PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 75(89) (2004), 53–57

ASYMPTOTICS OF THE ADMISSIBLE GROWTH OF THE COEFFICIENT OF QUASICONFORMALITY AT INFINITY AND INJECTIVITY OF IMMERSIONS OF RIEMANNIAN MANIFOLDS

V. A. Zorich

ABSTRACT. We present sharp asymptotics of the admissible growth for the quasiconformality coefficient in the general global homeomorphism theorem. The theorem applies to noncompact Riemannian manifolds of conformally parabolic type and of dimension greater than two.

1. Introduction

Locally invertible quasiconformal mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ *is globally invertible, provided* $n > 2$. This statement was formulated by Lavrentiev [1] for $n = 3$ and later was proved in [2]. Then it was developed in several directions (see survey paper [3]). In particular, the asymptotics of admissible growth of the coefficient of quasiconformality at infinity (sharp in a sense) which still guarantees the global invertibility of arbitrary immersion $f : \mathbb{R}^n \to \mathbb{R}^n$ was indicated in [4].

The passage from immersions of Euclidean space to the general case of immersions of Riemannian manifolds [5], [6] and the analysis of the proof in [2] stimulated the following conformally invariant form of the initial theorem related to asymptotic geometry and conformal classification of Riemannian manifolds [7].

For the global invertibility of a quasiconformal immersion $f : M^n \to N^n$ *of a Riemannian manifold* M^n *to a simply connected Riemannian manifold* N^n *of the same dimension* $n > 2$ *it suffice the manifold* M^n *be of conformally parabolic type.*

In the present paper we show how do the asymptotics of admissible growth of coefficient of quasiconformality looks like in a general theorem on the global invertibility for immersions of Riemannian manifolds of conformally parabolic type. (For the notion of the conformal type of a manifold see, for example, [7], [8]).

Recall (see [8]) that, for instance, the divergence of the integral $\int^{\infty} S^{1/(1-n)}$ may serve as a sufficient condition of conformal parabolicity of a noncompact complete Riemannian manifold M^n. Here $S = S(r)$ is a surface ($(n-1)$-measure) of the sphere of radius r with a fixed center $o \in M^n$.

2000 *Mathematics Subject Classification*: Primary 30C65; Secondary 53C42.

Partially supported by the RBRF, project 02-01-01291. The author is also indebted to FIM-ETH Zürich for the hospitality and prepublication of the present research. A short communication appeared in Russian Math. Surveys **58**(3) (2003).

This assertion follows immediately from the following useful (and used below) general estimate of the conformal capacity of the condenser $R_{r_0}^{r_1}$ bounded by geodesic spheres of radiuses r_0 and r_1 (see [9], [10], [8]):

$$\operatorname{cap} R_{r_0}^{r_1} \leqslant \left(\int_{r_0}^{r_1} S^{1/1-n}\right)^{1-n}.$$

2. Formulation of the main theorem

By the coefficient of quasiconformality of a mapping defined in a domain we mean the least upper bound of linear distortions over all points of this domain. The linear distortion in a point is measured by the ratio of the maximal dilatation to the minimal one. (We skip a detailed discussion of this definition which is actually applicable to nonsmooth mappings as well.)

Let $k(r)$ be the coefficient of quasiconformality of immersion $f : M^n \to N^n$ within the ball of radius r with a fixed center $o \in M^n$. If the mapping f is quasiconformal, the function $k(r)$ is bounded. In the general case (of not a quasiconformal immersion) the magnitude $k(r)$ can grows infinitely with r. The following theorem gives the asymptotics of admissible rate of growth of function $k(r)$ for which the immersion $f : M^n \to N^n$ is still globally injective.

THEOREM. *Suppose that M^n and N^n are Riemannian manifolds of dimension $n > 2$; N^n is simply connected, and M^n is a complete noncompact manifold of conformally parabolic type. If the immersion $f : M^n \to N^n$ satisfies*

$$\int^{\infty} \frac{dr}{\left(kS^{1/(n-1)}\right)(r)} = \infty,$$

then f is injective and the Hausdorff dimension of the set $N^n \setminus f(M^n)$ is equal to zero.

In particular, if $M^n = \mathbb{R}^n$ then $S(r) \asymp r^{n-1}$, and we obtain the result of [4].

3. Proof

The following hint is actually sufficient. Morally, the proof follows the one in [4] where we replace the estimates of conformal capacities of rings in $\mathbb{R}^n$ by the above mentioned general estimate of the value $\operatorname{cap} R_{r_0}^{r_1}$ on a Riemannian manifold M^n. Some more one can find below.

LEMMA. *Let M^n be complete Riemannian noncompact manifold, D unbounded domain in M^n, and $\Gamma = \{\gamma\}$ a family of curves (paths) in D, ending at infinity (i.e., they finally leave any compact set). Suppose that $\varphi : D \to N^n$ is a homeomorphic mapping of the domain D to a Riemannian manifold N^n and $\tilde{\Gamma} := \varphi(\Gamma)$. Under this conditions if $\operatorname{mod} \tilde{\Gamma} > 0$, then*

$$\int^{\infty} \frac{dr}{\left(kS^{1/(n-1)}\right)(r)} < \infty.$$

Here, as before, $S = S(r)$ is a surface of the sphere of radius r with a fixed center $o \in M^n$, but now $k = k_\varphi(r)$ is a coefficient of quasiconformality of mapping $\varphi : D \to N^n$ in the bounded part of the domain D cuted off by this sphere.

PROOF. Notice that the function $k = k_\varphi(r)$ is nondecreasing, therefore one may assume that it is everywhere finite (otherwise the lemma is trivial).

Let d be the metric in M^n generated by the Riemannian structure of the manifold. Take some monotone unbounded sequence $0 < r_0 < r_1 < \cdots < r_m < \cdots$ of values of radius r. Consider the sequence of M^n-rings $R_m = R_{r_m}^{r_{m+1}} = \{x \in M^n \mid r_m < d(o,x) < r_{m+1}\}$, and the corresponding sequence of D-rings $D_m := D \cap R_m$.

First suppose that all curves of the family Γ intersect the sphere of radius r_0. Then on each curve $\gamma \in \Gamma$ one can choose the following sequence of simple arcs γ_m: each arc is situated in the corresponding domain D_m and connects boundary spheres of the ring R_m. In this way we obtain a sequence $\Gamma_m := \bigcup_{\gamma \in \Gamma} \gamma_m$ of families of curves.

The curves of the family Γ_m form a part of all possible curves in the ring R_m connecting its boundary spheres. Therefore $\operatorname{mod} \Gamma_m$ is not greater than the conformal module $\operatorname{cap} R_{r_m}^{r_{m+1}}$ of this general family. Thus, by the estimate of conformal capacity of a geodesic ring mentioned in Section 1, we have

$$\operatorname{mod} \Gamma_m \leqslant \left(\int_{r_m}^{r_{m+1}} S^{1/(1-n)} \right)^{1-n}.$$

The families Γ_m are situated in disjoint domains D_m and each of them minorises the initial family Γ (i.e., for each curve $\gamma \in \Gamma$ there exists a curve $\gamma_m \in \Gamma_m$ such that $\gamma_m \subset \gamma$). The same holds for their images $\tilde{\Gamma}_m$ and $\tilde{\Gamma}$.

Thus, by virtue of classical principles of Grötzsch

$$\operatorname{mod}^{1/(1-n)} \tilde{\Gamma} \geqslant \sum_{m=0}^{\infty} \operatorname{mod}^{1/(1-n)} \tilde{\Gamma}_m.$$

The mapping under consideration is $(k(r_{m+1}) := k_\varphi(r_{m+1}))$-quasiconformal within the domain D_m. Therefore

$$\operatorname{mod} \tilde{\Gamma}_m \leqslant k^{n-1}(r_{m+1}) \operatorname{mod} \Gamma_m.$$

From the last three inequalities we obtain

$$\operatorname{mod}^{1/(1-n)} \tilde{\Gamma} \geqslant \sum_{m=0}^{\infty} k^{-1}(r_{m+1}) \left(\int_{r_m}^{r_{m+1}} S^{1/(1-n)} \right).$$

Because of the condition $\operatorname{mod} \tilde{\Gamma} > 0$, the last series must converge, and the convergence does not depend on the choice of the sequence $r_1 < \cdots < r_m < \cdots$ of values of radius r. This implies the convergence of the integral indicated in the lemma. In fact, this becomes clear, e.g., after the following estimate concerning any interval $[a, b]$ on r (with keeping in mind that we are free to choose any partition of the interval):

$$
\begin{aligned}
0 &\leqslant \sum \left(k^{-1}(r_m) - k^{-1}(r_{m+1})\right) \left(\int_{r_m}^{r_{m+1}} S^{1/(1-n)}\right) \\
&\leqslant \left(k^{-1}(r_{m_a}) - k^{-1}(r_{m_b})\right) \max_{m_a \leqslant m \leqslant m_b} \left(\int_{r_m}^{r_{m+1}} S^{1/(1-n)}\right).
\end{aligned}
$$

To complete the proof we should eliminate our additional assumption that all curves of the family Γ intersect a sphere of fixed radius r_0.

The family Γ can be presented as a countable union of the following families L_m. Curves of the family L_m do not intersect the sphere of radius m but intersect the sphere of radius $m+1$. Since the family Γ is not exceptional, one of the families L_m must be nonexceptional too. It can be admit for Γ. $\square$

Now we can prove the theorem. Notice that the construction used in [7] for the investigation of quasiconformal immersion $f : M^n \to N^n$, besides the condition $n > 2$, used only two things: conformal module of any family Γ of curves on M^n going to infinity within the domain $D \subset M^n$ where the mapping f (or, to say more correctly, the restriction $f|_D = \varphi$) is injective, is equal to zero; and the image of such a family also has the conformal module equal to zero.

In our case the condition $\operatorname{mod} \Gamma = 0$ is ensured by the conformal parabolicity of the manifold M^n, and the condition $\operatorname{mod} \tilde{\Gamma} = 0$ holds by virtue of the Lemma if the integral indicated in the theorem diverges.

4. Comments

Already Hadamard [11] observed that a locally invertible mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ is globally invertible if and only if $f(x) \to \infty$ when $x \to \infty$. Obviously, this is equivalent to the condition that the mapping f is a proper one (i.e., the pre-image of a compact set is a compact set); thus f is a covering of a simply connected manifold, and is a homeomorphism.

Certainly, this reasoning and the condition of Hadamard remain valid for immersion $f : M^n \to N^n$ of any manifold to a simply connected one of the same dimension.

Proofs of the global invertibility property for mappings of special classes, e.g., quasiconformal in the Lavrentiev's problem or polynomial in the (still unsolved) Jacobian problem, reduce in a sense (although, certainly, not literally) to checking the condition of Hadamard.

In the form convenient for application to quasiconformal mappings the condition of Hadamard can be reformulated as follows.

Let Γ_∞ be the set of all paths on M^n ending at infinity, and let $\operatorname{mod} f(\Gamma_\infty)$ be the conformal module of its image under an immersion $f : M^n \to N^n$. If M^n is a manifold of conformally parabolic type, then, certainly, $\operatorname{mod} \Gamma_\infty = 0$. In this setting the condition of Hadamard is equivalent to the relation $\operatorname{mod} f(\Gamma_\infty) = 0$ provided $n > 2$.

If the mapping f is quasiconformal, then the image of any exceptional family of curves (e.g., the family Γ_∞) is exceptional. The condition mentioned in the above formulated theorem related the admissible rate of growth of the function $k(r)$ do guaranties the necessary equality $\operatorname{mod} f(\Gamma_\infty) = 0$ if $\operatorname{mod} \Gamma_\infty = 0$.

The sharpness of the asymptotics indicated in the theorem can be confirmed by the example constructed in the paper [4].

Notice that the restrictions imposed here on the mapping can be weaken. One can consider mappings quasicinformal in a more general sense of integral mean. In this connection see [12] and the most recent paper [13].

Notice finally that we considered here immersions of manifolds of conformally parabolic type. However there are serious reasons to expect the global invertibility property for quasiconformal immersions in a much more general settings. As a model example in this context one can recall the problem formulated in [3] relating the removability of a segment of singular points.

References

[1] M. A. Lavrentieff, *Sur un critère differentiel des transformations homéomorphes des domains à trois dimensions*, Dokl. Acad. Nauk SSSR, **20** (1938), 241–242.

[2] V. A. Zorich, *M. A. Lavrent'ev's theorem on quasiconformal space maps*, Math. USSR, Sb. **3** (1967), 389–403; translation from Mat. Sb., (N.S.) **74**(116) (1967), 417–433.

[3] V. A. Zorich, *The global homeomorphism theorem for space quasiconformal mappings, its development and related open problems*, Lecture Notes in Mathematics 1508, Springer-Verlag, 1992, pp. 132–148.

[4] V. A. Zorich, *Admissible rate of growth of the characteristic of quasiconformality in M. A. Lavrent'ev's theorem*, Sov. Math., Dokl. **8** (1968), 1492–1494; translation from Dokl. Akad. Nauk SSSR **181** (1968), 530–533.

[5] M. Gromov, *Hyperbolic manifolds, groups and actions*, in: *Proceedings of the 1978 Stony Brook Conference*, Ann. of Math. Studies **97** (1981), 183–213, Princeton Univ. Press; Princeton, N.J.

[6] M. Gromov with appendices by M. Katz, P. Pansu, and S. Semmes, *Metric Structures for Riemannian and non-Riemannian Spaces*, Birkhäuser, Boston, Basel, Berlin, 1999.

[7] V. A. Zorich, *Quasiconformal immersions of Riemannian manifolds and a Picard type theorem*, Funct. Anal. Appl. **34**:3 (2000), 188–196; translation from Funktsional. Anal. Prilozhen. **34**:3 (2000), 37–48.

[8] V. A. Zorich and V. M. Kesel'man, *On the conformal type of a Riemannian manifold*, Funct. Anal. Appl. **30**:2 (1996), 106–117; translation from Funktsional. Anal. Prilozhen. **30**:2 (1996), 40–55.

[9] V. G. Maz'ya, *Sobolev Spaces*, Springer-Verlag, 1985.

[10] A. Grigor'yan, *Isoperimetric inequality and capacities on Riemannian manifilds*, Operator Theory: Advances and Application, **109**, 139–153, Birkhäuser, Basel, 1999.

[11] J. Hadamard, *Sur les transformations ponctuelles*, Bull. Soc. Math. France. **34** (1906), 71–84.

[12] M. Perović, *On the global homeomorphism of the mean quasiconformal mappings*, Dokl. Acad. Nauk SSSR, **230** (1976), 781–784; English transl. in Soviet Math. Dokl. (1976).

[13] I. Holopainen and P. Pankka, *Mappings of finite distortion: global homeomorphism theorem*, Reports of the Department of Mathematics, University of Helsinki, Preprint 339, 2002.

Department of Mathematics (Mech-Math)
Moscow State University
119992 Moscow
Russia
vzor@online.ru