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RATE OF CONVERGENCE OF THE SZASZ–KANTOROVITCH–BEZIER OPERATORS FOR BOUNDED VARIATION FUNCTIONS

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ABSTRACT. We introduce the Szasz-Kantorovitch-Bezier operators $\hat{S}_{n,\alpha}$ which is the modified form of Szasz-Kantorovitch operators and study the rate of convergence of bounded variation functions for these operators.

1. Introduction

For a function defined on the infinite interval $[0,\infty)$, the Szasz–Mirakyan operators S_n applied to f are

$$S_n(f,x)=\sum_{k=0}^{\infty}p_{n,k}(x)f(k/n),\quad p_{n,k}(x)=e^{-nx}\frac{(nx)^k}{k!}$$

and the Kantorovitch variant is defined by

$$\hat{S}_n(f,x)=n\sum_{k=0}^{\infty}p_{n,k(x)}\int_{I_k}f(t)\,dt,\quad I_k=[k/n,(k+1)/n] \tag{1}$$

Some approximation properties for Szasz–Kantorovitch operators defined by (1) are studied by Totik [5], Aniol [1] and Razi and Umar [3] etc. We now introduce the Bezier variant of the operators (1) as follows:

$$\hat{S}_{n,\alpha}(f,x)=n\sum_{k=0}^{\infty}Q_{n,k}^{(\alpha)}(x)\int_{I_k}f(t)\,dt \tag{2}$$

where $Q_{n,k}^{(\alpha)}(x)=J_{n,k}^{\alpha}(x)-J_{n,k+1}^{\alpha}(x)$, $\alpha\geqslant 1$ and $J_{n,k}(x)=\sum_{j=k}^{\infty}p_{n,j}(x)$ are the Szasz–Bezier basis function. It is obvious that $S_{n,\alpha}(f,x)$ are linear positive operators and $\hat{S}_{n,\alpha}(1,x)=1$. If $\alpha=1$, $\hat{S}_{n,\alpha}(f,x)$ reduces to the operator $\hat{S}_n(f,x)$,

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defined by (1). Some basic properties of the basis function $J_{n,k}(x)$, which are useful for our study, are as follows:

(i) $J_{n,k}(x) - J_{n,k+1}(x) = p_{n,k}(x)$, $k = 0, 1, 2, 3, \ldots$
(ii) $J'_{n,k}(x) = np_{n,k-1}(x)$, $k = 1, 2, 3, \ldots$
(iii) $J_{n,k}(x) = n\int_0^x p_{n,k-1}(u)\,du$, $k = 1, 2, 3, \ldots$
(iv) $\sum_{k=1}^{\infty} J_{n,k}(x) = n\int_0^x \sum_{k=1}^{\infty} p_{n,k}(u)\,du = nx$
(v) $J_{n,0}(x) > J_{n,1}(x) > J_{n,2}(x) > \cdots > J_{n,k}(x) > J_{n,k+1}(x) > \cdots$

$J_{n,k}(x)$ increases strictly on $[0,\infty)$ and $0 \leqslant J_{n,k}(x) < 1$, $k \in N$.

Rates of convergence for functions of bounded variation by different operators were studied in [1], [2], [4] and [6] etc. In [8] Zeng has introduced Szasz–Bezier operators and estimated the rate of convergence for functions of bounded variation. In the present paper we estimate the rate of convergence by the generalized Szasz–Kantorovitch operators for functions of bounded variation. It is also observed here that the second central moment of Szasz–Kantorovitch operators was wrongly estimated in [3], which leads to a major mistake in the main results of [3].

Our main theorem can be stated as follows:

THEOREM 1. *Let f be a function of bounded variation on every finite subinterval of $[0,\infty)$. If $\alpha \geqslant 1$, $x \in (0,\infty)$, $r \in N$ and $\lambda > 1$ are given, then for $f(t) = O(t^r)$, $t \to \infty$, there exists a constant $M(f,\alpha,r,x)$, such that for n sufficiently large*

$$
\left|\hat{S}_{n,\alpha}(f,x) - \frac{1}{2^\alpha} f(x+) - \left(1 - \frac{1}{2^\alpha}\right) f(x-)\right| \tag{3}
$$
$$
\leqslant \frac{\alpha|f(x+) - f(x-)|}{\sqrt{nx}}\left[H(j) + \sqrt{1+3x}\right] + \frac{2\alpha\lambda + x}{nx}\sum_{k=1}^{n} \bigvee_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x) + \frac{M(f,\alpha,r,x)}{n^r},
$$

where

$$
g_x(t) = \begin{cases} f(t) - f(x-), & 0 \leqslant t < x \\ 0, & t = x \\ f(t) - f(x+), & x < t < \infty \end{cases}
$$

and $\bigvee_a^b(g_x)$ is the total variation of g_x on $[a,b]$.

2. Auxiliary results

We need the following results for proving our main theorem.

It is well known that the basis function $p_{n,k}(x)$ corresponds to the Poisson distribution in probability theory. Using Berry Esseen theorem Gupta and Pant [2] recently obtained the inequality:

$$
p_{n,k}(x) \leqslant \frac{32x^2 + 24x + 5}{2\sqrt{nx}}, \quad x \in (0,\infty).
$$

Very recently Zeng and Zhao [7] obtained the exact bound as follows:

LEMMA 1. *Let* $H(j) = \dfrac{(j+1/2)^{j+1/2}}{j!}e^{-(j+1/2)}$. *Then for* $k \geqslant j$ *and* $x \in (0,\infty)$, *we have*

$$\sqrt{x}p_{n,k}(x) \leqslant H(j)\frac{1}{\sqrt{n}},$$

where the coefficient $H(j) = \dfrac{(j+1/2)^{j+1/2}}{j!}e^{-(j+1/2)}$ *and the estimate order* $n^{-1/2}$ *are best possible.*

Using Lemma 1, we have

$$Q_{n,k}^{(\alpha)}(x) \leqslant \alpha p_{n,k}(x) < \frac{\alpha H(j)}{\sqrt{nx}}. \tag{4}$$

LEMMA 2. *For each fixed* $x \in (0,\infty)$, *we have* $\hat{S}_n((t-x)^2, x) = \dfrac{1+3nx}{3n^2}$.

PROOF. Using the fact that $\sum_{k=0}^{\infty} p_{n,k}(x) = 1$, it can be easily verified by simple computation that

$$\hat{S}_n(1,x) = 1, \quad \hat{S}_n(t,x) = \frac{1+2nx}{2n} \quad \text{and} \quad \hat{S}_n(t^2,x) = \frac{1+3n^2x^2+6nx}{3n^2}.$$

By linearity property of the operators $\hat{S}_n$, the required result follows. For sufficiently large n, there exists a $\lambda > 1$ such that

$$\hat{S}_n((t-x)^2, x) = \lambda x/n. \tag{5}$$

Further for each $x \in [0,\infty)$, $\hat{S}_n((t-x)^m, x) = O(n^{-[(m+1)/2]})$, $n \to \infty$. □

REMARK 1. We may note here that the Lemma 3.1 of [3] is not correct. In [3] the authors get

$$\hat{S}_n((t-x)^2, x) = A/n, \tag{6}$$

where A is a positive constant independent of n and $x \in [0,\infty)$. Hence due to this major mistake the main results of [3] are not estimated correctly.

Throughout the paper let

$$K_{n,\alpha}(x,t) = n\sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x)\chi_{n,k}(t), \tag{7}$$

where $\chi_{n,k}$ is the characteristic function of the interval $[k/n, (k+1)/n]$ with respect to $I \equiv [0,\infty)$. Thus with this definition it is obvious that

$$\hat{S}_{n,\alpha}(f,x) = \int_0^{\infty} f(t)K_{n,\alpha}(x,t)\,dt$$

LEMMA 3. *Let* $x \in (0,\infty)$, *then for sufficiently large* n, *we have*

$$\beta_{n,\alpha}(x,y) = \int_0^y K_{n,\alpha}(x,t)dt \leqslant \frac{\alpha\lambda x}{n(x-y)^2}, \quad 0 \leqslant y < x \tag{8}$$

and

$$1 - \beta_{n,\alpha}(x,z) = \int_z^{\infty} K_{n,\alpha}(x,t)\,dt \leqslant \frac{\alpha\lambda x}{n(z-x)^2}, \quad x < z < \infty \tag{9}$$

PROOF. We first prove (8). We have

$$\int_0^y K_{n,\alpha}(x,t)\,dt \leqslant \int_0^y K_{n,\alpha}(x,t)\frac{(x-t)^2}{(x-y)^2}dt \leqslant (x-y)^{-2}\hat{S}_{n,\alpha}((t-x)^2,x)$$
$$\leqslant \alpha(x-y)^{-2}\hat{S}_n((t-x)^2,x) = \frac{\alpha\lambda x}{n(x-y)^2}, \quad 0 \leqslant y < x$$

where we have applied (5). The proof of (9) is similar. □

LEMMA 4. [8] *For $x \in (0,\infty)$, we have*

$$\left|\sum_{k>nx} p_{n,k}(x) - \frac{1}{2}\right| \leqslant \frac{0.82\sqrt{1+3x}}{\sqrt{nx}} < \frac{\sqrt{1+3x}}{\sqrt{nx}}.$$

3. Proof of the main theorem

PROOF. Making use of identity for all n, we have

$$f(t) = \frac{1}{2^\alpha}f(x+) + \left(1-\frac{1}{2^\alpha}\right)f(x-) + g_x(t) + \frac{f(x+)-f(x-)}{2^\alpha}\operatorname{sign}_x(t)$$
$$+\,\delta_x(t)\Big[f(x) - \frac{1}{2^\alpha}f(x+) - \Big(1-\frac{1}{2^\alpha}\Big)f(x-)\Big]$$

where

$$\begin{cases} \operatorname{sign}_x(t) = 2^\alpha - 1, & t > x; \\ \operatorname{sign}_x(t) = 0, & t = x; \\ \operatorname{sign}_x(t) = -1, & t < x \end{cases} \qquad \text{and} \qquad \delta_x(t) = \begin{cases} 1, & x = t \\ 0, & x \neq t \end{cases}.$$

It follows that

$$(10) \quad \left|\hat{S}_n(f,x) - \frac{1}{2^\alpha}f(x+) - \Big(1-\frac{1}{2^\alpha}\Big)f(x-)\right| \leqslant |\hat{S}_{n,\alpha}(g_x,x)|$$
$$+\left|\frac{f(x+)-f(x-)}{2^\alpha}\hat{S}_{n,\alpha}(\operatorname{sign}(t-x),x) + \Big[f(x) - \frac{1}{2^\alpha} - \Big(1-\frac{1}{2^\alpha}\Big)f(x-)\Big]\hat{S}_{n,\alpha}(\delta_x,x)\right|$$

For the operators $\hat{S}_{n,\alpha}$ it is obvious that

$$(11) \qquad \hat{S}_{n,\alpha}(\delta_x,x) = 0$$

First we estimate $\hat{S}_{n,\alpha}(\operatorname{sign}(t-x),x)$. Let us choose k' such that $x \in \left[\frac{k'}{n}, \frac{(k'+1)}{n}\right]$, then

$$\hat{S}_{n,\alpha}(\operatorname{sign}(t-x),x) = \sum_{k=0}^{k'-1}(-1)Q_{n,k}^{(\alpha)}(x) + \left(\frac{Q_{n,k'}^{(\alpha)}}{\int_{I'_k}dt}\right)\int_{k'/n}^x(-1)\,dt$$
$$+\left(\frac{Q_{n,k'}^{(\alpha)}(x)}{\int_{I'_k}dt}\right)\int_x^{(k'+1)/n}(2^\alpha-1)\,dt + \sum_{k=k'+1}^{\infty}(2^\alpha-1)Q_{n,k}^{(\alpha)}(x)$$
$$= \sum_{k=k'+1}^{\infty}2^\alpha Q_{n,k}^{(\alpha)}(x) + \left(\frac{Q_{n,k'}^{(\alpha)}(x)}{\int_{I'_k}dt}\right)\int_{I'_k}^{(k'+1)/n}2^\alpha dt - 1$$

Note that

$$0 \leqslant \left(\frac{Q^{(\alpha)}_{n,k'}(t)}{\int_{I'_k} dt}\right) \int_x^{(k'+1)/n} 2^\alpha dt \leqslant 2^\alpha Q^{(\alpha)}_{n,k'}(x),$$

we conclude

$$\begin{aligned}|\hat{S}_{n,\alpha}(\operatorname{sign}(t-x),x)| &\leqslant \left|\sum_{k=k'+1}^{\infty} 2^\alpha Q^{(\alpha)}_{n,k}(x) - 1\right| + 2^\alpha Q^{(\alpha)}_{n,k'}(x)\\ &= |2^\alpha J^\alpha_{n,k'+1}(x) - 1| + 2^\alpha Q^{(\alpha)}_{n,k'}(x)\end{aligned}$$

Applying the inequality $|a^\alpha - b^\alpha| \leqslant \alpha|a-b|$ for $0 \leqslant a, b \leqslant 1$ and $\alpha \geqslant 1$ yields

$$|2^\alpha J^\alpha_{n,k'+1}(x) - 1| \leqslant \alpha 2^\alpha \left|J_{n.k'+1}(x) - \frac{1}{2}\right| = a2^\alpha \left|\sum_{k>nx} p_{n,k}(x) - \frac{1}{2}\right|.$$

Therefore by (4) and Lemma 4, we get

$$\left|\hat{S}_{n,\alpha}(\operatorname{sign}(t-x),x)\right| \leqslant \alpha 2^\alpha \frac{\sqrt{1+3x}}{\sqrt{nx}} + \alpha 2^\alpha \frac{H(j)}{\sqrt{nx}} = \frac{\alpha 2^\alpha}{\sqrt{nx}}\left[H(j) + \sqrt{1+3x}\right] \tag{12}$$

Next we estimate $\hat{S}_{n,\alpha}(g_x,x)$. By (7), we have

$$\begin{aligned}\hat{S}_{n,\alpha}(g_x,x) &= \int_0^\infty g_x(t) K_{n,\alpha}(x,t)\,dt\\ &= \left(\int_0^{x-x/\sqrt{n}} + \int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} + \int_{x+x/\sqrt{n}}^{\infty}\right) K_{n,\alpha}(x,t) g_x(t) dt \qquad (13)\\ &= E_1 + E_2 + E_3, \quad \text{say.}\end{aligned}$$

We start with E_2. For $t \in [x - x/\sqrt{n}, x + x/\sqrt{n}]$, we have

$$|E_2| \leqslant \bigvee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_x) \leqslant \frac{1}{n}\sum_{k=1}^{n} \bigvee_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x). \tag{14}$$

Next we estimate E_1. Setting $y = x - x/\sqrt{n}$ and integrating by parts, we have

$$E_1 = \int_0^y g_x(t) d_t(\beta_{n,\alpha}(x,t)) = g_x(y)\beta_{n,\alpha}(x,y) - \int_0^y \beta_{n,\alpha}(x,t) d_t(g_x(t))$$

Since $|g_x(y)| \leqslant \mathrm{V}_y^x(g_x)$, we conclude

$$|E_1| \leqslant \bigvee_y^x(g_x)\beta_{n,\alpha}(x,y) + \int_0^y \beta_{n,\alpha}(x,t) d_t\left(-\bigvee_t^x(g_x)\right)$$

Also $y = x - x/\sqrt{n} \leqslant x$, (8) of Lemma 3 implies for n sufficiently large

$$|E_1| \leqslant \frac{\alpha\lambda x}{n(x-y)^2}\bigvee_y^x(g_x) + \frac{\alpha\lambda x}{n}\int_0^y \frac{1}{(x-t)^2} d_t\left(-\bigvee_t^x(g_x)\right)$$

Integrating by parts the last integral, we obtain

$$|E_1| \leqslant \frac{\alpha\lambda x}{n}\left(x^{-2}\bigvee_0^x(g_x) + 2\int_0^y \frac{\mathrm{V}_t^x(g_x) dt}{(x-t)^3}\right)$$

Replacing the variable y in the last integral by $x - x/\sqrt{n}$, we get

$$\int_0^{x-x/\sqrt{n}} \bigvee_t^x (g_x)(x-t)^{-3}dt = \sum_{k=1}^{n-1} \int_{x-x/\sqrt{k}}^{x+x/\sqrt{k}} \bigvee_{x-t}^x (g_x)t^{-3}dt \leqslant \frac{1}{2x^2} \sum_{k=1}^{n} \bigvee_{x-x/\sqrt{k}}^{x} (g_x)$$

Hence

$$|E_1| \leqslant \frac{2\alpha\lambda}{nx} \sum_{k=1}^{n} \bigvee_{x-x/\sqrt{k}}^{x} (g_x) \tag{15}$$

Finally we estimate E_3, we put

$$\hat{g}_x(t) = \begin{cases} g_x(t), & 0 \leqslant t \leqslant 2x \\ g_x(2x), & 2x < t < \infty \end{cases}$$

and divide $E_3 = E_{31} + E_{32}$, where

$$E_{31} = \int_{x+x/\sqrt{n}}^{\infty} K_{n,\alpha}(x,t)\hat{g}_x(t)\,dt, \quad \text{and} \quad E_{32} = \int_{2x}^{\infty} K_{n,\alpha}(x,t)[g_x(t) - g_x(2x)]\,dt$$

with $y = x + x/\sqrt{n}$ the first integral can be written in the form

$$E_{31} = \lim_{R\to+\infty} \Big\{ g_x(y)[1 - \beta_{n,\alpha}(x,y)] + \hat{g}_x(R)[\beta_{n,\alpha}(x,R) - 1] \\ + \int_y^R [1 - \beta_{n,\alpha}(x,t)]\, d_t\hat{g}_x(t) \Big\}$$

By (9) of Lemma 3, we conclude for each $\lambda > 1$ and n sufficiently large

$$|E_{31}| \leqslant \frac{\alpha\lambda x}{n} \lim_{R\to+\infty} \left\{ \frac{\mathrm{V}_x^y(g_x)}{(y-x)^2} + \hat{g}_x(R)(R-x)^2 + \int_y^R \frac{1}{(t-x)^2} d_t\left(\bigvee_x^t (\hat{g}_x)\right) \right\}$$
$$= \frac{\alpha\lambda x}{n} \left\{ \frac{\mathrm{V}_x^y(g_x)}{(y-x)^2} + \int_y^{2x} \frac{1}{(t-x)^2} d_t\left(\bigvee_x^t (g_x)\right) \right\}$$

Using the similar method as above, we get

$$\int_y^{2x} \frac{1}{(t-x)^2} d_t\left(\bigvee_x^t (g_x)\right) \leqslant x^{-2} \bigvee_x^{2x} (g_x) - \frac{\mathrm{V}_x^y(g_x)}{(y-x)^2} + x^{-2} \sum_{k=1}^{n-1} \bigvee_x^{x+x/\sqrt{k}} (g_x)$$

which implies the estimate

$$|E_{31}| \leqslant \frac{2\alpha\lambda}{nx} \sum_{k=1}^{n} \bigvee_x^{x+x/\sqrt{k}} (g_x) \tag{16}$$

Finally we estimate E_{32}. By assumption there exists an integer r such that $f(t) = O(t^{2r}), t \to \infty$. Thus for certain constant $M > 0$ depending only on f, x, r, we have

$$|E_{32}| \leqslant Mn \sum_{k=0}^{\infty} Q_{n,k}^{(\alpha)}(x) \int_{2x}^{\infty} \chi_{n,k}(t)t^{2r}dt$$

By Lemma 2, we have

$$|E_{32}| \leqslant \alpha 2^r M\hat{S}_n((t-x)^{2r}, x) = O(n^{-r}), \quad n \to \infty \tag{17}$$

Finally collecting the estimates of (10)–(17), we get (3). This completes the proof of the theorem. □

REMARK 2. For $\alpha = 1$, Theorem 1 gives the improved estimate over the result of Aniol [1]. In [1, p. 13] the author has used $\beta(x) \leqslant 8x^3 + 6x^2 + x$, which can be improved using $\beta(x) \equiv E|\xi_1 - a_1|^3 \leqslant \sqrt{E(\xi_1 - a_1)^4 E(\xi_1 - a_1)^2} \leqslant x\sqrt{(1+3x)}$.

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