PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 71(85) (2002), 113–122

CONVERGENCE OF RATIOS AND DIFFERENCES OF TWO ORDER STATISTICS

Jef L. Teugels and Giovanni Vanroelen

ABSTRACT. We study some direct and converse results for functions involving ratios or differences of two ascending order statistics. In our proofs, we rely heavily on techniques from the theory of regular variation and its adaptations.

1. Introduction

Let $X_1, X_2, \ldots, X_n$ be a sample of independent random variables with common distribution F and tail quantile function U

$$U(x) := \inf\{u \in \mathbb{R} : F(u) \geqslant 1 - 1/x\}, \qquad \text{for } x > 1.$$

For $n \geqslant 1$, let $X_1^* \leqslant X_2^* \leqslant \cdots \leqslant X_n^*$ be the corresponding order statistics. Further, denote by $x_+ := \sup\{x : F(x) < 1\}$ the upper endpoint of F; similarly, $x_- := \inf\{x : F(x) > 0\}$ is the lower endpoint of F. Although this requirement is not always needed, we will assume for simplicity that F is continuous.

In the sixties, several authors Arov and Bobrov [**2**], Breiman [**5**], Dwass [**6**], Lamperti [**12**] and Rossberg [**13, 14**] recognized the importance of a concept from real analysis within the realm of extreme value theory. The concept of *regular variation* had already been introduced by Karamata in 1930, [**11**]. However, its use in a probabilistic context was not fully recognized. While Gnedenko [**9**] gave a version for the solution of the domain of attraction problem for the maximum X_n^*, the formulation is still in the form of a relation like

$$\lim_{x\to\infty} \frac{1 - F(xu)}{1 - F(x)} = u^{-\rho} \qquad \text{for all } u > 0\,. \tag{1.1}$$

Nowadays we say that $1 - F(x)$ varies regularly with order $-\rho \leqslant 0$. The crowning piece of this development within extreme value theory is due to de Haan [**10**] who showed the overall use of functions of regular variation in the solution of the domain of attraction problem.

Going one step further, Dwass [**6**], Lamperti [**12**] and Rossberg [**13**] showed the relevance of regular variation for the limiting distribution of successive quotients

2000 *Mathematics Subject Classification.* Primary 62G30; Secondary 26A12.
Key words and phrases. order statistics; regular variation; selection principle.

of order statistics $(X^*_{n-j}/X^*_{n-j+1})^m_{j=1}$, m fixed. In particular, they showed that under the condition (1.1), the quotients $(X^*_{n-j}/X^*_{n-j+1})^m_{j=1}$ become asymptotically independent with marginal distribution functions $(t^{j\rho})^m_{j=1}$, $0<t<1$. That regular variation was actually not only sufficient but also necessary has been shown in 1975 by Smid and Stam [**15**]. They proved the following result:

PROPOSITION 1. *If for some $j \geqslant 1$, $t \in (0,1)$ and $\rho \geqslant 0$,*

$$\lim_{n\to\infty} P\left(\frac{X^*_{n-j}}{X^*_{n-j+1}} \leqslant t\right) = t^{j\rho},$$

then $1-F(x)$ varies regularly of order $-\rho$ as $x \to \infty$.

Bingham and Teugels [**3**] extended this result in two ways. First they showed that the explicit form of the limit on the right was not needed. Secondly, the distance between the indices of the order statistics could be taken arbitrarily as long as it was fixed. More precisely:

PROPOSITION 2. *Let $s \in \{0,1,2,\ldots\}$, $r \in \{1,2,\ldots\}$ be fixed integers. Let F be concentrated on the positive half-line. If X^*_{n-r-s}/X^*_{n-s} converges in distribution to a non-degenerate limit, then for some $\rho>0$, $1-F(x)$ varies regularly of order $-\rho$ as $x \to \infty$.*

In section 3, we present a similar converse result for distribution functions with a finite upper endpoint. Especially when $x_+ = 0$ a new situation seems to appear when a quotient of order statistics converges to a non-degenerate limit distribution. More precisely, we will show how the convergence of the ratios (or their reciprocals)

$$U_n(r,s) := \frac{x_+ - X^*_{n-s}}{x_+ - X^*_{n-r-s}} \quad \text{for } s \in \{0,1,\ldots\} \text{ and } r \in \{1,2,\ldots\}, \tag{1.2}$$

to a non-degenerate limit is linked to the regular variation of $1-F(x_+ - 1/x)$ as $x \to \infty$. One way of looking at the above results is the following: in Propositions 1 and 2 the value of one specific large order statistic is normalized by the value of another large order statistic. A similar result for the quantity $U_n(r,s)$ means that the distance between a large order statistic and the right end point of the distribution can be normalized by a similar random distance induced by another large order statistic.

In contrast with the above, section 4 deals with the quotient as a single quantity. We introduce appropriate normalizing constants $\{d_n\}$ such that for $n \to \infty$

$$Q_n(r,s) := \frac{X^*_{n-s}/X^*_{n-r-s} - 1}{d_n}, \tag{1.3}$$

converges to a non-degenerate limit. For these results, it will be necessary to assume that F belongs to the so-called *extremal-type class* $\mathcal{C}_\gamma(g)$, by which we mean that there exists a real-valued constant γ and an ultimately positive auxiliary function g such that

$$\lim_{x\to\infty} \frac{U(ux)-U(x)}{g(x)} = \int_1^u v^{\gamma-1}\mathrm{d}v := h_\gamma(u) \quad \text{for all } u>0. \tag{1.4}$$

For $\gamma = 0$, $h_0(u) = \log u$. Actually, the limit on the right of (1.4) does not have to be specified. From the mere existence of a non-trivial limit, it follows that g has to be regularly varying with an *extreme value index* coined γ and that then the limit has to be $h_\gamma(.)$.

Moreover it is known from extreme value theory that if $\gamma \geqslant 0$, then $g(x)/U(x) \to \gamma$, while if $\gamma \leqslant 0$, then $g(x)/(x_+ - U(x)) \to -\gamma$. These two side conditions tell us that for $\gamma \neq 0$ condition (1.4) reduces to Karamata's classical regular variation. The only novelty appears in the case $\gamma = 0$ where the condition (1.4) leads to the *de Haan-class* of slowly varying functions, a subclass of the one introduced by Karamata.

In what follows we will show that there is a close connection between the asymptotic behaviour of $Q_n(r,s)$ and the asymptotic behaviour of normalized differences of two ascending order statistics. Finally, we discuss a converse theorem for these differences. But before doing that, we collect some useful auxiliary properties in section 2.

2. Preliminaries

For our purposes, it is useful to have a couple of equivalent forms of the $\mathcal{C}_\gamma$-condition (1.4) at our disposal.

PROPOSITION 3. *Let $\gamma \in \mathbb{R}$ be a real-valued constant and g an ultimately positive auxiliary function. Then the following assertions are equivalent:*

(i) $F \in \mathcal{C}_\gamma(g)$;

(ii) *for all u for which $1 + \gamma u > 0$,*

$$\lim_{x \uparrow x_+} \frac{1 - F(x + uh(x))}{1 - F(x)} = \frac{1}{h_\gamma^{-1}(u)} := \eta_\gamma(u) \tag{2.1}$$

where $h \circ U = g$ and h is self-neglecting, i.e., for $x \uparrow x_+$

$$\frac{h(x + vh(x))}{h(x)} \to 1 + \gamma v;$$

(iii) *for all u for which $1 + \gamma u > 0$,*

$$x\left\{1 - F(U(x) + ug(x))\right\} \to \eta_\gamma(u), \tag{2.2}$$

as $x \to \infty$.

Notice that in the above $\eta_\gamma(u) = (1 + \gamma u)^{-1/\gamma}$ where for $\gamma = 0$ we read $\eta_0(u) = e^{-u}$.

For the proof of this result, see for example de Haan [**10**], Bingham et al. [**4**] or Embrechts et al. [**7**]. Is is useful to remark that the explicit forms of the right hand sides in the equations (2.1) and (2.2) are not needed; they follow from the existence of a non-trivial limit as was already illustrated before.

Along with the selection principle, a crucial role in the proofs will be played by the following lemma whose proof can be found in [**3**]:

LEMMA 1. *Assume that for* $m \in \{1, 2, \ldots\}$

$$\int_0^\infty e^{-mu} f(u) du = C m^{-\nu}$$

where $0 < C < \infty$, $\nu > 0$ *and* $f(u) \geqslant 0$. *Then* $f(u) = \frac{C}{\Gamma(\nu)} u^{\nu-1}$.

For later reference, we write down an explicit expression for the expectation of functions of two order statistics. Let $s \in \{0, 1, 2, \ldots\}$, $r \in \{1, 2, \ldots\}$ be fixed integers. Let k be any measurable function of two variates. If it exists, then the joint expectation of the quantity $k(X^*_{n-r-s}, X^*_{n-s})$ is given by

(2.3)
$$J_n^{r,s}(k) := \mathrm{E}[k(X^*_{n-r-s}, X^*_{n-s})] = \frac{c_{n,r,s}}{(r-1)!s!} \int_{x_-}^{x_+} W_n^{r,s}(x) [F(x)]^{n-r-s-1} \mathrm{d}F(x)$$

where

$$c_{n,r,s} := \frac{n!}{(n-r-s-1)!}$$

and

$$W_n^{r,s}(x) := \int_x^{x_+} k(x,y) [F(y) - F(x)]^{r-1} [1 - F(y)]^s \mathrm{d}F(y). \tag{2.4}$$

The above expression can be easily derived from a combinatorial argument. See for example [**1**] or [**2**].

We first change x into u by the substitution $F(x) = 1 - u/n$ or in terms of the tail quantile function by $x = U(n/u)$. Secondly, we change y into v by $F(y) = 1 - (uv)/n$ or by $y = U(n/(uv))$. Then (2.4) can be written in terms of the beta distribution

$$B_{p,q}(v) = \frac{(p+q-1)!}{(p-1)!(q-1)!} \int_0^v t^{p-1} (1-t)^{q-1} \mathrm{d}t.$$

where $p > 0$, $q > 0$ and $0 \leqslant v \leqslant 1$. Indeed,

$$W_n^{r,s}\left(U\left(\frac{n}{u}\right)\right) = \frac{(r-1)!s!}{(r+s)!} \left(\frac{u}{n}\right)^{r+s} H_n^{r,s}(u),$$

where in turn

$$H_n^{r,s}(u) := \int_0^1 k\left(U\left(\frac{n}{u}\right), U\left(\frac{n}{uv}\right)\right) \mathrm{d}B_{s+1,r}(v)\ . \tag{2.5}$$

Therefore, we conclude that (2.3) is also equal to

$$J_n^{r,s}(k) = \frac{c_{n,r,s}}{(r+s)!n^{r+s+1}} \int_0^\infty H_n^{r,s}(u) \left[1 - \frac{u}{n}\right]^{n-r-s-1} u^{r+s} I_{[0,n]}(u) \mathrm{d}u, \tag{2.6}$$

where I_A is the indicator function of the set A. This relation will be the starting point of the subsequent analysis.

3. Convergence of ratios of two order statistics with finite upper endpoint

The case of an infinite endpoint has been treated in [**3**]. We therefore treat the converse result for the ratio $U_n(r,s)$ as introduced in (1.2) for the case of finite x_+.

THEOREM 1. *Let $s \geqslant 0$, $r \geqslant 1$ be fixed integers. Let F be a distribution with finite upper endpoint x_+. If $U_n(r,s)$ converges in distribution to a non-degenerate limit, then for some $\rho > 0$, $1-F(x_+-1/x)$ varies regularly of order $-\rho$ as $x \to \infty$.*

PROOF. Take $0 \leqslant t \leqslant 1$ and choose $k(x,y) = I_{[0,t]}\left(\frac{x_+-y}{x_+-x}\right)$. Then (2.5) simplifies to

$$H_n^{r,s}(u) = \int_0^1 I_{[0,t]}\left(\frac{x_+ - U(n/(uv))}{x_+ - U(n/u)}\right) \mathrm{d}B_{s+1,r}(v).$$

Replace the expression in between the large brackets of the integrand by w. Then

$$H_n^{r,s}(u) = \int_0^1 I_{[0,t]}(w)\mathrm{d}B_{s+1,r}(\tau_n(w;u)) = B_{s+1,r}(\tau_n(t;u)),$$

where

$$\tau_n(t;u) := \frac{n}{u}\left\{1 - F\left[x_+ - \left(x_+ - U\left(\frac{n}{u}\right)\right)t\right]\right\}.$$

Consequently, the distribution function of $U_n(r,s)$, say $J_n^{r,s}(t)$, is given by

(3.1)
$$J_n^{r,s}(t) = \frac{c_{n,r,s}}{(r+s)!n^{r+s+1}}\int_0^\infty B_{s+1,r}(\tau_n(t;u))\left[1-\frac{u}{n}\right]^{n-r-s-1} u^{r+s} I_{[0,n]}(u)\mathrm{d}u.$$

For every natural n, $\tau_n(t;u)$ is bounded and monotonically non-decreasing in u and t. Therefore, by a bivariate extension of the selection principle (see for example Widder [**16**]) there exists a set of integers $n_1 < n_2 < \cdots$ and a bounded function $\tau^{(i)}(t;u)$ such that for all $u \geqslant 0$ and $t \in [0,1]$

$$\tau^{(i)}(t;u) = \lim_{i\to\infty} \tau_{n_i}(t;u).$$

We follow the procedure as in [**3**]. Take any natural integer m. First replace n by $m\,n_i$ in the above formula. Then replace the variable of integration u by vm. A little algebra transforms (3.1) into the form

$$J_{mn_i}^{r,s}(t) = \frac{c_{mn_i,r,s}}{(r+s)!n_i^{r+s+1}}\int_0^\infty B_{s+1,r}(\tau_{mn_i}(t;vm))\left[1-\frac{v}{n_i}\right]^{n_im-r-s-1} \times v^{r+s} I_{[0,mn_i]}(vm)\mathrm{d}v.$$

However, by its definition $\tau_{mn_i}(t,vm) = \tau_{n_i}(t,v)$ while also $I_{[0,mn_i]}(vm) = I_{[0,n_i]}(v)$. We now let $n_i \to \infty$ to find that for every $m \in \{1,2,\ldots\}$

$$J^{r,s}(t) = \frac{m^{r+s+1}}{(r+s)!}\int_0^\infty e^{-mu} B_{s+1,r}(\tau^{(i)}(t;u))u^{r+s}\mathrm{d}u,$$

where, by assumption, $J^{r,s}(t)$ can be taken to be the non-degenerate limit distribution of $J_n^{r,s}(t)$. Thanks to lemma 1, we can solve this integral equation and obtain that, independent of the subsequence,

$$B_{s+1,r}(\tau^{(i)}(t;u)) = J^{r,s}(t). \tag{3.2}$$

Both sides of this equation are distributions on $[0,1]$. Therefore, $\tau^{(i)}(t,u)$ does neither depend on the subsequence nor on the quantity u. Therefore, the sequence $\tau_n(t;u)$ itself has to converge to a function $a(t)$, independent of u. If we take $u=1$, we get for all $t \in [0,1]$ and for $n \to \infty$

$$\frac{1-F(x_+ - (x_+ - U(n))t)}{1-F(x_+ - (x_+ - U(n)))} \to a(t).$$

Putting $a_n = (x_+ - U(n))^{-1}$ and $t = y^{-1} (y \geqslant 1)$ we get for $n \to \infty$ that $a_n \to \infty$ and

$$\frac{1-F\left(x_+ - 1/(a_n y)\right)}{1-F\left(x_+ - 1/a_n\right)} \to a\left(\frac{1}{y}\right).$$

By the monotonicity of F and the theory of regular variation, $a(1/y) = y^{-\rho}$ for some $\rho \geqslant 0$ and $1 - F\left(x_+ - 1/x\right) = x^{-\rho} l(x)$ with l slowly varying. Since $J^{r,s}(t)$ is non-degenerate, $\rho > 0$. This finishes the proof. □

Notice that from (3.2) we find that $J^{r,s}(t) = B_{s+1,r}(a(t)) = B_{s+1,r}(t^\rho)$, where $J^{r,s}(t)$ is the limit distribution of $U_n(r,s)$.

4. Convergence of differences of two order statistics

We first formulate the direct theorem for the ratio $Q_n(r,s)$ as introduced in (1.3).

THEOREM 2. *Let $s \in \{0,1,\ldots\}$, $r \in \{1,2,\ldots\}$ be fixed integers. Assume $F \in \mathcal{C}_\gamma(g)$. Then there exist real constants $d_n = g(n)/U(n)$ such that $Q_n(r,s)$ converges in distribution to a non-degenerate limit, say $Y_{\gamma,r,s}$ where for $t > 0$*

$$P(Y_{\gamma,r,s} > t) = \frac{1}{(r+s)!} \int_{(-t\gamma)^{-1/\gamma}}^{\infty} e^{-u} u^{r+s} \left(\int_0^{(1+\gamma t u^\gamma)^{-1/\gamma}} \frac{(1-v)^{r-1} v^s}{B(s+1,r)} dv \right) du$$

if $\gamma < 0$ and $x_+ \neq 0$ and where

$$P(Y_{\gamma,r,s} > t) = \int_0^{(1+\gamma t)^{-1/\gamma}} \frac{(1-v)^{r-1} v^s}{B(s+1,r)} dv = B_{s+1,r}(\eta_\gamma(t)), \quad (1+\gamma t > 0)$$

if $\gamma \geqslant 0$ or $x_+ = 0$.

PROOF. We follow the classical Helly-Bray approach. Let k be an arbitrary bounded and continuous function. Then (2.6) becomes

$$\mathrm{E}\{k(Q_n(r,s))\} = \frac{c_{n,r,s}}{(r+s)! n^{r+s+1}} \int_0^\infty H_n^{r,s}(u) \left[1 - \frac{u}{n}\right]^{n-r-s-1} u^{r+s} I_{[0,n]}(u) \mathrm{d}u\ . \tag{4.1}$$

Here $H_n^{r,s}(u)$ can be written in an alternative form thanks to the symmetry property of the beta distribution in that $B_{s+1,r}(1-v) = 1 - B_{r,s+1}(v)$. Therefore,

$$H_n^{r,s}(u) = \int_0^1 k\left(\frac{\frac{U(n/(u(1-v))}{U(n/u)} - 1}{d_n}\right) \mathrm{d}B_{r,s+1}(v).$$

Replace the expression between brackets by a new variable w. Then

$$H_n^{r,s}(u) = \int_0^{a_n(u)} k(w)\mathrm{d}B_{r,s+1}(\mu_n(w;u)),$$

where

$$a_n(u) = \frac{x_+ - U(n/u)}{d_n U(n/u)}$$

and where

$$\mu_n(w;u) = 1 - \frac{n}{u}\left\{1 - F\left[U\left(\frac{n}{u}\right)(1 + d_n w)\right]\right\}.$$

Notice that for every $u > 0$, $\mu_n(.;u)$ is a probability distribution function on $[0,\infty)$.

Now define $f(x) = g(x)/U(x)$. Then $d_n = f(n)$ and

$$\mu_n(w;u) = 1 - \frac{n}{u}\left\{1 - F\left[U\left(\frac{n}{u}\right) + g\left(\frac{n}{u}\right)\frac{f(n)}{f(n/u)}w\right]\right\}.$$

Recall that our assumption on $F \in \mathcal{C}_\gamma(g)$ implies something on the behaviour of $g(x)$ in comparison with $U(x)$ or with $x_+ - U(x)$ when $x \to \infty$. It takes a bit of calculations to check that for every $\lambda > 0$

$$\begin{aligned}\lim_{x\to\infty}\frac{f(\lambda x)}{f(x)} &= \lim_{x\to\infty}\frac{g(\lambda x)}{g(x)}\left\{\frac{U(\lambda x) - U(x)}{g(x)}\frac{g(x)}{U(x)} + 1\right\}^{-1} \\ &= \begin{cases}\lambda^\gamma, & \text{if } \gamma < 0 \text{ and } x_+ \neq 0 \\ 1, & \text{if } \gamma \geqslant 0 \text{ or } x_+ = 0.\end{cases}\end{aligned}$$

Notice the somewhat unexpected discontinuity for $x_+ = 0$. The equation above shows that f is regularly varying with index γ if $\gamma < 0$ and $x_+ \neq 0$, or with index 1 if $\gamma \geqslant 0$ or $x_+ = 0$.

To finish the analysis, we better distinguish three cases:

Case 1. $x_+ = \infty$. Use part (*iii*) of Proposition 3 to see that for $n \to \infty$

$$\mu_n(w;u) \to 1 - \eta_\gamma(w),$$

which is known in the literature as a *generalized Pareto distribution.* This implies that for $n \to \infty$

$$H_n^{r,s}(u) \to H_\infty^{r,s}(u) := \int_0^\infty k(w)\mathrm{d}B_{r,s+1}(1 - \eta_\gamma(w)).$$

Case 2. $x_+ = 0$. Again, $\mu_n(w;u)$ converges to a generalized Pareto distribution concentrated on the interval $[0, -1/\gamma]$ where $\gamma < 0$ as $x_+ < \infty$. Indeed, for $n \to \infty$, $a_n(u) = -\frac{U(n)}{g(n)} \to -\frac{1}{\gamma}$. We find the same expression as in case 1.

Case 3. $x_+ < \infty$ and $x_+ \neq 0$. Now, $\mu_n(w;u) \to 1 - \eta_\gamma(u^\gamma w)$ as $n \to \infty$ and

$$\lim_{n\to\infty} a_n(u) = \lim_{n\to\infty} \frac{x_+ - U(n/u)}{g(n/u)} \frac{f(n/u)}{f(n)} = -\frac{1}{\gamma} u^{-\gamma}.$$

This implies that for $n \to \infty$

$$H_n^{r,s}(u) \to H_\infty^{r,s}(u) := \int_0^{-\frac{1}{\gamma}u^{-\gamma}} k(w)\mathrm{d}B_{r,s+1}(1 - \eta_\gamma(u^\gamma w)) \ .$$

For the three cases at hand we see that, under our conditions, $H_n^{r,s}(u)$ converges properly to $H_\infty^{r,s}(u)$. To finish the limiting procedure it suffices to perform the usual analysis to the expression (4.1) leading to

$$\mathrm{E}\{k(Q_n(r,s))\} \to \frac{1}{(r+s)!} \int_0^\infty e^{-u} u^{r+s} H_\infty^{r,s}(u) \mathrm{d}u.$$

Finally, put $k(w) = I_{[t,\infty)}(w)$ and perform a little algebra to find the expression in the statement of the theorem. This finishes the proof. □

The above result can also be cast in the form of a difference of normalized order statistics to which the classical Fisher–Tippett theorem [**8**] can be applied. For assume $F \in \mathcal{C}_\gamma(g)$. Then we can take the positive constants $a_n = g(n)$ and real constants $b_n = U(n)$ such that

$$Q_n(r,s) = \left\{ \frac{X_{n-s}^* - b_n}{a_n} - \frac{X_{n-r-s}^* - b_n}{a_n} \right\} \left\{ \frac{a_n}{b_n} \frac{X_{n-r-s}^* - b_n}{a_n} + 1 \right\}^{-1}$$

converges in distribution to a non-degenerate limit with tail distribution as described in theorem2. Recall that $g(n)/U(n) \to \gamma \geqslant 0$ while otherwise the limit is zero. Therefore, unless $\gamma > 0$, the asymptotic behaviour of $Q_n(r,s)$ is the same as the asymptotic behaviour of a normalized difference of two large order statistics.

In general, for these differences, the following converse result can be proven.

THEOREM 3. *Let $s \in \{0,1,2,\ldots\}$, $r \in \{1,2,\ldots\}$ be fixed integers. Let $X_1, X_2, \ldots$ be a sample of independent r.v. with common distribution F and tail quantile function U. Let $\{a_n\}$ be a regularly varying sequence of positive constants. If $D_n(r,s) := (X_{n-s}^* - X_{n-r-s}^*)/a_n$ converges in distribution to a non-degenerate limit, then for some auxiliary function g, $F \in \mathcal{C}_\gamma(g)$.*

PROOF. Take $t \in \mathbb{R}_+$ and choose $k(x,y) = I_{[0,t]}(\frac{y-x}{a_n})$ in (2.5). Then, by the same operation as before, (2.5) is equal to

$$\begin{aligned} H_n^{r,s}(u) &= \int_0^1 I_{[0,t]} \left(\frac{U(n/(u(1-v)) - U(n/u)}{a_n} \right) \mathrm{d}B_{r,s+1}(v) \\ &= \int_0^{c_n(u)} I_{[0,t]}(w) \mathrm{d}B_{r,s+1}(\nu_n(w;u)), \end{aligned}$$

where

$$c_n(u) = \frac{x_+ - U(n/u)}{a_n} \tag{4.2}$$

and

$$\nu_n(w;u) = 1 - \frac{n}{u}\left\{1 - F\left(U\left(\frac{n}{u}\right) + a_n w\right)\right\}. \tag{4.3}$$

Notice that for every $u > 0$, $\nu_n(.;u)$ is a probability distribution function on $[0,\infty)$. This means that for every $t \geqslant 0$, $H_n^{r,s}(u) = B_{r,s+1}(\nu_n(t;u))$. Moreover, the distribution of $D_n(r,s)$, say $G_n^{r,s}(t)$, is given by

$$G_n^{r,s}(t) = \frac{c_{n,r,s}}{(r+s)!n^{r+s+1}}\int_0^\infty B_{r,s+1}(\nu_n(t;u))\left[1-\frac{u}{n}\right]^{n-r-s-1} u^{r+s} I_{[0,n]}(u)\mathrm{d}u.$$

Perform the same operations as in the proof of Theorem 2. Then

$$\begin{aligned}&G_{mn_i}^{r,s}\left(\frac{a_{n_i}}{a_{mn_i}}t\right)\\ &= \frac{c_{mn_i,r,s}m^{r+s+1}}{(r+s)!(mn_i)^{r+s+1}}\int_0^\infty B_{r,s+1}\left(\nu_{n_i}(t;u)\right)\left[1-\frac{u}{n_i}\right]^{mn_i-r-s-1} u^{r+s} I_{[0,n_i]}(u)\mathrm{d}u.\end{aligned}$$

We can take the limit $i \to \infty$ on the right hand side. On the left the sequence of distributions converges by assumption. Also the argument converges by the regular variation of the sequence a_n. If we put $a_{mn_i}/a_{n_i} \to p_m$, then $p_m = m^\sigma$ for some $\sigma \in \mathbb{R}$. In particular, $p_1 = 1$. We conclude that for every $m \in \{1,2,\ldots\}$

$$G^{r,s}(t/p_m) = \frac{m^{r+s+1}}{(r+s)!}\int_0^\infty e^{-mu} B_{r,s+1}(\nu^{(i)}(t;u))u^{r+s}\mathrm{d}u, \tag{4.4}$$

where $G^{r,s}(t)$ is the non-degenerate limit distribution of $G_n^{r,s}(t)$. Now, divide both sides of (4.4) by $G^{r,s}(t)$ and change u into $-\log x$. Then for all $m \in \{1,2,\ldots\}$ and $t > 0$

$$\begin{aligned}\frac{G^{r,s}(t/p_m)}{m^{r+s+1}G^{r,s}(t)} &= \int_0^1 x^{m-1}\frac{B_{r,s+1}(\nu^{(i)}(t;-\log x))(-\log x)^{r+s}}{(r+s)!G^{r,s}(t)}\mathrm{d}x\\ &:= \int_0^1 x^{m-1} f^{r,s}(t;x)\mathrm{d}x.\end{aligned}$$

If we put $m = 1$, then we see that $f^{r,s}(t;x)$ is a density function on $[0,1]$ with moments given by

$$\int_0^1 x^m f^{r,s}(t;x)\mathrm{d}x = \frac{G^{r,s}(t/p_{m+1})}{(m+1)^{r+s+1}G^{r,s}(t)}.$$

These moments uniquely determine the density. Transforming back we find that the limit function $\nu^{(i)}(t;u)$ does not depend on the subsequence we have used. Therefore, the whole sequence $\nu_n(t;u)$ has to converge to a function $\nu(t;u)$. Now, define for all $u > 0$: $g(u) = a_{[u]}$. Then, putting $u = 1$, we have that for all $t > 0$ and for $n \to \infty$

$$1 - n\{1 - F(U(n) + g(n)t)\} \to \nu(t,1).$$

So, using part (iii) of Proposition 3, we can conclude that $F \in \mathcal{C}_\gamma(g)$. This finishes the proof. $\square$

From (4.3) and (4.2), we find that for all $t > 0$ and for $n \to \infty$

$$\nu_n(t;u) = 1 - \frac{n}{u}\left\{1 - F\left(U\left(\frac{n}{u}\right) + g\left(\frac{n}{u}\right)\frac{g(n)}{g(n/u)}t\right)\right\} \to \nu(t;u) = 1 - \eta_\gamma(u^{-\gamma}t)$$

and

$$c_n(u) = \frac{x_+ - U(n/u)}{g(n/u)}\frac{g(n/u)}{g(n)} \to c(u) = -\frac{1}{\gamma}u^{-\gamma}$$

if $x_+ < \infty$. This means that, if $x_+ < \infty$, $\nu(t;u)$ is a probability distribution on $[0, c(u)]$. Finally, we conclude that

$$G^{r,s}(t) = \frac{1}{(r+s)!}\int_0^\infty B_{r,s+1}(\nu(t;u))e^{-u}u^{r+s}\mathrm{d}u,$$

where $G^{r,s}(t)$ is the limit distribution function of $D_n(r,s)$.

References

[1] B. C. Arnold, N. Balakrishnan and H. N. Nagaraja (1992), *A First Course in Order Statistics*, Wiley, New York.

[2] D.Z. Arov and A.A. Bobrov (1960), *The extreme terms of a sample and their role in the sum of independent random variables*, Theory Prob. Appl. **5**, 377–396.

[3] N.H. Bingham and J.L. Teugels (1979), *Conditions implying domains of attraction*, In: *Proceedings of the Sixth Conference on Probability Theory* in Brasov/Romania, 23–34.

[4] N.H. Bingham, C.M. Goldie and J.L. Teugels (1987), *Regular Variation*, Cambridge University Press, Cambridge.

[5] L. Breiman (1965), *On some limit theorems similar to the arc-sine law*, Theory Prob. Appl. **10**, 323–331.

[6] M. Dwass (1964), *Extremal processes*, Ann. Math. Statist. **35**, 1718–1725.

[7] P. Embrechts, C. Klüppelberg and T. Mikosch (1997), *Modelling Extremal Events*, Springer-Verlag, Berlin.

[8] R.A. Fisher and L.H.C. Tippett (1928), *Limiting forms of the frequency distributions of the largest or smallest member of a sample*, Proc. Cambridge Phil. Soc. **24**, 180–190.

[9] B.V. Gnedenko (1943), *Sur la distribution limite du terme maximum d'une série aléatoire*, Annals Math. **44**, 423–453.

[10] L. de Haan (1970), *On Regular Variation and its Application to the Weak Convergence of Sample Extremes*, Math. Centre Tracts 32, Amsterdam.

[11] J. Karamata (1930), *Sur un mode de croissance régulière des fonctions*, Mathematica (Cluj) **4**, 38–53.

[12] J. Lamperti (1964), *On extreme order statistics*, Ann. Math. Statist. **35**, 1726–1736.

[13] H.J. Rossberg (1963), *Über das asymptotische Verhalten der Rand- und Zentralglieder einer Variationsreihe. I*, Publ. Math. Inst. Hung. Acad. Sci. Ser. A **8**, 463–468.

[14] H.J. Rossberg (1967), *Über das asymptotische Verhalten der Rand- und Zentralglieder einer Variationsreihe. II*, Publ. Math. **14**, 83–90.

[15] B. Smid and A.J. Stam (1975), *Convergence in distribution of quotients of order statistics*, Stochastic Proc. Appl. **3**, 287–292.

[16] D.V. Widder (1946), *The Laplace Transform*, Princeton University Press, Princeton.

Katholieke Universiteit Leuven
Belgium