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ON GAPS BETWEEN BOUNDED OPERATORS

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Abstract. We consider the distance between an arbitrary bounded operator on a Banach space X and the null operator. The distance is computed in terms of various gaps. Thus, we generalize the Habibi's result from [4]

1. Introduction

In [4] Habibi considered the spherical gap between a linear operator and the null operator on finite dimensional Hilbert spaces. We extend (see also [1]) his result to various gaps on arbitrary Banach spaces.

Let X, Y be arbitrary Banach spaces. We denote by $\mathcal{G}(X)$, and $\mathcal{B}(X, Y)$, respectively, the set of all closed subspaces of X and the set of all bounded linear operators from X to Y. The closed unit sphere of a Banach space X is denoted by $S(X)$. If x is a vector of a Banach space X and M is a subset of X we put $\operatorname{dist}(x, M) = \inf_{m\in M} \|x - m\|$. If M, N are subspaces of X, the set of all invertible operators C on X such that $C(M) = N$ is denoted by $\mathcal{B}(X; M, N)^{-1}$. Let $X \times Y$ be the space with the norm

$$\|(x, y)\| = (\|x\|^p + \|y\|^p)^{1/p}, \quad x \in X,\ y \in Y,\ p \geqslant 1.$$

Let $M, N \in \mathcal{G}(X)$. The spherical gap between M and N is defined by [2]

$$\widetilde{\Theta}(M, N) = \max\left\{\widetilde{\Theta}_0(M, N), \widetilde{\Theta}_0(N, M)\right\},$$

where

$$\widetilde{\Theta}_0(M, N) = \sup_{m\in S(M)} d(m, S(N)).$$

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The geometric gap between M and N is defined by [6]

$$\Theta(M,N) = \max\left\{\Theta_0(M,N), \Theta_0(N,M)\right\},$$

where

$$\Theta_0(M,N) = \sup_{m\in S(M)} d(m,N).$$

If $A, B \in \mathcal{B}(X,Y)$, then their graphs $G(A), G(B)$ are closed subspaces of $X\times Y$, so the gap between the operators A and B is defined as the gap between their graphs, i.e.,

$$\widetilde{\Theta}(A,B) = \widetilde{\Theta}(G(A),G(B)) \text{ and } \Theta(A,B) = \Theta(G(A),G(B)).$$

Also,

$$\widetilde{\Theta}_0(A,B) = \widetilde{\Theta}_0(G(A),G(B)) \quad \text{and} \quad \Theta_0(A,B) = \Theta_0(G(A),G(B)).$$

Obviously,

$$\widetilde{\Theta}(A,B) = \max\left\{\widetilde{\Theta}_0(A,B), \widetilde{\Theta}_0(B,A)\right\},$$
$$\Theta(A,B) = \max\left\{\Theta_0(A,B), \Theta_0(B,A)\right\}.$$

In [4] Habibi considered the spherical gap between operators on finite dimensional Hilbert spaces, assuming $p = 2$. We shall consider spherical and geometric gap of bounded operators on Banach spaces, where $p \geqslant 1$ is arbitrary. Thus, a generalization of Habibi's result is obtained.

2. Results

First we compute the spherical gap between a bounded operator and the null operator.

THEOREM 2.1. *If X, Y are Banach spaces and $A \in \mathcal{B}(X,Y)$, then*

$$\widetilde{\Theta}(A) = \widetilde{\Theta}(A,0) = \left(\left(1 - \frac{1}{(1+\|A\|^p)^{1/p}}\right)^p + \frac{\|A\|^p}{1+\|A\|^p}\right)^{1/p}.$$

Proof. Since

$$G(A) = \{(x,Ax) : x \in X\} \text{ and } G(0) = \{(x,0) : x\in X\},$$

it follows that

$$\begin{aligned}\widetilde{\Theta}_0(A,0) &= \sup_{(y,Ay)\in S(G(A))} \inf_{(x,0)\in S(G(0))} \|(x,0)-(y,Ay)\| \\ &= \sup_{\substack{y\in X \\ \|y\|^p+\|Ay\|^p=1}} \inf_{\substack{x\in X \\ \|x\|=1}} \left(\|x-y\|^p + \|Ay\|^p\right)^{1/p}.\end{aligned}$$

Consider the function

$$f(x,y) = (\|x-y\|^p + \|Ay\|^p)^{1/p}.$$

Suppose that $x \in X$, $y \in Y$ satisfy

$$\|x\| = 1, \ \|y\|^p + \|Ay\|^p = 1. \tag{1}$$

Then

$$f(x,y) \geqslant ((\|x\| - \|y\|)^p + \|Ay\|^p)^{1/p} = ((1-\|y\|)^p + \|Ay\|^p)^{1/p}.$$

Hence,

$$\inf_{\substack{x\in X\\ \|x\|=1}} f(x,y) \geqslant ((1-\|y\|)^p + \|Ay\|^p)^{1/p}. \tag{2}$$

Also,

$$\inf_{\substack{x\in X\\ \|x\|=1}} f(x,y) \leqslant f\left(\frac{y}{\|y\|}, y\right) = ((1-\|y\|)^p + \|Ay\|^p)^{1/p}. \tag{3}$$

So, from (2) and (3) it follows that

$$\inf_{\substack{x\in X\\ \|x\|=1}} f(x,y) = ((1-\|y\|)^p + \|Ay\|^p)^{1/p}. \tag{4}$$

According to the condition (1) we have

$$\|y\| \geqslant \frac{1}{(1+\|A\|^p)^{1/p}}.$$

Since the function $\phi(t) = (1-t)^p + 1 - t^p$ is decreasing for $0 \leqslant t \leqslant 1$, we have

$$\begin{aligned} \widetilde{\Theta}_0(A,0) &= \sup_{\substack{y\in X\\ \|y\|^p+\|Ay\|^p=1}} ((1-\|y\|)^p + 1 - \|y\|^p)^{1/p} \\ &= \left((1-\frac{1}{(1+\|A\|^p)^{1/p}})^p + \frac{\|A\|^p}{1+\|A\|^p}\right)^{1/p}. \end{aligned} \tag{5}$$

Let $y = cx$, where $c = \frac{1}{(1+\|Ax\|^p)^{1/p}}$. It is obvious that y satisfy the condition (1) and $f(x,y) = \left(\left(1-\frac{1}{(1+\|Ax\|^p)^{1/p}}\right)^p + \frac{\|Ax\|^p}{1+\|Ax\|^p}\right)^{1/p}$.

So,

$$\inf_{\substack{y\in Y\\ \|y\|^p+\|Ay\|^p=1}} f(x,y) \leqslant \left(\left(1-\frac{1}{(1+\|Ax\|^p)^{1/p}}\right)^p + \frac{\|Ax\|^p}{1+\|Ax\|^p}\right)^{1/p}. \tag{6}$$

The function $\psi(t) = (1 - \frac{1}{(1+t)^{1/p}})^p + \frac{t}{1+t}$ is increasing, so from (6) it follows that

$$\begin{aligned}\widetilde{\Theta}_0(0,A) &\leqslant \sup_{\substack{x\in X\\ \|x\|=1}} \left(\left(1 - \frac{1}{(1+\|Ax\|^p)^{1/p}}\right)^p + \frac{\|Ax\|^p}{1+\|Ax\|^p}\right)^{1/p}\\ &= \left(\left(1 - \frac{1}{(1+\|A\|^p)^{1/p}}\right)^p + \frac{\|A\|^p}{1+\|A\|^p}\right)^{1/p}. \qquad (7)\end{aligned}$$

Finally, from (5) and (7) we have that

$$\begin{aligned}\widetilde{\Theta}(A) &= \max\left\{\widetilde{\Theta}_0(A,0), \widetilde{\Theta}_0(0,A)\right\} = \widetilde{\Theta}_0(A,0)\\ &= \left(\left(1 - \frac{1}{(1+\|A\|^p)^{1/p}}\right)^p + \frac{\|A\|^p}{1+\|A\|^p}\right)^{1/p} \qquad \square\end{aligned}$$

From the proof of Theorem 2.1 it follows that

$$\widetilde{\Theta}_0(0,A) \leqslant \widetilde{\Theta}_0(A,0) = \left((1 - \frac{1}{(1+\|A\|^p)^{1/p}})^p + \frac{\|A\|^p}{1+\|A\|^p}\right)^{1/p}.$$

Notice that Habibi's main result in [4] is a special case of our Theorem 2.1 for $p = 2$.

Now, we consider the geometric gap.

THEOREM 2.2. *If X, Y are Banach spaces and $A \in \mathcal{B}(X,Y)$, then*

$$\Theta_0(A,0) = \frac{\|A\|}{(1+\|A\|^p)^{1/p}}.$$

Proof. It follows that

$$\begin{aligned}\Theta_0(A,0) &= \sup_{\substack{y\in X\\ \|y\|^p+\|Ay\|^p=1}} \inf_{x\in X} (\|x-y\|^p + \|Ay\|^p)^{1/p}\\ &= \sup_{\substack{y\in X\\ \|y\|^p+\|Ay\|^p=1}} \|Ay\| = \sup_{z\in X} \frac{\|Az\|}{(\|z\|^p + \|Az\|^p)^{1/p}} = \frac{\|A\|}{(1+\|A\|^p)^{1/p}} \qquad \square\end{aligned}$$

In the case $p = 2$ we can obtain the following Habibi's result.

THEOREM 2.3. *If X and Y are Banach spaces, $A \in \mathcal{B}(X,Y)$ and $p = 2$, then the following holds:*

$$\Theta(A,0) = \frac{\|A\|}{(1+\|A\|^2)^{1/2}}.$$

Proof. By Theorem 2.2 it is sufficient to prove

$$\Theta_0(0, A) \leqslant \frac{\|A\|}{(1 + \|A\|^2)^{1/2}}.$$

Actually, this is a Habibi's result from [3]. For reader's convenience, we give a complete proof. Consider the function $f(x, y) = (\|x - y\|^2 + \|Ay\|^2)^{1/2}$. Let $x \in S(X)$ and $y = \frac{x}{1+\|Ax\|^2}$. Then we have

$$\mathrm{dist}((x, 0), G(A)) \leqslant f(x, y) = \frac{\|Ax\|}{(1 + \|Ax\|^2)^{1/2}}.$$

Since the function $\mu(t) = \frac{t}{1+t}$ is increasing, it follows that

$$\begin{aligned}\Theta_0(0, A) &= \sup_{\|x\|=1} \mathrm{dist}((x, 0), G(A)) \leqslant \sup_{\|x\|=1} \frac{\|Ax\|}{(1 + \|Ax\|^2)^{1/2}} \\ &= \frac{\|A\|}{(1 + \|A\|^2)^{1/2}}.\end{aligned} \qquad \square$$

Finally, we can prove the following result.

THEOREM 2.4. *Let X, Y be Banach spaces and $X \times Y$ is the space with the norm*

$$\|(x, y)\| = (\|x\|^2 + \|y\|^2)^{1/2}, \quad x \in X,\ y \in Y.$$

If $A, B \in B(X, Y)$, then

$$\Theta(A, B) \leqslant 2(1 + \min\{\|A\|, \|B\|\}^2)\frac{\|A - B\|}{(1 + \|A - B\|^2)^{1/2}}.$$

Proof. From [5],Theorem 2.17, page 204, it follows that

$$\Theta(A, B) = \Theta((A - B) + B, 0 + B) \leqslant 2(1 + \|B\|^2)\Theta(A - B, 0),$$

and

$$\Theta(A, B) = \Theta(0 + A, (B - A) + A) \leqslant 2(1 + \|A\|^2)\Theta(B - A, 0).$$

Now, by Theorem 2.3 we have that

$$\Theta(A, B) \leqslant 2(1 + \|B\|^2)\frac{\|A - B\|}{(1 + \|A - B\|^2)^{1/2}},$$

and

$$\Theta(A, B) \leqslant 2(1 + \|A\|^2)\frac{\|A - B\|}{(1 + \|A - B\|^2)^{1/2}},$$

i.e.,

$$\Theta(A, B) \leqslant 2(1 + \min\{\|A\|, \|B\|\}^2)\frac{\|A - B\|}{(1 + \|A - B\|^2)^{1/2}}. \quad \square$$

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