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# ON SOLUTIONS OF THE BELTRAMI EQUATION. II

**Melkana A. Brakalova and James A. Jenkins**

ABSTRACT. We study the existence of solutions of the generalized Beltrami equation $f_{\bar{z}} = \mu(z) f_z$, $\|\mu(z)\|_\infty = 1$, in a plane domain $\Delta$, under general conditions that include previously known results.

## 1. Introduction

Let $\mu(z)$ be a measurable complex valued function. In our previous paper [2] we treated the question of existence and uniqueness of solutions for the Beltrami equation

$$f_{\bar{z}}(z) = \mu(z) f_z(z), \tag{1}$$

assuming that $|\mu(z)|$ satisfies a subexponential integrability condition. In the present paper we treat the existence problem under general conditions which include previous results.

## 2. Main results

Let $h(x)$ be a convex, increasing function defined on $[1,\infty)$ such that $h(x) \geqslant C_\lambda x^\lambda$ for any $\lambda > 1$ with $C_\lambda > 0$. From now on we will assume also that

$$\int_1^\infty \frac{1}{t h^{-1}(t)} dt = \infty. \tag{2}$$

MAIN THEOREM. *Let $\Delta$ be a plane domain, $\mu(z)$ a measurable function defined a.e. in $\Delta$, with $\|\mu\|_\infty \leqslant 1$. Suppose that for every bounded measurable set $B \subset \Delta$ there exists a positive constant $\Phi_B$ such that*

$$\iint_B h\left(\frac{1}{1-|\mu|}\right) dA < \Phi_B. \tag{3}$$

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*Then there exists an ACL homeomorphism $f(z)$ of $\Delta$ into the plane, which satisfies the Beltrami equation a.e., with partials $f_z$ and $f_{\bar{z}}$, locally in $L^q$, for $0 < q < 2$. The partials are also distributional derivatives. The inverse $g(w) = f^{-1}(w)$ is ACL in $f(\Delta)$ , and has partials $g_w$ and $g_{\bar{w}}$ locally in $L^2$.*

Theorem A. (the case of the plane) *If $\Delta$ is the plane and if, in addition to* (3), *$\mu(z)$ satisfies*

$$\iint_{\{|z|<R\}} \frac{1}{1-|\mu|} dA = O(R^2), \quad R \to \infty.$$

*then there exists an ACL homeomorphism $f$ which maps the plane onto itself with all the properties listed in the Main Theorem.*

## 3. Auxiliary Results and an Equivalent Statement

Let $h(x)$ be the function defined in Section 2. Denote by $\theta(x) = \ln(h(x))$ for $x$ greater than some constant $c \geqslant 1$, such that $h(c) > e$. $\theta(x)$ is a positive increasing function in $[\ln h(c), \infty)$. Next we show that the following conditions

$$(4) \qquad \int_{c_1}^{\infty} \frac{dx}{xh^{-1}(x)} = \infty, \qquad\qquad (5) \qquad \int_{c_2}^{\infty} \frac{\theta(x)}{x^2} dx = \infty.$$

hold simultaneously, where $c_1$ and $c_2$ are suitable constants. The result can be stated as:

Lemma 1. *Conditions* (4) *and* (5) *are equivalent.*

*Proof.* Make a change of variables in $\int\limits_{c_1}^{\infty} \dfrac{dx}{xh^{-1}(x)}$ by using the substitution $y = \ln(x)$. Then the last integral becomes $\int\limits_{c^*}^{\infty} \dfrac{dx}{\theta^{-1}(x)}$, where $c^* = \ln c_1$. Since $\dfrac{1}{\theta^{-1}(x)}$ and $\theta\left(\dfrac{1}{x}\right)$ are inverses of each other, it follows that $\int\limits_{c^*}^{\infty} \dfrac{dx}{\theta^{-1}(x)}$ is divergent iff $\int\limits_{0}^{c_*} \theta\left(\dfrac{1}{x}\right) dx$ is for some suitable constant $c_* < 1$. After another substitution $y = \dfrac{1}{x}$ we obtain that the divergence of the last integral is equivalent to the divergence of $\int\limits_{c_3}^{\infty} \dfrac{\theta(x)}{x^2} dx$, where $c_3 = \dfrac{1}{c_*}$.

From the auxiliary results above follows a statement equivalent to the Main Theorem.

Theorem B. *Let $\Delta$ be a plane domain, $\mu(z)$ a measurable function defined a.e. in $\Delta$, with $\|\mu\|_\infty \leqslant 1$. Suppose that for every bounded measurable set $B \subset \Delta$ there exists a positive constant $\Phi_B$ such that*

$$\iint_B \exp\left(\theta\left(\frac{1}{1-|\mu|}\right)\right) dA < \Phi_B.$$

*If*

$$\int_1^\infty \frac{\theta(x)}{x^2}dt = \infty,$$

*there exists an ACL homeomorphism $f(z)$ of $\Delta$ into the plane, which satisfies the Beltrami equation a.e., with partials $f_z$ and $f_{\bar{z}}$, locally in $L^q$, for $0 < q < 2$. The partials are also distributional derivatives . The inverse $g(w) = f^{-1}(w)$ is ACL in $f(\Delta)$ , and has partials $g_w$ and $g_{\bar{w}}$ locally in $L^2$.*

## 4. Construction of the solution $f(z)$

Here we assume that $\mu(z)$ satisfies condition (3), with $h(z)$ satisfying (2). In $\Delta$ we define $\mu_n$, $n = 1, 2, \ldots$, so that

$$\mu_n(z) = \begin{cases} \mu(z), & \text{if } |\mu(z)| \leqslant 1 - 1/n \\ 0, & \text{if } |\mu(z)| > 1 - 1/n. \end{cases}$$

From the theory of quasiconformal mappings we know that there exist q.c. mappings $f_n$, $n = 1, 2, \ldots$, of $\Delta$ into the plane with complex dilatations $\mu_n$, $n = 1, 2, \ldots$.

Let $z_0$ be a fixed point in the plane. For $r_2 > r_1 > 0$ denote by $A$ the circular ring $A = \{z : r_1 < |z - z_0| < r_2\}$, and by $M_n(r_1, r_2)$ the module of its image under $f_n$.

Proposition 1. *For any point $z_0$ and circular ring $A = \{r_1 < |z - z_0| < r_2\}$, the module $M_n(r_1, r_2)$ of the image of $A$ under $f_n$ tends uniformly to $\infty$ as $r_1 \to 0$.*

*Proof.* The module $M_n(r_1, r_2)$ can be estimated from below in terms of the complex dilatation $\mu_n$, where $\mu_n = \mu_n(z) = \mu_n(z_0 + re^{i\theta})$, as follows (see [4]):

$$M_n(r_1, r_2) \geqslant \int_{r_1}^{r_2} \frac{1}{\int_0^{2\pi} \frac{|1 - e^{-2i\theta}\mu_n|^2}{1 - |\mu_n|^2} d\theta} \frac{dr}{r}.$$

Using this we obtain:

$$M_n(r_1, r_2) \geqslant \frac{1}{4} \int_{r_1}^{r_2} \frac{1}{\int_0^{2\pi} \frac{1}{1 - |\mu|} d\theta} \frac{dr}{r}.$$

For any $z_0$ in a compact subset $T$ of the plane containing the disc $|z - z_0| < r_2$

$$\int_{r_1}^{r_2} r^2 \int_0^{2\pi} h\left(\frac{1}{1 - |\mu|}\right) d\theta \frac{dr}{r} \leqslant C,$$

where $C$ depends only on the compact subset $T$ and the choice of $r_2$.

Now we have

$$r^2 \int_0^{2\pi} h\left(\frac{1}{1-|\mu|}\right) d\theta < \frac{2C}{\log \dfrac{r_2}{r_1}}$$

on a set $E$ of logarithmic measure $\dfrac{1}{2} \log \dfrac{r_2}{r_1}$. Thus

$$\frac{1}{2\pi} \int_0^{2\pi} h\left(\frac{1}{1-|\mu|}\right) d\theta < \frac{C}{\pi r^2 \log \dfrac{r_2}{r_1}} \text{ on } E.$$

Using the convexity of $h(x)$, we have

$$h\left(\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{1-|\mu|} d\theta\right) < \frac{C}{\pi r^2 \log \dfrac{r_2}{r_1}} \quad \text{on } E$$

and

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{1}{1-|\mu|} d\theta < h^{-1}\left(\frac{C}{\pi r^2 \log \dfrac{r_2}{r_1}}\right) \quad \text{on } E.$$

From the estimates of the module and monotonicity properties of $h(x)$ we have

$$M_n(r_1, r_2) \geqslant \frac{1}{8\pi} \int_{r_1}^{r_2} \frac{1}{h^{-1}\left(\dfrac{C}{\pi r^2 \log \frac{r_2}{r_1}}\right)} \frac{dr}{r} \geqslant \frac{1}{8\pi} \int_{r_1}^{\sqrt{r_1 r_2}} \frac{1}{h^{-1}\left(\dfrac{C}{\pi r^2 \log \frac{r_2}{r_1}}\right)} \frac{dr}{r}.$$

Now we consider a monotonically decreasing sequence $\{s_k\}_{k=1}^{\infty}$ of positive numbers tending to 0 such that each interval $[s_{k+1}, s_k]$ has the same logarithmic length, where $\dfrac{s_k}{s_{k+1}} = c$. By a ring decomposition we mean a family of rings $r_1^{(j)} < |z - z_0| < r_2^{(j)}$ with $r_2^{(j+1)} \leqslant r_1^{(j)}$ and $r_1^{(j)}$ and $r_2^{(j)} \to 0$ as $j \to \infty$.

We take two ring decompositions with

$$r_1^{(j)} = s_{2j+1}, r_2^{(j)} = s_{2j-1}$$

$$\hat{r}_1^{(j)} = s_{2j+2}, \hat{r}_2^{(j)} = s_{2j}.$$

Now

$$\sum_{j=1}^{\infty} M_n\left(r_1^{(j)}, r_2^{(j)}\right) \geqslant \frac{1}{8\pi} \sum_{j=1}^{\infty} \int_{s_{2j+1}}^{s_{2j}} \frac{1}{h^{-1}\left(\dfrac{C}{\pi r^2 \log c}\right)} \frac{dr}{r},$$

while

$$\sum_{j=1}^{\infty} M_n\left(\hat{r}_1^{(j)}, \hat{r}_2^{(j)}\right) \geqslant \frac{1}{8\pi} \sum_{j=1}^{\infty} \int_{s_{2j+2}}^{s_{2j+1}} \frac{1}{h^{-1}\left(\dfrac{C}{\pi r^2 \log c}\right)} \frac{dr}{r},$$

so

$$\sum_{j=1}^{\infty} M_n\left(r_1^{(j)}, r_2^{(j)}\right) + \sum_{j=1}^{\infty} M_n\left(\hat{r}_1^{(j)}, \hat{r}_2^{(j)}\right) \geqslant \frac{1}{8\pi}\int_0^{s_1} \frac{1}{h^{-1}\left(\dfrac{C}{\pi r^2 \log c}\right)}\frac{dr}{r}.$$

Making the change of variables $t = \dfrac{C}{\pi r^2 \log c}$ this last term becomes equal to $\dfrac{1}{8\pi}\int_{\star}^{\infty} \dfrac{1}{th^{-1}(t)}dt$, with a well defined lower limit. Thus at least one of the ring decompositions has module sum bounded below by $\dfrac{1}{16\pi}\int_{\star}^{\infty} \dfrac{1}{th^{-1}(t)}dt$ and therefore approaches $\infty$ uniformly with respect to $n$ and $z_0$. From the superadditivity property of the module it follows that $\lim\limits_{r_1 \to 0} M_n(r_1, r_2) = \infty$, uniformly with respect to $z_0$ and $n$.

From now on we shalll assume that the quasiconformal mappings $\{f_n(z)\}$, $n = 1, 2\ldots$ have two fixed points $a_1$ and $a_2$, with $d = |a_2 - a_1|$. The following proposition was proved in [2]:

PROPOSITION 2. *If* $\lim\limits_{r_1 \to 0} M_n(r_1, r_2) = \infty$, *uniformly with respect to* $z_0$ *and* $n$, *then the family of quasiconformal mappings* $\{f_n(z)\}$, $n = 1, 2, \ldots,$ *is uniformly equicontinuous on each compact subset* $T$ *of* $\Delta$.

Thus from this proposition and the Arzela-Ascoli's theorem follows:

PROPOSITION 3. *For the sequence* $\{f_n(z)\}$ *there exists a subsequence of functions, which converges uniformly to a function* $f(z)$ *on compact subsets.*

We follow the statements in [2] to show the properties of $f(z)$ outlined in the Main Theorem.

## 5. $f(z)$ is a homeomorphism

In the same manner as in [2], one can prove that:

PROPOSITION 4. *The function* $f(z)$ *constructed in Proposition 3 is a homeomorphism of* $\Delta$ *into the plane.*

## 6. Differentiability properties of $f(z)$

In the same manner as in [2], one can prove that:

PROPOSITION 5. *The function f(z) is ACL.*

PROPOSITION 6. *The partials* $f_z$ *and* $f_{\bar{z}}$ *of* $f(z)$ *are in* $L^q$ *on compact subsets of* $\Delta$ *for every* $q < 2$.

Thus $f(z)$ has generalized $L^q$-derivatives according to the terminology introduced in [3].

## 7. $f(z)$ satisfies the Beltrami equation

Using the same methods as in [2], one can prove that:

Proposition 7. *The function $f(z)$ satisfies the Beltrami equation.*

## 8. The inverse function $g(w)$ of $f(z)$

In the same manner as in [2], one can prove that:

Proposition 8. *The function $g$ is ACL and $g_w$ and $g_{\bar{w}}$ are locally in $L^2$.*

So far we have proved the Main Theorem and Theorem B.

## 9. The case of mapping the plane onto itself

In the same manner as in [2], one can prove that

Proposition 9. *If*

$$\iint\limits_{|z|<R} \frac{1}{1-|\mu|} dA = O(R^2) \qquad \text{as } R \to \infty,$$

*then $f_n(z)$ converges uniformly to $\infty$, as $z \to \infty$.*

This proposition and the rest of the results imply Theorem A. This concludes the proofs of the Main Theorem, Theorem A and Theorem B.

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Institute for Mathematical Sciences, SUNY, Stony Brook, NY 11794-3660, USA
melkana@math.sunysb.edu, or

Mathematics Department Fordham University Rose Hill Campus 441 East Fordham Rd, Bronx, NY 10458
brakalova@fordham.edu

Department of Mathematics, Washington University, Campus Box 1146, St. Louis, MO 63130-4899, USA (James A. Jenkins)