PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 72(86) (2002), 29–38

# THE CATEGORY OF COMPACT METRIC SPACES AND ITS FUNCTIONAL ANALYTIC DUALS

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ABSTRACT. A Lipschitz algebra $\mathrm{Lip}(X, d_X)$ over a compact metric space $(X, d_X)$ consists of all complex valued continuous functions on $(X, d_X)$ which are Lipschitz with respect to $d_X$ and the standard metric on the complex plane $\mathbb{C}$ (absolute value). The norm on $\mathrm{Lip}(X, d_X)$ is given by $\|f\| = \sup\{|f(x)| : x \in X\} + \sup\{|f(x) - f(y)|/d_X(x, y) : x, y \in X \;\&\; x \neq y\}$. We show that the category CLip in which objects are Lipschitz algebras and morphisms are algebra homomorphisms is dual to the category CMet in which objects are compact metric spaces and morphisms are Lipschitz maps. Let $(X, d)$ be any metric space, and let $Y = \{(x, y) \in X \times X : x \neq y\}$. De Leeuw derivation defined by the metric $d$ is the operator $D : C_b(X) \to C_b(Y)$ be defined by $(Df)(x, y) = (f(y) - f(x))/d(x, y)$ for $(x, y) \in Y$. We consider the category CDer in which objects are pairs $(C(X), D_X)$, where $(X, d_X)$ is a compact metric space and $D_X$ is the corresponding de Leeuw derivation, and morphisms are all homomorphisms $\nu : C(X) \to C(Y)$ for which $f \in \mathrm{Domain}(D_X)$ implies $\nu f \in \mathrm{Domain}(D_Y)$. We show that CDer is equivalent to CLip, and that CDer is dual to CMet.

## 1. Introduction and definitions

It is well known that the following two categories are dual: (1) the category in which objects are compact Hausdorff spaces and morphisms are continuous maps; (2) the category in which objects are commutative unital $C^*$-algebras and morphisms are homomorphisms. Recent work in noncommutative geometry (**[2, 7]**) has prompted a search for functional analytic representation of metric spaces and

2000 *Mathematics Subject Classification.* Primary 18B99; Secondary 18B30, 46J10, 46L89, 46M15.

*Key words and phrases.* Lipschitz algebras; de Leeuw derivations; dual and equivalent categories.

Work on this paper began while the author was a Post-Doctoral Fellow at the University of Victoria, Canada, with Professors John Phillips and Ian Putnam, where the research was supported by their NSERC grants.

This research was supported in part by the Ministry of Science and Technology of Serbia, through the Institute of Mathematics of the Serbian Academy of Sciences and Arts, project # 04M03.

for a functional analytic characterization of their geometric properties. The goal of this paper is to show that similar to how topological spaces have their functional analytic counterpart in $C^*$-algebras, metric spaces have their functional analytic counterpart in Lipschitz algebras. We also want to show that just as there is a connection between a metric and the underlying topological space (if $X$ is a compact Hausdorff space, metric is a continuous function on $X \times X$ satisfying certain conditions), there likewise exists a connection between a Lipschitz algebra and the underlying $C^*$-algebra $A$ via the de Leeuw derivation on $A$.

Definition 1.1. A map $f : X \to Y$ from a metric space $(X, d_X)$ to a metric space $(Y, d_Y)$ is said to be *Lipschitz* if there exists a constant $M$ such that for all $x, y$ in $X$

$$d_Y(f(x), f(y)) \leqslant M\, d_X(x, y).$$

The smallest such constant is called the *Lipschitz constant* of $f$.

The Lipschitz constant of $f$, $\mathrm{p}(f)$, can be expressed explicitly as

$$\begin{aligned}\mathrm{p}(f) &= \inf\{M > 0 : d_Y(f(x), f(y)) \leqslant M\, d_X(x, y)\ \ \forall x, y \in X\}\\ &= \sup\{d_Y(f(x), f(y))/d_X(x, y) : x, y \in X \ \&\ x \neq y\}.\end{aligned}$$

When $Y$ is a normed space $\mathrm{p}(f)$ is also called the *Lipschitz norm* of $f$ (it is in fact a semi-norm).

If $(X, d_X)$, $(Y, d_Y)$, and $(Z, d_Z)$ are metric spaces, and $f : X \to Y$ and $g : Y \to Z$ are Lipschitz maps, then $g \circ f$ is Lipschitz with $\mathrm{p}(g \circ f) \leqslant \mathrm{p}(f)\,\mathrm{p}(g)$. Since the composition of two Lipschitz maps is Lipschitz, and the identity map on $X$ is the identity morphism in the categorical sense, it follows that compact metric spaces (as objects) and Lipschitz maps (as morphisms) form a category. We denote it by CMet.

Let $(X, d_X)$ be a metric space. We denote by $\mathrm{Lip}(X, d_X)$ the set of all bounded complex valued continuous functions on $(X, d_X)$ which are Lipschitz with respect to $d_X$ and the standard metric on the complex plane $\mathbb{C}$ (absolute value). Let $\|f\|_\infty = \sup\{|f(x)| : x \in X\}$. Define a norm on $\mathrm{Lip}(X, d_X)$ by $\|f\| = \|f\|_\infty + \mathrm{p}(f)$. With respect to pointwise operations $\mathrm{Lip}(X, d_X)$ is a self-adjoint Banach $*$-algebra over $X$ ($\|a^*\| = \|a\|$, $a \in$ the algebra, and the $^*$-operation is complex conjugation).

Definition 1.2. A commutative Banach $*$-algebra $A$ is called *Lipschitz* if there exists a metric space $(X, d_X)$ such that $A = \mathrm{Lip}(X, d_X)$.

If $(X, d_X)$ is compact, then $\mathrm{Lip}(X, d_X) = \{f : f \in C(X) \text{ and } \mathrm{p}(f) < \infty\}$, and it is a unital, natural, regular, self-adjoint Banach function algebra over $X$ [**1, 6, 8**]. Clearly, Lipschitz algebras over compact metric spaces (as objects) and unital*-homomorphisms (as morphisms) form a category, which we denote by CLip.

Let $(X, d)$ be any metric space. Let $Y = \{(x, y) \in X \times X : x \neq y\}$, and let $C_b(Y)$ denote the space of all bounded continuous complex valued functions on $Y$. For $f \in C_b(X)$ and $b \in C_b(Y)$ let $f.b$ and $b.f$ be defined by $(f.b)(x, y) = f(x)b(x, y)$ and $(b.f)(x, y) = f(y)b(x, y)$, $(x, y) \in Y$. Then $C_b(Y)$ is a $C_b(X)$-bimodule (for this and the subsequent definition see [**3**] and [**7**]).

DEFINITION 1.3. Let $(X,d)$ be any metric space, and let $Y = \{(x,y) \in X \times X : x \neq y\}$. Let $D : C_b(X) \to C_b(Y)$ be defined by

$$(Df)(x,y) = \frac{f(y) - f(x)}{d(x,y)}$$

for $(x,y) \in Y$. We say that $D$ is the *de Leeuw derivation* defined by metric $d$.

It is easy to see that $D$ is indeed a derivation, and that $\mathrm{Dom}(D) = \mathrm{Lip}(X,d) = \{f \in C_b(X) : \|Df\| < \infty\}$ (see [**7**]). Clearly $\|Df\| = \mathrm{p}(f)$, for $f \in \mathrm{Dom}(D)$.

DEFINITION 1.4. Let $(X, d_X)$ and $(Y, d_Y)$ be compact metric spaces, and let $(C(X), D_X)$ and $(C(Y), D_Y)$ be the corresponding algebras of continuous functions with de Leeuw derivations defined by the corresponding metrics. We say that a homomorphism $\nu : C(X) \to C(Y)$ is *Lipschitz* if the following condition holds:

$$f \in \mathrm{Dom}(D_X) \text{ implies } \nu f \in \mathrm{Dom}(D_Y). \tag{1.1}$$

Since the composition of two Lipschitz homomorphisms is Lipschitz, and the identity homomorphism on $C(X)$ is identity morphism in categorical sense, it follows that commutative unital $C^*$-algebras with de Leeuw derivations as objects, and Lipschitz homomorphisms as morphisms form a category. We denote it by CDer.

NOTE 1.1. All homomorphisms of algebras which we encounter here are automatically continuous, since each of them is from a Banach algebra into another commutative semisimple Banach algebra.

The main goal of this paper is to prove the following result, which is a direct consequence of Theorem 2.2 and Theorem 3.1.

THEOREM 1.1. *The category* CLip *is equivalent to* CDer, *and* CLip *and* CDer *are the duals of* CMet.

This result shows that Lipschitz algebras are indeed a functional analytic counterpart of compact metric spaces. The general direction in which this research aims is to find a reasonable noncommutative analog of a metric space. This result shows that we need a good definition of a noncommutative Lipschitz algebra. So we need a functional analytic characterization of (commutative) Lipschitz algebras in the $C^*$-algebra setting. This characterization should be similar to the characterization of $C(X)$, the algebras of all continuous functions over compact Hausdorff spaces, as the unital commutative $C^*$-algebras. This is where the importance of the category CDer comes from. Suppose we can find a characterization of de Leeuw derivations in $C^*$-algebraic terms, i.e., as derivations from commutative $C^*$-algebras satisfying certain conditions. Lipschitz algebras would then be characterized as domains of such derivations. Then we can define the noncommutative de Leeuw derivation as a derivation from any $C^*$-algebra (not necessarily commutative) satisfying the same conditions, and we can simply define the noncommutative Lipschitz algebra as the domain of such a de Leeuw derivation. At the moment we do not know of such a desired characterization of de Leeuw derivations. However, it is clear that such a characterization would have to rely on the results of [**7**], which state the conditions which any operator from a $C^*$-algebra, which defines a metric on the state space of that algebra, has to satisfy.

## 2. The categories CMet and CLip

We will use the following relation between two metrics on a space.

DEFINITION 2.1. Two metrics $d_1$ and $d_2$ on a space $X$ are said to be *boundedly equivalent metrics* if there exist constants $C_1 > 0$ and $C_2 > 0$ such that

$$C_1 d_1(x,y) \leqslant d_2(x,y) \leqslant C_2 d_1(x,y)$$

for any $x, y \in X$.

Note that this is the same as saying that the identity map on $X$, $\mathrm{I}_X$, is Lipschitz both as a map from $(X, d_1)$ to $(X, d_2)$ and as a map from $(X, d_2)$ to $(X, d_1)$. It is easily seen that the topology on $X$ induced by $d_2$ is the same as the one induced by $d_1$. The Lipschitz algebras over $(X, d_1)$ and $(X, d_2)$ are also similarly related.

PROPOSITION 2.1. [**8**, Corollary 3.5] *Let $d_1$ and $d_2$ be bounded metrics on $X$. Then* $\mathrm{Lip}(X, d_1)$ *and* $\mathrm{Lip}(X, d_2)$ *have the same elements if and only if $d_1$ and $d_2$ are boundedly equivalent.*

Similar relationship holds for their norms.

PROPOSITION 2.2. *Let $d_1$ and $d_2$ be bounded metrics on $X$. The norms on* $\mathrm{Lip}(X, d_1)$ *and on* $\mathrm{Lip}(X, d_2)$ *are equivalent if and only if $d_1$ and $d_2$ are boundedly equivalent.*

PROOF. Suppose that $d_1$ and $d_2$ are boundedly equivalent. Then for $f \in \mathrm{Lip}(X, d_1)$

$$\begin{aligned}
\mathrm{p}_1(f) &= \sup\Big\{\frac{|f(x)-f(y)|}{d_1(x,y)} : x,y \in X \ \&\ x \neq y\Big\} \\
&= \sup\Big\{\frac{|f(x)-f(y)|}{d_2(x,y)}\ \frac{d_2(x,y)}{d_1(x,y)} : x,y \in X \ \&\ x \neq y\Big\} \\
&\leqslant \sup\Big\{\frac{|f(x)-f(y)|}{d_2(x,y)} : x,y \in X \ \&\ x \neq y\Big\} \times \\
&\qquad \sup\Big\{\frac{d_2(x,y)}{d_1(x,y)} : x,y \in X \ \&\ x \neq y\Big\} = \mathrm{p}_2(f)\ \mathrm{p}_{12}(\mathrm{I}_X)
\end{aligned}$$

where $\mathrm{p}_{12}(\mathrm{I}_X)$ is the Lipschitz constant of $\mathrm{I}_X$ as a map from $(X, d_1)$ to $(X, d_2)$. Similarly, $\mathrm{p}_2(f) \leqslant \mathrm{p}_1(f)\,\mathrm{p}_{21}(\mathrm{I}_X)$, where $\mathrm{p}_{21}(\mathrm{I}_X)$ is the Lipschitz constant of $\mathrm{I}_X$ as a map from $(X, d_2)$ to $(X, d_1)$. Hence

$$\begin{aligned}
\|f\|_1 = \|f\|_\infty + \mathrm{p}_1(f) &\leqslant \|f\|_\infty + \mathrm{p}_1(f)\ \mathrm{p}_{12}(\mathrm{I}_X) \\
&\leqslant \max\{1, \mathrm{p}_{12}(\mathrm{I}_X)\}(\|f\|_\infty + \mathrm{p}_2(f)) = \max\{1, \mathrm{p}_{12}(\mathrm{I}_X)\}\,\|f\|_2.
\end{aligned}$$

Similarly, $\|f\|_2 \leqslant \max\{1, p_{21}(\mathrm{I}_X)\}\|f\|_1$.

Conversely, suppose that the norms are equivalent, that is, there are constants $C$ and $K$ such that for any $f \in \mathrm{Lip}(X, d_1)$ we have $\mathrm{p}_1(f) \leqslant C\,\mathrm{p}_2(f)$ and $\mathrm{p}_2(f) \leqslant K\,\mathrm{p}_1(f)$. Let $x, y \in X$. Let the function $f_1$ be defined by $f_1(z) = d_1(z, y)$, and let the function $f_2$ be defined by $f_2(z) = d_2(z, y)$. Then $\mathrm{p}_1(f_1) = 1$, $\mathrm{p}_2(f_1) \leqslant K$, $\mathrm{p}_2(f_2) = 1$ and $\mathrm{p}_1(f_2) \leqslant C$. We obtain

$$\frac{d_1(x,y)}{d_2(x,y)} = \frac{|f_1(x) - f_1(y)|}{d_2(x,y)} \leqslant \mathrm{p}_2(f_1) \leqslant K$$

and

$$\frac{d_2(x,y)}{d_1(x,y)} = \frac{|f_2(x) - f_2(y)|}{d_1(x,y)} \leqslant \mathrm{p}_1(f_2) \leqslant C,$$

which shows that the two metrics are boundedly equivalent. □

In order to show that CMet is dual to CLip we also need the following result which we quote from [**8**].

THEOREM 2.1. [**8**, Theorem 5.1] *Let $A_i = \mathrm{Lip}(X_i, d_i)$ where $(X_i, d_i)$ is compact, $i = 1, 2$. Then every homomorphism $\nu : A_1 \to A_2$ is of the form*

$$(\nu f)(x) = f(F(x)) \quad f \in A_1,\ x \in X_2 \tag{2.1}$$

*where $F : X_2 \to X_1$ satisfies*

$$d_1(F(x), F(y)) \leqslant K d_2(x, y) \quad x, y \in X_2 \tag{2.2}$$

*for some positive constant $K$. Conversely, if $\nu$ is defined on $A_1$ by the equation* (2.1) *where $F : X_2 \to X_1$ satisfies the condition* (2.2)*, then $\nu$ is a homomorphism of $A_1$ into $A_2$. $\nu$ is one-to-one if and only if $F(X_2) = X_1$. $\nu$ takes $A_1$ onto $A_2$ if and only if $F$ satisfies the additional condition*

$$K^{'} d_2(x, y) \leqslant d_1(F(x), F(y)) \quad x, y \in X$$

*for some positive constant $K^{'}$.*

We single out the following easy fact which we need later.

PROPOSITION 2.3. *Let $(X, d_X)$ be a metric space and let $Y$ be a topological space homeomorphic to $X$ via $\tau : X \to Y$. Then $d_Y$ defined by $d_Y(y_1, y_2) = d_X(\tau^{-1}(y_1), \tau^{-1}(y_2))$ is a metric on $Y$ such that the metric and the original topology on $Y$ coincide. Furthermore, both $\tau$ and $\tau^{-1}$ are Lipschitz maps between $(X, d_X)$ and $(Y, d_Y)$ of Lipschitz constants $\mathrm{p}(\tau) = \mathrm{p}(\tau^{-1}) = 1$.*

PROOF. Let $U$ be any open subset of $Y$. Then $\tau^{-1}(U)$ is open in $X$ and there exists an open ball $B(x_0, r) = \{x \in X : d_X(x_0, x) < r\} \subset \tau^{-1}(U)$. Moreover, $\tau(B(x_0, r)) = \{\tau(x) : x \in B(x_0, r)\} = \{\tau(x) : x \in X \ \&\ d_X(x_0, x) < r\} = \{y \in Y : d_Y(y, \tau(x_0)) < r\} = B(\tau(x_0), r)$ is an open ball which is contained in $U$.

Conversely, by the same argument as above, any open ball in $(Y, d_Y)$ is the image of an open ball in $(X, d_X)$ which is an open set in $X$, and thus so is its $\tau$-image in $Y$. □

We denote by $\mathrm{I}_A$ the identity map (morphism) on an object $A$. For example: $\mathrm{I}_{\mathrm{CMet}}$ is the identity functor on the category CMet; $\mathrm{I}_X$ is the identity map on the set $X$, i.e., $\mathrm{I}_X(x) = x$ for all $x \in X$; and $\mathrm{I}_A$ is the identity automorphism of an algebra $A$, i.e., $\mathrm{I}_A f = f$ for all $f \in A$.

THEOREM 2.2. *Let* $(X, d_X)$ *and* $(Y, d_Y)$ *be compact metric spaces, i.e., objects in* CMet*, and let* $F : (X, d_X) \to (Y, d_Y)$ *be a Lipschitz map, that is a morphism in* CMet*, i.e.,* $F \in \operatorname{Hom}((X, d_X), (Y, d_Y))$*. Let* $T : \mathrm{CMet} \to \mathrm{CLip}$ *be defined by* $T(X, d_X) = \operatorname{Lip}(X, d_X)$*, and let* $TF : \operatorname{Lip}(Y, d_Y) \to \operatorname{Lip}(X, d_X)$ *be defined by* $(TF)g = g \circ F$ *for* $g \in \operatorname{Lip}(Y, d_Y)$*.*

*Let* $A = \operatorname{Lip}(X, d_X)$ *and* $B = \operatorname{Lip}(Y, d_Y)$ *be objects in* CLip*, and let* $\nu : B \to A$ *be a homomorphism, i.e.,* $\nu \in \operatorname{Hom}(B, A)$*. Let* $S : \mathrm{CLip} \to \mathrm{CMet}$ *be defined by* $SA = (X_{SA}, d_{SA})$*, where* $X_{SA}$ *is the character space of* $A$*, and* $d_{SA}$ *is a metric on* $X_{SA}$ *defined by*

$$d_{SA}(\xi, \eta) = \sup\{|\xi f - \eta f| : f \in A \ \&\ \|f\| \leqslant 1\}, \tag{2.3}$$

*for* $\xi, \eta \in X_{SA}$*; let* $S\nu : SA \to SB$ *be defined by* $(S\nu)(\xi)g = \xi(\nu g)$ *for* $\xi \in X_{SA}$ *and* $g \in B = \operatorname{Lip}(Y, d_Y)$*. Then:*

(a) $T$ *is a contravariant functor from* CMet *to* CLip*;*

(b) $S$ *is a contravariant functor from* CLip *to* CMet*;*

(c) $S \circ T$ *is naturally isomorphic to* $\mathrm{I}_{\mathrm{CMet}}$ *and* $T \circ S$ *is naturally isomorphic to* $\mathrm{I}_{\mathrm{CLip}}$*.*

*We conclude that* CMet *and* CLip *are dual categories.*

PROOF. (a) By the quoted Theorem 2.1, $TF \in \operatorname{Hom}(\operatorname{Lip}(Y, d_Y), \operatorname{Lip}(X, d_X))$ for every $F \in \operatorname{Hom}((X, d_X), (Y, d_Y))$. Clearly $T\,\mathrm{I}_X = \mathrm{I}_{TX}$, since $(T\,\mathrm{I}_X)f = f \circ \mathrm{I}_X = f$, for all $f \in \operatorname{Lip}(X, d_X)$. Let $G \in \operatorname{Hom}((Y, d_Y), (Z, d_Z))$. Then for $h \in \operatorname{Lip}(Z, d_Z)$ we have

$$\begin{aligned}(T(G \circ F))(h) &= h \circ (G \circ F) = (h \circ G) \circ F \\ &= TG(h) \circ F = TF(TG(h)) = (TF \circ TG)(h),\end{aligned}$$

which means that $T(G \circ F) = TF \circ TG$. So $T$ is indeed a contravariant functor.

(b) Since $A = \operatorname{Lip}(X, d_X) \in \mathrm{CLip}$ is a unital natural commutative Banach algebra, by Gelfand theory $X_{SA}$ is a compact Hausdorff space homeomorphic to $X$ via $\tau_X : X \to X_{SA}$ defined by $(\tau_X(x))f = f(x)$ for $x \in X$ and $f \in A$. Let $\xi, \eta \in X_{SA}$ be such that $\xi = \tau_X(x)$ and $\eta = \tau_X(y)$ for some $x, y \in X$. Then $d_{SA}(\xi, \eta) = d_{SA}(\tau_X(x), \tau_X(y))$ and so

$$\begin{aligned}d_{SA}(\xi, \eta) &= \sup\{|\xi f - \eta f| : f \in A \ \&\ \|f\| \leqslant 1\} \\ &= \sup\{|(\tau_X(x))f - (\tau_X(y))f| : f \in A \ \&\ \|f\| \leqslant 1\} \\ &= \sup\{|f(x) - f(y)| : f \in A \ \&\ \|f\| \leqslant 1\} = \frac{d_X(x, y)}{1 + d_X(x, y)}.\end{aligned} \tag{2.4}$$

To check the validity of last equality in the above equation, note that the value $d_X(x, y)/(1 + d_X(x, y))$ is achieved by the function $f$ defined by $f(z) = \min\{d(x, z), d(x, y)\}/(1 + d_X(x, y))$. On the other hand, if $f(x) = 0$ and $f(y) > d_X(x, y)/(1 + d_X(x, y))$, then $\mathrm{p}(f) > 1/(1 + d_X(x, y))$ and $\|f\|_\infty > d_X(x, y)/(1 + d_X(x, y))$, so that $\|f\| > 1$, which should not happen. The important thing is that the metric $d_{1X}$ on $X$ defined by $d_{1X}(x, y) = d_{SA}(\tau(x), \tau(y)) = d_X(x, y)/(1 + d_X(x, y))$ is boundedly equivalent to $d_X$. In fact, we have

$$\frac{d_X(x,y)}{1+\mathrm{Diam}(X,d_X)} \leqslant d_{1X}(x,y) = \frac{d_X(x,y)}{1+d_X(x,y)} \leqslant d_X(x,y) \tag{2.5}$$

for all $x, y \in X$, where $\mathrm{Diam}(X,d_X) = \sup\{d_X(x,y) : x, y \in X\}$. By Proposition 2.3 the metric $d_{SA}$ defined by (2.3) defines a topology on $X_{SA}$ agreeing with the Gelfand topology. Thus $(X_{SA}, d_{SA}) \in$ CMet. To prove that $S\nu$ is a morphism in CMet, we need to show that it is a Lipschitz map. For $\xi, \eta \in X_{SA}$ we have

$$\begin{aligned} d_{SB}(S\nu\xi, S\nu\eta) &= \sup\{|(S\nu\xi)g - (S\nu\eta)g| : g \in B \;\&\; \|g\| \leqslant 1\} \\ &= \sup\{|\xi(\nu g) - \eta(\nu g)| : g \in B \;\&\; \|g\| \leqslant 1\} \\ &\leqslant \sup\{|\xi f - \eta f| : f \in \nu B \;\&\; \|f\| \leqslant \|\nu\|\} \\ &\leqslant \|\nu\| \sup\{|\xi f - \eta f| : f \in A \;\&\; \|f\| \leqslant 1\} = \|\nu\| d_{SA}(\xi, \eta). \end{aligned}$$

This shows that $\mathrm{p}(S\nu) \leqslant \|\nu\|$. Clearly $S\,\mathrm{I}_A = \mathrm{I}_{X_{SA}}$, since for $\xi \in X_{SA}$ $(S\,\mathrm{I}_A(\xi))f = \xi(\mathrm{I}_A f) = \xi f$ for all $f \in A$, so $S\,\mathrm{I}_A(\xi) = \xi$. For $\nu \in \mathrm{Hom}(B,A)$, $\mu \in \mathrm{Hom}(C,B)$, $\xi \in X_{SA}$ and $h \in C$ we have

$$\begin{aligned} (S(\nu \circ \mu)(\xi))h &= \xi((\nu \circ \mu)h) = \xi(\nu(\mu h)) \\ &= (S\nu(\xi))(\mu h) = (S\mu(S\nu(\xi)))h = (S\mu \circ S\nu)(\xi)h, \end{aligned}$$

which means that $S(\nu \circ \mu) = S\mu \circ S\nu$. So $S$ is indeed a contravariant functor.

(c) To see that $S \circ T$ is naturally isomorphic to $\mathrm{I}_{\mathrm{CMet}}$, let $(X, d_X) \in$ CMet, $A = T(X,d_X) = \mathrm{Lip}(X,d_X)$ and $(S \circ T)(X,d_X) = TA = (X_{SA}, d_{SA})$, and let $\tau_X : X \to X_{SA}$ be as in (b). We show that $\tau_X$ is a natural isomorphism, i.e., that $\tau_X$ is an invertible morphism in CMet. Since by Gelfand theory, as pointed out in (b), $\tau_X$ is a homeomorphism, we only need to show that $\tau_X$ and $\tau_X^{-1}$ are both Lipschitz. But, that clearly follows from (2.4) and (2.5), and in fact $\mathrm{p}(\tau_X) = \sup\{d_{1X}(x,y)/d_X(x,y) : x,y \in X \;\&\; x \neq y\} = \sup\{1/(1+d_X(x,y)) : x,y \in X \;\&\; x \neq y\} = 1$ and $\mathrm{p}(\tau_X^{-1}) = \sup\{d_X(x,y)/d_{1X}(x,y) : x,y \in X \;\&\; x \neq y\} \leqslant 1 + \mathrm{Diam}(X,d_X)$. Thus $\tau_X$ is an invertible morphism in CMet, therefore $\mathrm{I}_{\mathrm{CMet}}$ is naturally isomorphic to $S \circ T$.

To see that $T \circ S$ is naturally isomorphic to $\mathrm{I}_{\mathrm{CLip}}$, let $A = \mathrm{Lip}(X,d_X) \in$ CLip. Define a map $\sigma_A : A \to (T \circ S)A = \mathrm{Lip}(X_{SA}, d_{SA})$ by $\sigma_A f = f \circ \tau_X^{-1}$, for $f \in A$, i.e., $(\sigma_A f)(\xi) = f(\tau_X^{-1}(\xi))$ for $\xi \in X_{SA}$. Clearly, $g = \sigma_A f \in C(X_{SA})$ and $\|\sigma_A f\|_\infty = \|f\|_\infty$. Since by Proposition 2.3 $\tau_X^{-1}$ is Lipschitz with $\mathrm{p}(\tau_X^{-1}) = 1 + \mathrm{Diam}(X,d_X)$, we have that $g = f \circ \tau_X^{-1}$ is Lipschitz, and $\mathrm{p}(g) = \mathrm{p}(f \circ \tau_X^{-1}) \leqslant \mathrm{p}(f)\,\mathrm{p}(\tau_X^{-1}) = \mathrm{p}(f)\,(1 + \mathrm{Diam}(X,d_X))$. Clearly, $\sigma_A$ is one-to-one. It is also onto, since for any $g \in \mathrm{Lip}(X_{SA}, d_{SA})$, by the same arguments as above $f = g \circ \tau_X \in \mathrm{Lip}(X,d_X)$, and $\sigma_A f = g$. Note that the situation is similar to the one in Proposition 2.2. Thus $\sigma_A$ is an invertible morphism in CLip, which means that $\sigma_A$ is a natural isomorphism. Therefore $T \circ S$ is naturally isomorphic to $\mathrm{I}_{\mathrm{CLip}}$. □

Some remarks concerning the functors $T$ and $S$ are in order.

Remark 2.1. Since every Lipschitz algebra $A$ is $\mathrm{Lip}(X,d_X)$ for some $(X,d_X)$, $T$ is onto as a map of objects. Let $F, G \in \mathrm{Hom}((X,d_X),(Y,d_Y))$, and suppose that

$TF = TG$. That means that for all $g \in \mathrm{Lip}(Y, d_Y)$, $TF(g) = TG(g)$, i.e., that $g \circ F = g \circ G$. If $F \neq G$, then there exists $x \in X$ with $F(x) \neq G(x)$. But then there exists $g \in \mathrm{Lip}(Y, d_Y)$ with $g(F(x)) = 1$ and $g(G(x)) = 0$, so that we get $g \circ F \neq g \circ G$. Contradiction. Thus $T$ is one-to-one on morphisms. It is also onto on morphisms, since by the quoted Theorem 2.1 every homomorphism of Lipschitz algebras arises from some Lipschitz map. However, we cannot still claim that CMet and CLip are in fact contravariantly isomorphic categories, since we do not have an explicit inverse for $T$. Note that the functor $S$ (the would-be inverse of $T$) does not map $\mathrm{Lip}(X, d_X)$ to $(X, d_X)$ but to $(X_{SA}, d_{SA})$.

REMARK 2.2. There are other choices for defining metric $d_{SA}$ than the one given by (2.3). For example, one can define $d_{SA}$ by

$$d_{SA}(\xi, \eta) = \sup\{|\xi f - \eta f| : f \in A \;\&\; \mathrm{p}(f) \leqslant 1\}, \tag{2.6}$$

for $\xi, \eta \in X_{SA}$. The difference between (2.3) and (2.6) is that in (2.6) we are taking $\mathrm{p}(f) \leqslant 1$ instead of $\|f\| \leqslant 1$ as in (2.3). However, we have to be sure that we can compute $\mathrm{p}(f)$ from the given data. This can be done using the following two steps: (1) obtain $\|f\|_\infty$ by $\|f\|_\infty = \sup\{|\xi f| : \xi \in X_{SA}\}$; (2) take $\mathrm{p}(f) = \|f\| - \|f\|_\infty$. This approach has the advantage that when the metric $d_{1X}$ on $X$ is defined by $d_{1X}(x, y) = d_{SA}(\tau(x), \tau(y))$, we obtain that $d_{1X} = d_X$, that is, we get back our original metric. This also means that we do not need to use the results concerning boundedly equivalent metrics. We leave it to the taste of the reader to decide which of these two choices we have given is the more suitable one.

## 3. Category CDer and its relation to CLip and CMet

We now turn our attention to the category CDer.

THEOREM 3.1. *Let $(X, d_X), (Y, d_Y) \in$ CMet, and $F \in \mathrm{Hom}((X, d_X), (Y, d_Y))$. Let $R$ : CMet $\to$ CDer be defined by $R(X, d_X) = (A_{RX}, D_{RX})$, where $A_{RX} = C(X)$ and $D_{RX}$ is the de Leeuw derivation defined by $d_X$, i.e., $(D_{RX} f)(x, y) = (f(y) - f(x))/d_X(x, y)$ for $x, y \in X$, $x \neq y$, and $f \in \mathrm{Dom}(D_{RX}) = \{f \in C(X) : \|D_{RX} f\|_\infty < \infty\}$. Let $RF : A_{RY} = C(Y) \to A_{RX} = C(X)$ be defined by $(RF)g = g \circ F$ for $g \in C(Y)$.*

*Let $(A, D_A), (B, D_B) \in$ CDer, and let $\nu \in \mathrm{Hom}((A, D_A), (B, D_B))$. Let $Q$ : CDer $\to$ CLip be defined by $QA = \mathrm{Dom}(D_A)$ equipped with the norm $\|f\|_{QA} = \|f\| + \|D_A f\|$ and let $Q\nu : QA \to QB$ be defined by $Q\nu = \nu|_{QA}$. Then:*

(a) *$R$ is a contravariant functor from* CMet *to* CDer*;*

(b) *$Q$ is a covariant functor from* CDer *to* CLip*;*

(c) *$P = R \circ S$ : CLip $\to$ CDer is a covariant functor;*

(d) *$P \circ Q$ is naturally isomorphic to $\mathrm{I}_{\mathrm{CDer}}$ and $Q \circ P$ is naturally isomorphic to $\mathrm{I}_{\mathrm{CLip}}$.*

*We conclude that* CDer *is equivalent to* CLip *and that it is dual to* CMet.

PROOF. (a) Clearly, $(A_{RX}, D_{RX})$ is an object in CDer. Since $\mathrm{Dom}(D_{RX}) = \mathrm{Lip}(X, d_X)$, by (a) of Theorem 2.2, $RF$ is a morphism in CDer, and everything else is as for the functor $T$.

(b) Clear, since if $A = C(X)$ and $D_A = D_{d_X}$, the derivation defined by $d_X$, $QA = \mathrm{Lip}(X, d_X)$, and by the condition (1.1) in Definition 1.4, $(Q\nu)f \in \mathrm{Lip}(Y, d_Y)$, so that $Q\nu$ is a homomorphism from $\mathrm{Lip}(X, d_X)$ into $\mathrm{Lip}(Y, d_Y)$.

(c) Clear.

(d) To see that $P \circ Q = R \circ S \circ Q$ is naturally isomorphic to $\mathrm{I}_{\mathrm{CDer}}$, let $(A, D_A) \in$ CDer where $A = C(X)$ and $D_A = D_{d_X}$. Then $QA = \mathrm{Lip}(X, d_X)$, $(S \circ Q)A = (X_{SA}, d_{SA})$, and $(R \circ S \circ Q)A = (C(X_{SA}), D_{d_{SA}})$. Define $\theta_A : A \to C(X_{SA})$ by $\theta_A f = f \circ \tau_X^{-1}$, i.e., $(\theta_A f)(\xi) = f(\tau_X^{-1}(\xi))$ for $\xi \in X_{SA}$. Note that $\theta_A$ is defined similarly as $\sigma_A$ from the part (c) of the proof of Theorem 2.2. Clearly, $g = \theta_A f \in C(X_{SA})$ and $\|\theta_A f\|_\infty = \|f\|_\infty$, and $\theta_A$ is an isomorphism of $A$ and $C(X_{SA})$. We need to show that $\theta_A$ and $\theta_A^{-1}$ are Lipschitz homomorphisms, that is, that they satisfy the condition (1.1) in Definition 1.4. But, $\mathrm{Dom}(D_A) = \mathrm{Lip}(X, d_X)$ and $\mathrm{Dom}(D_{d_{SA}}) = \mathrm{Lip}(X_{SA}, d_{SA})$, so by the proof of (c) of Theorem 2.2, this is satisfied. Hence $\theta_A$ is an invertible morphism in CDer, which means that $\theta_A$ is a natural isomorphism. Therefore $\mathrm{I}_{\mathrm{CDer}}$ is naturally isomorphic to $P \circ Q$.

To see that $Q \circ P = Q \circ R \circ S$ is naturally isomorphic to $\mathrm{I}_{\mathrm{CLip}}$, it is enough to observe that $Q \circ R = T$. Thus $Q \circ P = T \circ S$, which we already know from (c) of Theorem 2.2 to be naturally isomorphic to $\mathrm{I}_{\mathrm{CLip}}$. □

REMARK 3.1. A similar remark holds concerning the functors $Q$, $P$ and $R$ as for the functors $T$ and $S$ as in Remark 2.1. The functor $Q$ is almost an isomorphism of CDer and CLip, but we do not have its explicit inverse, as the functor $P$ is not the one. Likewise, the functor $R$ is almost a contravariant isomorphism from CMet to CDer, but the functor $S \circ Q$ is not its inverse.

Here is an easy, but interesting consequence of the above considerations.

PROPOSITION 3.1. *Let $A = \mathrm{Lip}(X, d_X)$ and $B = \mathrm{Lip}(Y, d_Y)$ be Lipschitz algebras over compact metric spaces, and let $\nu : B \to A$ be a homomorphism. Then $\nu$ is continuous as a map from $C(Y)$ to $C(X)$ and there exists a unique homomorphism $\nu_1 : C(Y) \to C(X)$ such that $\nu = \nu_1|_B$.*

PROOF. Let $F : X \to Y$ be defined by $F = \tau_Y^{-1} \circ S\nu \circ \tau_X$, where $\tau_X$, $\tau_Y$ and $S$ are defined in Theorem 2.2. Clearly, $F$ is a continuous map, and so $\nu_1 : C(Y) \to C(X)$ defined by $\nu_1 g = g \circ F$ for $g \in C(Y)$ is a homomorphism. We need to show that $\nu_1|_B = \nu$. So let $g \in B$. Then

$$\begin{aligned}(\nu_1 g)(x) &= g(F(x)) = g(\tau_Y^{-1} \circ S\nu \circ \tau_X(x)) \\ &= (S\nu \circ \tau_X(x))g = \tau_X(x)(\nu g) = (\nu g)(x),\end{aligned}$$

since by definition $g(\tau_Y^{-1}(\eta)) = \eta g$ for $\eta \in X_{SB}$ and $\tau_X(x)f = f(x)$.

To prove the uniqueness of the extended homomorphism $\nu_1 : C(Y) \to C(X)$, note first that $\nu_1$ is automatically continuous. Furthermore, $B = \mathrm{Lip}(Y, d_Y)$ is dense in $C(Y)$, since it is a point-separating, self-adjoint subalgebra of $C(Y)$. Thus, if there existed another extension $\nu_2 : C(Y) \to C(X)$ of $\nu$, it would also be continuous and it would coincide with $\nu_1$ on the dense subalgebra $B = \mathrm{Lip}(Y, d_Y)$. Obviously, it then has to be equal to $\nu_1$. □

In other words, every homomorphism of Lipschitz algebras is a restriction of some Lipschitz homomorphism of the underlying algebras of continuous functions. To prove the existence of the extension $\nu_1$ of $\nu$ without using the approach taken here, one would have to show that the homomorphism $\nu$ is continuous as a map from continuous functions on $Y$ to continuous functions on $X$. Then, one could get the extended homomorphism $\nu_1$ by continuity, using the fact that $\mathrm{Lip}(Y, d_Y)$ is dense in $C(Y)$.

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(Received 13 03 2002)
(Revised 09 02 2003)