel's Theorem [5]. *Let $E \subset \mathcal{C}$ be a closed denumerably infinite set. If E has an accumulation point at $z = 0$ but not at $z = \infty$ (or at $z = \infty$ but not at $z = 0$), then f is an extremal mapping among the class of quasiconformal mappings of $\mathcal{C}$ onto itself that agree with f on E and are homotopic to f, but f is not uniquely extremal. If E has accumulation points both at $z = 0$ and at $z = \infty$, then f is the uniquely extremal quasiconformal mapping of $\mathcal{C}$ onto itself that agrees with f on E and is homotopic to f.*

A surprising aspect of Strebel's theorem (as he points out) is that the density of E near 0 or ∞ plays no role in determining whether or not extremality or unique extremality of f holds. Our purpose will be to give an "analytic" proof of both parts of Strebel's theorem. This may be compared with Strebel's "geometric" proof which makes strong use of a modulus theorem of Teichmüller's. The analytic proof turns out to be somewhat shorter than the original proof.

There is evidently no loss of generality in assuming that E has an accumulation point at $z = 0$. If E is bounded, then it is easy to see that f cannot be uniquely extremal by merely replacing f outside a circle of sufficiently large radius by a multiple of the identity.

Note that we can write

$$\nu(z) = \frac{\overline{\varphi_o(z)}}{|\varphi_o(z)|}, \quad \text{with } \varphi_o(z) = \frac{1}{z^2}\,. \tag{2.2}$$

For $\Omega = \mathcal{C} \setminus E$, we set

$$\|\varphi\| = \iint_\Omega |\varphi(z)|\,dx\,dy, \quad \varphi \in L^1(\Omega),$$

$$\Lambda[\varphi] = \iint_\Omega \nu(z)\varphi(z)\,dx\,dy,\ \lambda[\varphi] = \operatorname{Re}\Lambda[\varphi], \quad \varphi \in L^1_a(\Omega).$$

Neglecting the unimportant factor k in the definition of $\delta\{\varphi\}$, set $\delta\{\varphi\} = \|\varphi\| - \lambda[\varphi]$. It is clear that $\delta\{\varphi\} \geqslant 0$ for all $\varphi \in L^1_a(\Omega)$.

It follows from well known principles that, with a given set E, f is extremal if and only if $\|\Lambda\| = 1$; that is, if and only if there exist $\varphi_n \in L^1_a(\Omega)$, such that

$$\lim_{n\to\infty} \frac{\lambda[\varphi_n]}{\|\varphi_n\|} = 1. \tag{2.3}$$

According to Theorem B, we have the following:

A sufficient condition that f is uniquely extremal is that there exist $\varphi_n \in L^1_a(\Omega)$, such that (2.4), (2.5), *and* (2.6) *hold; namely*

$$\lim_{n\to\infty} \varphi_n(z) = 1/z^2, \ \textit{pointwise a.e. in } \Omega, \tag{2.4}$$

$$\{\delta\{\varphi_n\}\} \ \textit{is a bounded sequence}, \tag{2.5}$$

$$\lim_{t\to 0} \iint_{\Omega(n,t)} |\varphi_n(z)|\, dx\, dy = 0 \ \textit{uniformly with respect to } n, \tag{2.6}$$

where $\Omega(n,t) = \{z \in \Omega : |\varphi_n(z)| > 1/t|z^2|\}$, $(t > 0)$.

Of conditions (2.4)–(2.6), conditions (2.4) and (2.5) have to be given first priority; if we are lucky with our choice of $\{\varphi_n\}$, the hope is that (2.6) is satisfied automatically.[3] In our proof of Strebel's theorem we will choose $\varphi_n(z)$ as

$$\varphi(a, A, z) = \frac{A}{z(z-a)(A-z)}, \tag{2.7}$$

where a and A, with $0 < |a| < 1 < |A|$, will be selected for $n = 1, 2, 3, \ldots$ as certain elements of E. It is clear that $\varphi \in L^1_a(\Omega)$ for any such choice of a and A. We proceed to the details that will lead to the selection. For the function $\varphi(a, A, z)$ of (2.7) we have, setting $\gamma = a/A$, $(0 < |\gamma| < 1)$,

$$\Lambda[\varphi] = \iint_{\mathcal{C}} \frac{1}{\bar{z}(z-1)(1-\gamma z)}\, dx\, dy = \frac{2\pi}{1-\gamma} \log \frac{1}{|\gamma|}, \tag{2.8}$$

$$\|\varphi\| = \iint_{\mathcal{C}} \frac{1}{|z(z-1)(1-\gamma z)|}\, dx\, dy\,. \tag{2.9}$$

It will alternately be useful to write

$$\begin{aligned}\|\varphi\| = \iint_{|z|<1} \frac{1}{|z(z-a)(1-z/A)|}\, dx\, dy + \iint_{|z|<1} \frac{1}{|z(z-1/A)(1-az)|}\, dx\, dy \\ = I_1 + I_2,\end{aligned} \tag{2.10}$$

and correspondingly,

$$\begin{aligned}\Lambda[\varphi] = \iint_{|z|<1} \frac{1}{\bar{z}(z-a)(1-z/A)}\, dx\, dy + \iint_{|z|<1} \frac{1}{\bar{z}(z-1/A)(1-az)}\, dx\, dy \\ = \Lambda_1 + \Lambda_2.\end{aligned} \tag{2.11}$$

[3] This matter is discussed further in Section 3.

LEMMA 2.1. *We have* $\delta\{\varphi(a,A,\cdot)\} < 200$, $(0 < |a| \leqslant 1/4,\ |A| \geqslant 4)$.

Proof. By (2.10), (2.11),

$$\delta\{\varphi(a,A,\cdot)\} = \delta_1 + \delta_2, \quad \delta_1 = I_1 - \operatorname{Re}\Lambda_1, \quad \delta_2 = I_2 - \operatorname{Re}\Lambda_2.$$

It is evidently enough to consider δ_1 since any bound for δ_1 on the postulated set of pairs (a,A) will also be a bound for δ_2. Replacing z by az in the integration, we can write

$$\delta_1(a,A) = \iint_{|z|<1/|a|} \left[\frac{1}{|\bar{z}(z-1)(1-\gamma z)|} - \operatorname{Re}\frac{1}{\bar{z}(z-1)(1-\gamma z)}\right] dx\,dy = \delta_{11} + \delta_{12},$$

where

$$\delta_{11} = \iint_{|z|<2}, \quad \delta_{12} = \iint_{2<|z|<1/|a|}.$$

Since $|\gamma| \leqslant 1/16$, we have

$$\delta_{11} \leqslant \frac{8}{7}\iint_{|z|<2} \frac{1}{|z(z-1)|}\,dx\,dy < 18. \tag{2.12}$$

In polar coordinates we can write

$$\delta_{12} = \int_2^{1/|a|} G(r)\,dr, \quad G(r) = \int_0^{2\pi}\left(\frac{1}{|w|} - \operatorname{Re}\frac{1}{w}\right) d\theta = \int_0^{2\pi} \frac{[\operatorname{Im} w]^2}{|w|^2[|w| + \operatorname{Re} w]}\,d\theta, \tag{2.13}$$

where

$$w = (r - e^{-i\theta})(1 - \gamma r e^{i\theta}) = r + \gamma r(1 - re^{i\theta}) - e^{-i\theta}, \quad (2 \leqslant r \leqslant 1/|a|).$$

During this interval for r, we have

$$|\gamma r| \leqslant \left|\frac{a}{A}\right| \frac{1}{|a|} = \frac{1}{|A|} \leqslant \frac{1}{4}.$$

Hence,

$$\operatorname{Re} w \geqslant r - \frac{1}{4}(1+r) - 1 = \frac{3}{4}r - \frac{5}{4} \geqslant \frac{1}{4}.$$

Therefore, by (2.13),

$$G(r) \leqslant \int_0^{2\pi} \frac{[\operatorname{Im} w]^2}{|w|^3}\,d\theta, \quad (2 \leqslant r \leqslant 1/|a|). \tag{2.14}$$

Now,

$$\operatorname{Im} w = \sin\theta + r\operatorname{Im}\gamma - r^2\operatorname{Im}(\gamma e^{i\theta}).$$

Therefore,

$$|\operatorname{Im} w| \leqslant 1 + |\gamma|(r^2 + r) \leqslant 1 + 2|\gamma|r^2, \quad (2 \leqslant r \leqslant 1/|a|).$$

Furthermore,

$$|w| \geqslant (r-1)(1-|\gamma|r) \geqslant \left(\frac{r}{2}\right)\left(1 - \frac{1}{|A|}\right) \geqslant \frac{3r}{8}, \quad (2 \leqslant r \leqslant 1/|a|).$$

So, by (2.14),

$$G(r) \leqslant 120\,\frac{(1+2|\gamma|r^2)^2}{r^3}, \quad (2 \leqslant r \leqslant 1/|a|).$$

Going back to (2.13), this gives

$$\delta_{12} \leqslant 120\left[\frac{1}{8} + 4\,|\gamma|\log\frac{1}{|a|} + \frac{2}{|A|^2}\right] < 120\left[\frac{1}{8} + \frac{4}{|A|e} + \frac{2}{16}\right] < 75.$$

So, in view of (2.12), we get $\delta_1(a, A) < 18 + 75 < 100$, when $0 < |a| \leqslant 0.25$, $|A| \geqslant 4$. Since the same bound holds for $\delta_2(a, A)$, the proof is complete. □

Lemma 2.2. *For fixed A, we have*

$$\Lambda[\varphi] = 2\pi\log\frac{1}{|a|} + O(1), \quad \|\varphi\| = 2\pi\log\frac{1}{|a|} + O(1), \quad \textit{as } a \to 0. \tag{2.15}$$

Proof. The first relation follows immediately from (2.8). The second assertion follows from Lemma 2.1 and the fact that $\|\varphi\| \geqslant |\Lambda[\varphi]|$. □

We now proceed to what we will need in connection with condition (2.6). Let

$$J(a, A, t) = \iint_{\Omega(a,A,t)} |\varphi(a, A, z|\,dx\,dy, \quad (t > 0),$$

where $\Omega(a, A, t) = \left\{z \in \mathcal{C} : |\varphi(a, A, z)| > 1/t|z|^2\right\}.$

Lemma 2.3. *As $t \to 0$, $J(a, A, t) = 4\pi t + o(t)$, uniformly with repect to a, A, $0 < |a| \leqslant 1/4$, $|A| \geqslant 4$.*

Proof. Analogously to the development above, we can write $J(a, A, t) = J_1 + J_2$, where the integrand for J_1 is the same as the integrand for I_1 in (2.10), but the domain of integration for J_1 is $\{z \in \mathcal{C} : |z| < 1,\ z \in \Omega(a, A, t)\}$. J_2 is obtained from J_1 by interchanging a and $1/A$. Replacing the integration variable z by az again,

$$J_1(a, A, t) = \iint \frac{1}{|z(z-1)(1-\gamma z)|}\,dx\,dy\,, \tag{2.16}$$

where the domain of integration is now

$$\left\{z \in \mathcal{C} : |z| < \frac{1}{|a|}\,,\ \left|\frac{(z-1)(1-\gamma z)}{z}\right| < t\right\}. \tag{2.17}$$

Since the factor $|1-\gamma z|$ is uniformly bounded away from 0 in the domain of integration, one sees from (2.16) and (2.17) that, as $t \to 0$,

$$J_1(a, A, t) = \iint_{\{z:|(1-\gamma)(z-1)|<t\}} \frac{1}{|(1-\gamma)(z-1)|}\, dx\, dy + o(t) = 2\pi t + o(t),$$

uniformly with respect to a, A. Since the same holds for $J_2(a, A, t)$, the lemma follows. □

Proof of the theorem. For the case when both 0 and ∞ are accumulation points of E, we let $\varphi_n(z) = \varphi(a_n, A_n, z)$, where $\{a_n\}$, $\{A_n\}$ are subsequences of points of E with $\lim a_n = 0$, $\lim A_n = \infty$. It is obvious that (2.4) holds. By Lemma 2.1, condition (2.5) holds, and by Lemma 2.3, condition (2.6) holds, and hence unique extremality of f follows.

When we are only given that 0 is an accumulation point of E, it is clear that one can rescale the problem so that E contains at least one point Υ, with $|\Upsilon| \geqslant 4$. We then choose $\varphi_n(z) = \varphi(a_n, \Upsilon, z)$. By multiplying $\varphi_n(z)$ by appropriate unimodular complex constants we can insure that $\Lambda[\varphi_n] \geqslant 0$. By Lemma 2.2, it follows that (2.3) holds, thus guaranteeing that f is extremal. Since (2.3) holds, $\{\varphi_n\}$ constitutes what is known as a Hamilton sequence for μ_f. It may at first sight seem strange that $\lim \varphi_n(z)$ differs from $\varphi_o(z)$, but for a Hamilton sequence it is only the behavior of the elements of the sequence near points where the L^1-norm is large that matters. □

In the unique-extremality case, our sequence $\{\varphi_n\}$ has the property that

$$\liminf_{n\to\infty} \delta\{\varphi_n\} > 0.$$

Namely, by Fatou's Lemma,

$$\liminf_{n\to\infty} \delta\{\varphi_n\} \geqslant 2 \liminf_{\substack{a\to 0,\\ A\to\infty}} \delta_1(a, A) \geqslant \iint_{\mathcal{C}} \left[\frac{1}{|\bar{z}(z-1)|} - \operatorname{Re} \frac{1}{\bar{z}(z-1)}\right] dx\, dy > 0.$$

This shows that the proof of unique extremality, using our particular sequence $\{\varphi_n\}$, could not have been accomplished with Theorem A, which, although it would not have required hypothesis (2.6), would have required, not just that $\{\delta\{\varphi_n\}\}$ is bounded, but that $\lim \delta\{\varphi_n\} = 0$.

3. An open question

It is clearly of interest to know whether condition (1.10) can be omitted from the hypotheses of Theorem B. We do not know the answer. If the answer were yes, then Theorem B would be a stronger theorem than Theorem A, and it would provide a necessary and sufficient condition for unique extremality. We proceed with some remarks relevant to the basic special case when f is the affine stretch $\mathcal{A}_K$, where K is a constant, $K > 1$,

$$f(z) = \mathcal{A}_K(z) = Kx + iy, \quad (z = x + iy).$$

In this case, $\kappa(z) = \mu_f(z) \equiv k = (K-1)/(K+1) > 0$, $\alpha(z) \equiv -k$, $\varphi_o(z) \equiv 1$. The problem is therefore the following:

QUESTION. *Suppose there exists a sequence* $\varphi_n \in L^1_a(\Omega)$, $n = 1, 2, \dots$, *so that*
(i) $\lim_{n\to\infty} \varphi_n(z) = 1$ *locally uniformly in* Ω,
(ii) $\{\iint_\Omega \left[|\varphi_n(z)| - \operatorname{Re}\varphi_n(z)\right] dx\,dy\}$ *is a bounded sequence.*
Does it follow that $\mathcal{A}_K$ *is a uniquely extremal quasiconformal mapping of* Ω *for the boundary values induced by* $\mathcal{A}_K$*?*

We note that conditions (i) and (ii) do *not* imply that (1.10) holds. A simple counterexample is obtained by letting Ω be the unit square $S = \{(x, y) : 0 < x < 1, 0 < y < 1\}$, and setting

$$\varphi_n(z) = ne^{-nz} + 1, \quad z \in S. \tag{3.1}$$

Conditions (i) and (ii) are satisfied, but it is easy to check that, contrary to (1.10),

$$\iint_{\{z\in S: |\varphi_n(z)| > \sqrt{n}\}} |\varphi_n(z)|\, dx\, dy \to 1 \text{ as } n \to \infty.$$

A roughly positive answer to our question can be obtained by introducing the concept of a *tight* region. We shall say that Ω is a tight region if Ω does not contain any region congruent to $\Omega_s = \{z = x + iy : x > |y|^s\}$ whenever $1 \leqslant s < 3$, i.e., whenever $s_o < s < 3$, where s_o is close to 3. Examples of tight regions are regions Ω of finite area, and regions Ω_s with $s \geqslant 3$. Another example is the region consisting of the plane punctured at the integral lattice points $\{m + in\}$ of [2]. For all these examples it is known that $\mathcal{A}_K$ is uniquely extremal. On the other hand, among examples of non-tight regions are the regions Ω_s when $1 < s < 3$, as well as Strebel's famous chimney region $\{z : |\operatorname{Im} z| < 1\} \cup \{\operatorname{Re} z < 0\}$. For these examples of non-tight regions it is known that $\mathcal{A}_K$ is extremal but not uniquely extremal. Of course there are also many examples of non-tight regions, for example the half plane, for which $\mathcal{A}_K$ is not even extremal. So it seems that, at least roughly speaking, the class of regions Ω for which $\mathcal{A}_K$ is uniquely extremal resembles the class of regions Ω that are tight in the foregoing sense.

THEOREM 3.1. *Suppose there exists a sequence* $\varphi_n \in L^1_a(\Omega)$, $n = 1, 2, \dots$, *such that conditions* (i) *and* (ii) *hold. Then* Ω *is a tight region.*

Proof. The proof is by contradiction. Suppose Ω contained a region Ω_s for some s, $1 \leqslant s < 3$. By the first part of Theorem 1.1 of [4], if $\varphi_n \in L^1_a(\Omega)$ and if condition (i) holds, then

$$\lim_{n\to\infty} \iint_\Omega \left[|\varphi_n(z)| - \operatorname{Re}\varphi_n(z)\right] dx\, dy = +\infty$$

This contradicts condition (ii). $\square$

In contrast to the procedure used for proving Theorem A or Theorem B, the Main Inequality (1.2) alone is not powerful enough to settle the question. This can be seen by taking $\Omega = S$ and defining φ_n by (3.1) again. With $f = \mathcal{A}_K$, Inequality (1.2) becomes

$$(1-k)\,\mathrm{Re}\iint_S \frac{k^2 - |\beta(z)|^2 + (1-k)(k+\beta(z))}{1-|\beta(z)|^2}|\varphi_n(z)|\,dx\,dy$$
$$\leqslant \mathrm{Re}\iint_S \frac{(1+k\overline{\beta(z)})(k+\beta(z))}{1-|\beta(z)|^2}\Big(|\varphi_n(z)| - \varphi_n(z)\Big)\,dx\,dy. \tag{3.2}$$

Since S has finite area, $\mathcal{A}_K$ is obviously uniquely extremal. If this followed from (3.2) alone, where φ_n is given by (3.1), then (3.2) would have to imply that $\beta(z) \equiv \alpha(z) \equiv -k$. In fact, however, substituting $\beta(z) \equiv 0$ in (3.2) in order to see if it might satisfy (3.2), we are led to the inequality

$$k\iint_S |ne^{-nz}+1|\,dx\,dy \geqslant \mathrm{Re}\iint_S (ne^{-nz}+1)\,dx\,dy, \quad n = 1,2,\ldots \tag{3.3}$$

Since the left side of (3.3) goes to $2k$ as $n \to \infty$ and the right side to 1, we see that if we choose, say $k = 0.7$, then $\beta(z) \equiv 0$ will satisfy (3.2) for all sufficiently large n. Hence (3.2) is unfortunately not strong enough to imply that $\alpha = \beta$ with the above sequence $\{\varphi_n\}$. So, *if* the answer to the open question is *yes*, some facts beyond the Main Inequality will have to be used to show it.

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