PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 72(86) (2002), 23–28

VERIFICATION OF ATIYAH'S CONJECTURE FOR SOME NONPLANAR CONFIGURATIONS WITH DIHEDRAL SYMMETRY

Dragomir Ž. Đoković

Communicated by Žarko Mijajlović

Abstract. To an ordered N-tuple of distinct points in the three-dimensional Euclidean space, Atiyah has associated an ordered N-tuple of complex homogeneous polynomials in two variables of degree $N-1$, each determined only up to a scalar factor. He has conjectured that these polynomials are linearly independent. In this note it is shown that Atiyah's conjecture is true if m of the points are on a line L and the remaining $n = N - m$ points are the vertices of a regular n-gon whose plane is perpendicular to L and whose centroid lies on L.

1. Introduction

Let $(x_1, \ldots, x_N)$ be an ordered N-tuple of distinct points in $\mathbf{R}^3$. Each ordered pair (x_i, x_j) with $i \neq j$ determines a point

$$\frac{x_j - x_i}{|x_j - x_i|}$$

on the unit sphere $S^2 \subset \mathbf{R}^3$. Identify S^2 with the complex projective line $\mathbf{CP}^1$ by using a stereographic projection. We obtain a point $(u_{ij}, v_{ij}) \in \mathbf{CP}^1$ and a nonzero linear form $l_{ij} = u_{ij}x + v_{ij}y \in \mathbf{C}[x, y]$. Define homogeneous polynomials $p_i \in \mathbf{C}[x, y]$ of degree $N - 1$ by

$$p_i = \prod_{j \neq i} l_{ij}(x, y), \quad i = 1, \ldots, N. \tag{1.1}$$

Conjecture 1.1. (Atiyah [**2**]) *The polynomials $p_1, \ldots, p_N$ are linearly independent.*

2000 *Mathematics Subject Classification.* Primary 51M04, 51M16; Secondary 70G25.

The author was supported in part by the NSERC Grant A-5285.

Atiyah [**1, 2**] observed that his conjecture is true if the points $x_1, \dots, x_N$ are collinear. He also verified the conjecture for $N = 3$. Then Eastwood and Norbury [**5**] verified it for $N = 4$. In our previous note [**4**] we verified this conjecture for two special planar configurations of N points. For additional information on the conjecture (further conjectures, generalizations, and numerical evidence) see [**2, 3**].

Apart from the above mentioned result for arbitrary four points, there are no results known for nonplanar configurations. In this note we prove Atiyah's conjecture for the infinite family of nonplanar configurations described in the abstract.

2. Preliminaries

Identify $\mathbf{R}^3$ with $\mathbf{R} \times \mathbf{C}$ and denote the origin by O. Following Eastwood and Norbury [**5**], we make use of the Hopf map $h : \mathbf{C}^2 \setminus \{O\} \to (\mathbf{R} \times \mathbf{C}) \setminus \{O\}$ defined by:

$$h(z, w) = ((|z|^2 - |w|^2)/2, z\bar{w}).$$

This map is surjective and its fibers are the circles $\{(zu, wu) : u \in S^1\}$, where S^1 is the unit circle in $\mathbf{C}$. If $h(z, w) = (a, v)$, we say that (z, w) is a *lift* of (a, v).

Let $x_i = (a_i, z_i)$. For the sake of simplicity, we assume that if $i < j$ and $z_i = z_j$ then $a_i < a_j$. As the lift of the vector $x_j - x_i$, $i < j$, we choose

$$\lambda_{ij}^{-1/2} \left(\lambda_{ij}, \bar{z}_j - \bar{z}_i\right),$$

where

$$\lambda_{ij} = a_j - a_i + \sqrt{(a_j - a_i)^2 + |z_j - z_i|^2}.$$

Then

$$\lambda_{ij}^{-1/2} \left(z_i - z_j, \lambda_{ij}\right),$$

is a lift of $x_i - x_j$. The corresponding linear forms are

$$l_{ij}(x, y) = \lambda_{ij} x + (\bar{z}_j - \bar{z}_i) y, \quad i < j;$$
$$l_{ij}(x, y) = (z_j - z_i) x + \lambda_{ji} y, \quad i > j.$$

Define the binary forms p_i by using (1.1) and the above expressions for the l_{ij}'s. Atiyah's conjecture asserts that the $N \times N$ coefficient matrix of these forms is nonsingular.

3. Verification of the conjecture

We shall prove Atiyah's conjecture for the configurations of N points satisfying the following two conditions:

(i) The first m points $x_1, \dots, x_m$ lie on a line L.
(ii) The remaining $n = N - m$ points $y_j = x_{m+j+1}$ $(j = 0, 1, \dots, n-1)$ are the vertices of a regular n-gon whose plane is perpendicular to L, and whose centroid lies on L.

Without any loss of generality, we may assume that $L = \mathbf{R} \times \{0\}$ and that the y_j's lie on the unit circle in $\{0\} \times \mathbf{C}$. Write $x_i = (a_i, 0)$ for $i = 1, \dots, m$ and $y_j = (0, b_j)$ for $j = 0, 1, \dots, n-1$. We may also assume that $a_1 < a_2 < \cdots < a_m$ and that $b_j = -\zeta^j$, where $\zeta = e^{2\pi i/n}$.

Vectors	Index restrictions	Lifts	Linear forms
$x_r - x_i$	$i < r \leqslant m$	$(2(a_r - a_i))^{1/2}\,(1,0)$	x
$x_r - x_i$	$r < i \leqslant m$	$(2(a_i - a_r))^{1/2}\,(0,1)$	y
$y_s - y_j$	$s \neq j$	$\lvert b_s - b_j\rvert^{1/2}\left(\frac{b_s-b_j}{\lvert b_s-b_j\rvert}, 1\right)$	$\frac{b_s-b_j}{\lvert b_s-b_j\rvert}x + y$
$y_j - x_i$	$i \leqslant m$	$\lambda_i^{-1/2}(1, \lambda_i \bar{b}_j)$	$x + \lambda_i \bar{b}_j y$
$x_i - y_j$	$i \leqslant m$	$\lambda_i^{-1/2}(-\lambda_i b_j, 1)$	$y - \lambda_i b_j x$

TABLE 1. The lifts of the vectors $x_j - x_i$

The lifts of the nonzero vectors $x_j - x_i$, $i, j \in \{1, \ldots, N\}$ are given in Table 1, where we have set

$$\lambda_i = a_i + \sqrt{1 + a_i^2}.$$

The associated polynomials p_i (up to scalar factors) are given by:

$$p_i(x,y) = x^{m-i} y^{i-1}\,(x^n - \lambda_i^n y^n)\,, \quad 1 \leqslant i \leqslant m;$$

$$p_{m+j+1}(x,y) = \prod_{s \neq j} \left(x + \frac{\bar{b}_s - \bar{b}_j}{\lvert b_s - b_j \rvert} y \right) \cdot \prod_{i=1}^{m} (y - \lambda_i b_j x), \quad 0 \leqslant j < n.$$

We now give the proof of our result.

THEOREM 3.1. *Atiyah's conjecture is valid for configurations described above.*

PROOF. If $n = 1$ or 2, these configurations are planar and they have been dealt with in [**4**]. So, we assume that $n \geqslant 3$.

Note that

$$b_s - b_j = -2i\zeta^{j+s} \sin \frac{\pi(s-j)}{n}.$$

After dehomogenizing the polynomials p_i by setting $x = 1$, we obtain (up to scalar factors and ordering) the following polynomials:

$$y^{i-1}(1 - \lambda_i^n y^n), \quad 1 \leqslant i \leqslant m; \tag{3.1}$$

$$f(\zeta^j y), \quad 0 \leqslant j < n, \tag{3.2}$$

where

$$f(y) = \prod_{s=1}^{n-1} (y - ie^{\pi i s/n}) \cdot \prod_{i=1}^{m} (y + \lambda_i). \tag{3.3}$$

Denote by $\tilde{E}_k$ the k-th elementary symmetric function of the $N - 1$ numbers:

$$\lambda_i,\ (1 \leqslant i \leqslant m); \quad -ie^{\pi i s/n},\ (1 \leqslant s \leqslant n-1).$$

By convention we set $\tilde{E}_0 = 1$ and $\tilde{E}_k = 0$ if $k < 0$ or $k \geqslant N$. Then

$$f(y) = \sum_{k=0}^{N-1} \tilde{E}_{N-1-k} y^k.$$

By factorizing f over the real numbers, we see that all coefficients of f are positive.

Let P be the coefficient matrix of the polynomials (3.1) and (3.2). The top m rows of P form the submatrix

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & -\lambda_1^n & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & -\lambda_2^n & 0 & \dots & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 & 0 & -\lambda_3^n & \dots & 0 \\ \vdots \end{pmatrix}$$

and the bottom n rows the submatrix

$$\begin{pmatrix} \tilde{E}_{N-1} & \tilde{E}_{N-2} & \tilde{E}_{N-3} & \cdots & \tilde{E}_1 & \tilde{E}_0 \\ \tilde{E}_{N-1} & \tilde{E}_{N-2}\zeta & \tilde{E}_{N-3}\zeta^2 & \cdots & \tilde{E}_1\zeta^{N-2} & \tilde{E}_0\zeta^{N-1} \\ \tilde{E}_{N-1} & \tilde{E}_{N-2}\zeta^2 & \tilde{E}_{N-3}\zeta^4 & \cdots & \tilde{E}_1\zeta^{2(N-2)} & \tilde{E}_0\zeta^{2(N-1)} \\ \vdots \end{pmatrix}.$$

In order to compute $\det(P)$ we perform on P successively the following operations:

- Add the first column multiplied with λ_1^n to the $(n+1)$-st column.
- Add the second column multiplied with λ_2^n to the $(n+2)$-nd column.

$\vdots$

- Add the m-th column multiplied with λ_m^n to the N-th column.

By expanding the determinant of this new matrix along the first m rows, we obtain that

$$|\det(P)| = c\prod_{k=0}^{n-1} f_k,$$

where $c = n^{n/2}$ is the modulus of the determinant of the matrix (ζ^{rs}), $0 \leqslant r, s < n$, and

$$f_k = \tilde{E}_k + \lambda_{m-k}^n \tilde{E}_{k+n} + \lambda_{m-n-k}^n \lambda_{m-k}^n \tilde{E}_{k+2n} + \cdots, \quad 0 \leqslant k < n.$$

As the λ_i's and the $\tilde{E}_k$'s are positive, the proof is completed. □

4. Comments on Atiyah and Sutcliffe conjecture

Let us also state explicitly the stronger conjecture of Atiyah and Sutcliffe [**3**, Conjecture 2] for the case of our configurations:

$$n^{n/2}\prod_{k=0}^{n-1} f_k \geqslant 2^{\binom{n}{2}} \prod_{i=1}^{m} (1+\lambda_i^2)^n, \tag{4.1}$$

where, as in the proof above,

$$f_k = \sum_{s\geqslant 0} \left(\prod_{j=1}^{s} \lambda_{N-jn-k}^n \right) \tilde{E}_{k+sn}, \quad 0 \leqslant k < n.$$

Recall that $a_1 < a_2 < \cdots < a_m$ and, consequently, $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_m$.

The substitution $(a_1,\ldots,a_m)\to(-a_m,\ldots,-a_1)$ corresponds to the reflection in the plane $\{0\}\times\mathbf{C}$. Consequently, the function

$$\frac{\prod_{k=0}^{n-1} f_k}{\prod_{i=1}^{m}(1+\lambda_i^2)^n}$$

is invariant under the transformation

$$(\lambda_1,\ldots,\lambda_m)\to(\lambda_m^{-1},\ldots,\lambda_1^{-1}).$$

For $n=1$ the inequality (4.1) was proved in [**4**] in general, and for $n=2$ only in the special (limit) case when all λ_i's are equal. One expects the inequality (4.1) to be strict for all $n\geqslant 3$ (see [**3**, Section 4]).

Expand the two products in (3.3) separately:

$$\prod_{s=1}^{n-1}(y-ie^{\pi is/n})=\sum_{j=0}^{n-1}c_j y^{n-1-j},$$
$$\prod_{i=1}^{m}(y+\lambda_i)=\sum_{j=0}^{m}E_j y^{m-j}.$$

The coefficients c_j, $0\leqslant j<n$, and E_j, $0\leqslant j\leqslant m$, are all positive. We also set $E_j=0$ if $j<0$ or $j>m$. Then

$$\tilde{E}_k=\sum_{i=0}^{n-1}c_iE_{k-i}.$$

In the limit case, when all λ_k's are equal to some $\lambda>0$, the inequality (4.1) specializes to

$$n^{n/2}\prod_{k=0}^{n-1}\sum_{s\geqslant 0}\lambda^{2sn+k}\sum_{i=0}^{n-1}\binom{m}{k+sn-i}c_i\lambda^{-i}\geqslant 2^{\binom{n}{2}}(1+\lambda^2)^{mn}.$$

For $\lambda=0$ this gives

$$n^{n/2}\prod_{k=0}^{n-1}c_k\geqslant 2^{\binom{n}{2}}.$$

We conjecture that the following apparent strengthening of (4.1) is valid:

$$\prod_{k=0}^{n-1}\frac{f_k}{c_k}\geqslant\prod_{i=1}^{m}(1+\lambda_i^2)^n. \tag{4.2}$$

When all λ_k's are equal to some $\lambda>0$, this becomes:

$$\prod_{k=0}^{n-1}\sum_{s\geqslant 0}\lambda^{2sn+k}\sum_{i=0}^{n-1}\binom{m}{k+sn-i}c_i\lambda^{-i}\geqslant\left(\prod_{k=0}^{n-1}c_k\right)\cdot(1+\lambda^2)^{mn}. \tag{4.3}$$

If $n=3$ then $c_0=c_2=1$, $c_1=\sqrt{3}$ and the inequality (4.2) takes the form:

$$f_0f_1f_2\geqslant\sqrt{3}\prod_{i=1}^{m}(1+\lambda_i^2)^3, \tag{4.4}$$

where

$$f_k = \sum_{s\geqslant 0}\left(\prod_{j=1}^{s}\lambda_{N-3j-k}^3\right)(E_{3s+k}+\sqrt{3}E_{3s+k-1}+E_{3s+k-2}),$$

and (4.3) the form:

$$f_0 f_1 f_2 \geqslant \sqrt{3}(1+\lambda^2)^{3m},$$

where now

$$f_k = \sum_{s\geqslant 0}\lambda^{6s+k}\left[\binom{m}{3s+k}+\sqrt{3}\binom{m}{3s+k-1}\lambda^{-1}+\binom{m}{3s+k-2}\lambda^{-2}\right].$$

By using Maple, we have verified the last inequality for $m \leqslant 6$, and, by using the invariance property mentioned above, it is easy to verify (4.4) for $m = 2$.

References

[1] M. Atiyah, *The geometry of classical particles*, in: *Surveys in Differential Geometry, vol. VII*, S.-T. Yau, ed., International Press, 2000, pp. 1–15.

[2] M. Atiyah, *Configurations of points*, Phil. Trans. R. Soc. Lond. A **359** (2001), 1375–1387.

[3] M. Atiyah and P. Sutcliffe, *The geometry of point particles*, arXiv:hep-th/0105179 v2, 15 Oct 2001.

[4] D. Ž. Đoković, *Proof of Atiyah's conjecture for two special types of configurations*, arXiv:math.GT/0205221 v4, 11 June 2002; Electron. J. Linear Algebra **9** (2002), 132–137.

[5] M. Eastwood and P. Norbury, *A proof of Atiyah's conjecture on configurations of four points in Euclidean three-space*, Geometry and Topology **5** (2001), 885–893.

Department of Pure Mathematics
University of Waterloo
Waterloo, Ontario, N2L 3G1
Canada

djokovic@uwaterloo.ca

(Received 08 10 2002)