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GENERAL TAUBERIAN THEOREM IN S'_+

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ABSTRACT. We show how an idea developed by J. Karamata in [1] and [2] can be used to prove a Tauberian type theorem for generalized functions.

Introduction

Without doubt the mathematical word sets a high value on the results of J. Karamata, especially on those related to the summability and to the inverse processes of summability. One can find the most general idea of these problems, first of all in the paper [2] and in the book [1]. We cite a theorem which represents a part of it, and which we use in this paper.

THEOREM A. [2] *If the kernel $\psi(x,t)$ of the Ψ-summability*

$$\Psi(x) = \int_0^\infty \psi(x,t)s(t)\,dt, \quad (x \to \infty) \tag{I}$$

satisfies

$$\psi(x,t) \geqslant 0; \ \int_0^\infty \psi(x,t)\,dt = 1 \text{ for all } x > 0 \text{ and } t \geqslant 0; \tag{II}$$

$$\int_0^m \psi(x,t)\,dt \to 0, \quad x \to \infty \text{ for every } m > 0,$$

and if L is a function monotone increasing to infinity, which for a suitable choice of y satisfies

$$\int_0^\infty \psi(x,t)|\log(L(y)/L(t))|\,dt = O(1), \ x, y \to \infty, \tag{III}$$

then from the Ψ-summability of $s(t)$ follows the convergence of $s(t)$ to the same limit provided that

$$\int_0^x L(t)\,ds(t) = o(L(x)), \quad x \to \infty; \tag{IV}$$

or

$$s(t') - s(t) \to 0 \text{ for every } t \leqslant t' \leqslant T_\lambda(t), \quad t \to \infty, \tag{V}$$

where $T_\lambda(t) = V\{\lambda L(t)\}$, V is the inverse function of L and λ is a number, $\lambda > 0$.

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This form of Theorem A calls for some explanations. The function s is supposed to be of bounded variation on $(0,b)$ for every $b>0$. The Ψ-summability of s, denoted by (I), means that there exists a number α such that the integral in (I) exists for every $x>0$ and that

$$\lim_{x\to\infty}\int_0^\infty \psi(x,t)s(t)\,dt=\alpha.$$

The phrase: "which for a suitable choice of y..." means only that $y\to\infty$ together with x in a suitable combination.

This Theorem A gives in fact conditions to find a function L so that (IV) or (V) become a convergence condition of the Ψ-summability. Of course, the function L is not defined uniquely by (III). One has to find such an L to have the best condition for $s(t)$ in the given case.

In the meantime, to this day, a theory of integral transforms of generalized functions has been developed and in the last twenty years Tauberian type theorems have been proved (cf. [3–7]). At any rate V. S. Vladimirov and his pupils Yu. N. Drozhinov and B. I. Zavialov had a pioneering role in this field of mathematics.

Tauberian type theorems in fact give inverse processes for summability defined by the integral transforms.

In this paper we shall use only the idea exploited in Theorem A to prove a general Tauberian type theorem for integral transforms of generalized functions of the form

$$x\to\Psi(x)=\langle s(t),\psi(x,t)\rangle,\ \ s\in\mathcal{S}_+,$$

adjusting the assumptions and the method of the proof to the global character of distributions.

Certainly, this result can be improved. It could be extended to other spaces of generalized functions or treated in many dimensional case, as well. Our aim is only to show how the idea of J. Karamata has a general character and how it can be used to obtain results in the theory of generalized functions.

Notation and definitions

We denote by $\mathcal{S}([0,\infty))\equiv\mathcal{S}_+$ the basic space of rapidly diminishing functions. $f\in\mathcal{S}_+$ if $f\in\mathcal{C}^\infty([0,\infty))$ and

$$\|f\|_p=\sup_{x\geqslant 0,\,0\leqslant\alpha\leqslant p}(1+|x|^2)^{p/2}|f^{(\alpha)}(x)|<\infty,\ \ p=0,1,\dots;$$

$\|f\|_p$, $p=0,1,\dots$ are the norms in $\mathcal{S}_+$. Then by $\mathcal{S}'_+$ we denote the space of continuous linear functionals on $\mathcal{S}_+$, called the space of generalized functions of slow growth (space of tempered distributions).

In $\mathcal{S}'_+$ a convolution is defined, denoted by $*$: for f,g in $\mathcal{S}'_+$

$$\langle f*g,\varphi\rangle=\langle f(x)\times g(y),\xi(x)\eta(y)\varphi(x+y)\rangle \tag{1}$$

where $\times$ denotes the direct product and

$$\xi,\eta\in\mathcal{C}^\infty,\ \ |\xi^{(\beta)}(x)|\leqslant C_\beta,\ \ |\eta^{(\beta)}(x)|\leqslant C_\beta;$$

ξ, η are equal 1 in $(\operatorname{supp} f)^\epsilon$ and $(\operatorname{supp} g)^\epsilon$ and are equal 0 outside of $(\operatorname{supp} f)^{2\epsilon}$, $(\operatorname{supp} g)^{2\epsilon}$ respectively, $\epsilon > 0$ (cf. [8]).

$\mathcal{S}'_+$ is an algebra if for the operation of multiplication we take the convolution. The space $\mathcal{O}_M([0,\infty))$ is the space of functions of "slow growth"; $f \in \mathcal{O}_M([0,\infty))$ if the function and its derivatives have a majorant polynomial on $[0,\infty)$.

By $H(t)$ we denote the Heaviside function: $H(t) = 1$, $t \geqslant 0$; $H(t) = 0$, $t < 0$.

Tauberian type theorem for tempered distributions

THEOREM. *Let the function $\psi(x,t)$ satisfy the following conditions:*

$$\psi(x,t) \in \mathcal{S}_+,\ x > 0;\ \psi(rx,t) = \lambda(r)\psi(x,\lambda(r)t),\ t > 0,\ x > 0, \tag{2}$$

where $\lambda(r) > 0$, $r \geqslant r_0 > 0$ and $\lambda(r) \to \infty$, $r \to \infty$. Let $L \in \mathcal{O}_M([0,\infty))$, $L(t) \neq 0$, $t \geqslant 0$. Then every $s \in \mathcal{S}'_+$ can be written in the form:

$$s = LD + [H * (L^{(1)}D)], \tag{3}$$

where

$$D = L^{-2}[H * (Ls^{(-1)})] \in \mathcal{S}'_+. \tag{4}$$

Suppose that

$$L(t/\lambda(r))D(t/\lambda(r)) \to 0,\ r \to \infty,\ in\ \mathcal{S}'_+ \tag{5}$$

and that

$$\lambda^{-1}(r)L^{(1)}(t/\lambda(r))D(t/\lambda(r)) \to 0,\ r \to \infty,\ in\ \mathcal{S}'_+, \tag{6}$$

as well. Then from

$$\Psi(x) \equiv \langle s(t), \psi(x,t)\rangle \to s_0,\ x \to \infty, \tag{7}$$

it follows that

$$s(t/\lambda(r)) \to s_0, \quad r \to \infty, \quad in\ \mathcal{S}'_+. \tag{8}$$

PROOF. First we prove that

$$\langle H(t), \psi(x,t)\rangle = A, x > 0 \text{ where } A \text{ is a constant.} \tag{9}$$

By the properties of ψ:

$$\begin{aligned}\langle H(t), \psi(x,t)\rangle &= \lambda(x)\langle H(t), \psi(1, \lambda(x)t)\\ &= \langle H(t/\lambda(x)), \psi(1,t)\rangle\\ &= \langle H(t), \psi(1,t)\rangle = A, \quad x > 0.\end{aligned}$$

For simplicity we may assume that $A = 1$.

In the second step we prove that $s(x)$ can be written in the form (3), where $D \in \mathcal{S}'_+$ is given by (4). This will follow by properties of the derivative of the convolution when we substitute in $LD + (H * L^{(1)}D)$ the value of D given by (4):

$$\begin{aligned} LD + [H * (L^{(1)}D)] &= L^{-1}[H(t) * (Ls^{(1)})] + H * \{L^{(1)}L^{-2}[H * (Ls^{(1)})]\} \\ &= L^{-1}[H * (Ls)^{(1)}] - L^{-1}[H * (L^{(1)}s)] \\ &\quad + H * \{L^{(1)}L^{-2}[H * (Ls)^{(1)}]\} - (H * \{L^{(1)}L^{-2}[H * (L^{(1)}s)]\} \\ &= s - L^{-1}[H * (L^{(1)}s)] + H * (L^{(1)}L^{-1}s) \\ &\quad + H * \{L^{-1}[H * (L^{(1)}s)]\}^{(1)} - [H * (L^{-1}L^{(1)}s)] = s. \end{aligned}$$

In the third step we show that we can menage the proof of the theorem in the following way: Supposing that $\Psi(x) \to s_0$, $x \to \infty$, we prove that

$$\lim_{x,r\to\infty} \langle \Psi(x)H(y) - s(y/\lambda(r)), \varphi(y)\rangle = 0 \tag{10}$$

for every $\varphi \in \mathcal{S}_+$. In this case we may conclude that $s(y/\lambda(r)) \to s_0 H(y)$, $r \to \infty$, in $\mathcal{S}'_+$ because for every $\varphi \in \mathcal{S}_+$

$$\begin{aligned} &\langle \Psi(x)H(y) - s(y/\lambda(r)), \varphi(y)\rangle \\ &\qquad = \langle (\Psi(x) - s_0)H(y) - (s(y/\lambda(r)) - s_0H(y)), \varphi(y)\rangle \\ &\qquad = (\Psi(x) - s_0)\langle H(y), \varphi(y)\rangle - \langle s(y/\lambda(r)) - s_0 H(y), \varphi(y)\rangle. \end{aligned} \tag{11}$$

But by (11) it follows that we may assume that $s_0 = 0$ without loss of generality. We choose just this way of the proof. Let us consider

$$J(x,r) \equiv \langle \Psi(x)H(y) - s(y/\lambda(r)),\ \varphi(y)\rangle, \quad x > 0.$$

By (7), (9) and the properties of the direct product (cf. [8])

$$\begin{aligned} J(x,r) &= \langle H(y)\langle s(t/\lambda(x)), \psi(1,t)\rangle - \langle s(y/\lambda(r))H(t), \psi(1,t)\rangle, \varphi(y)\rangle \\ &= \langle H(y)\langle s(t/\lambda(x)), \psi(1,t)\rangle, \varphi(y)\rangle - \langle\langle s(y/\lambda(r))H(t), \psi(1,t)\rangle, \varphi(y)\rangle \\ &= \langle H(y), \varphi(y)\rangle\langle s(t/\lambda(x)), \psi(1,t)\rangle - \langle s(y/\lambda(r)), \varphi(y)\rangle. \end{aligned} \tag{12}$$

Substituting (3) into (12) we rewrite (12) as

$$\begin{aligned} J(x,r) &= \langle H(y), \varphi(y)\rangle\langle L(t/\lambda(x))D(t/\lambda(x)), \psi(1,t)\rangle \\ &\quad - \langle L(y/\lambda(r))D(y/\lambda(r)), \varphi(y)\rangle \\ &\quad + \langle H(y), \varphi(y)\rangle\langle G(t/\lambda(x)), \psi(1,t)\rangle - \langle G(y/\lambda(r)), \varphi(y)\rangle \\ &= J_1(x,r) + J_2(x,r), \end{aligned} \tag{13}$$

where $G = H * L^{(1)}D$. Let us consider the second part of (13) related to the generalized function G:

$$\begin{aligned}
J_2(x,r) &= \langle H(y),\varphi(y)\rangle\langle G(t/\lambda(x)),\psi(1,t)\rangle - \langle G(y/\lambda(r)),\varphi(y)\rangle\\
&= \langle H(y),\langle G(t/\lambda(x)),\psi(1,t)\varphi(y)\rangle\rangle - \langle G(y/\lambda(r)),\langle H(t),\psi(1,t)\rangle\varphi(y)\rangle\\
&= \langle H(y)G(t/\lambda(x)),\psi(1,t)\varphi(y)\rangle - \langle G(y/\lambda(r))H(t),\psi(1,t)\varphi(y)\rangle\\
&= \langle H(y)G(t/\lambda(x)) - H(t)G(y/\lambda(r)),\psi(1,t)\varphi(t)\rangle\\
&= \langle H(y)(H * L^{(1)}D)(t/\lambda(x)) - H(t)(H * L^{(1)}D)(y/\lambda(r),\psi(1,t)\varphi(y)\rangle\\
&= \lambda(x)\lambda(r)\langle H(y)(H*L^{(1)}D)(t) - H(t)(H*L^{(1)}D)(y),\psi(1,\lambda(x)t)\varphi(\lambda(r)y)\rangle\\
&= \lambda(x)\lambda(r)\big[\langle H(y)(H * L^{(1)}D)(t),\psi(1,\lambda(x)t)\varphi(\lambda(r)y)\rangle\\
&\qquad - \langle H(t)(H * L^{(1)}D)(y),\psi(1,\lambda(x)t)\varphi(\lambda(r)y)\rangle\big]\\
&= \lambda(x)\lambda(r)\big[\langle L^{(1)}(t)D(t),\langle H(y)H(u),\psi(1,\lambda(x)(t+u))\varphi(\lambda(r)y\rangle\rangle\\
&\qquad - \langle L^{(1)}(y)D(y),\langle H(t)H(v),\psi(1,\lambda(x)t)\varphi(\lambda(r)(y+v))\rangle\rangle\big]\\
&= \lambda^{-1}(x)\langle L^{(1)}(t/\lambda(x))D(t/\lambda(x)),\langle H(y)H(u),\psi(1,t+u)\varphi(y)\rangle\rangle\\
&\quad - \lambda^{-1}(r)\langle L^{(1)}(y/\lambda(r))D(y/\lambda(r)),\langle H(t)H(v),\psi(1,t)\varphi(y+v)\rangle\rangle.
\end{aligned}$$

Since

$$\langle H(y)H(u),\psi(1,t+u)\varphi(y)\rangle \equiv \alpha(t) \in \mathcal{S}_+$$

and

$$\langle H(t)H(v),\psi(1,t)\varphi(y+v)\rangle \equiv \beta(y) \in \mathcal{S}_+,$$

$J_2(x,r)$ is given by

$$\begin{aligned}
J_2(x,r) = \lambda^{-1}(x)\langle L^{-1}(t/\lambda(x))D(t/\lambda(x)),\alpha(t)\rangle\\
- \lambda^{-1}(r)\langle L^{(1)}(y/\lambda(r))D(y/\lambda(r)),\beta(y)\rangle.
\end{aligned}$$

Now we have in $\mathcal{S}'_+$

$$\lim_{x,r\to\infty} J(x,r) = \lim_{x,r\to\infty} J_1(x,r) + \lim_{x,r\to\infty} J_2(x,r) = 0$$

because in $\mathcal{S}'_+$

$$\lim_{x,r\to\infty} J_1(x,r) = 0 \text{ by (5) and } \lim_{x,r\to\infty} J_2(x,r) = 0$$

by (6), which completes the proof of the theorem. □

Condition (5) in the theorem has the role of a convergence condition, while (6) defines the function $L(t)$. Since (6) is not simple we prove:

COROLLARY. *If in the theorem the function L has the additional property: Let*

$$L^{(1)}(t/\lambda(r))L^{-1}(t/\lambda(r)) = \lambda^{\gamma}(r)W(t,r) \textit{ for an } \gamma \leqslant 1,$$

where $W(t,r)\varphi(t) \to V(t)\varphi(t)$ in $\mathcal{S}_+$ for every $\varphi \in \mathcal{S}_+$, then (6) is satisfied and the theorem holds.

Proof. For any $\varphi \in \mathcal{S}_+$

$$
\begin{aligned}
\lambda^{-1}(x)\langle L^{(1)}(t/\lambda(r))D(t/\lambda(r)),\varphi(t)\rangle \\
&= \lambda^{\gamma-1}(r)\langle L(t/\lambda(r))D(t/\lambda(r))W(t,r),\varphi(t)\rangle \\
&= \lambda^{\gamma-1}(r)\langle L(t/\lambda(r))D(t/\lambda(r)),V(t)\varphi(t)\rangle \\
&\quad + \lambda^{\gamma-1}(r)\langle L(t/\lambda(r))D(t/\lambda(r)),(W(t,r)-V(t))\varphi(t)\rangle \\
&= I_1(r)+I_2(r).
\end{aligned}
\tag{14}
$$

By (5), $I_1(r)\to 0$, $r\to\infty$. Since $\{L(t/\lambda(r))D(t/\lambda(r)) : r \geqslant r_0 > 0\}$ is a bounded set in $\mathcal{S}'_+$, we can use Schwartz's theorem for $I_2(r)$ which asserts that there exist $K \geqslant 0$ and $m \in \mathcal{N}_0$ such that

$$
\left|\langle L(t/\lambda(r))D(t/\lambda(r)),(W(t,r)-V(t))\varphi(t)\rangle\right| \leqslant K\|(W(t,r)-V(t))\varphi(t)\|_m \tag{15}
$$

(cf. [8]). Now, by (15) it follows that $I_2(r)\to 0$, $r\to\infty$, which proves (6). This completes the proof of the theorem. □

Since tempered distributions have been defined as continuous functionals on $\mathcal{S}_+$ we had to adjust the assumptions and the method of proof of our theorem to the global structure of them. Therefore, it is not easy to compare Theorem A and our theorem. Thus, the function $\dfrac{x}{(1+xt)^2}$ can be the kernel in Theorem A but not in our theorem. On the other hand $(x-t)e^{-t/x}$ satisfies the assumptions on the kernel in our theorem but not in Theorem A. Finally the function xe^{-xt} can be the kernel in both cases. We have a similar situation with the function L.

We illustrate the theorem by the Laplace transform. Let $\psi(x,t) = xe^{-xt}$, $x > 0$. Then $\psi(x,t)\in\mathcal{S}_+$ and satisfies (2) with $\lambda(r) = r$. One possibility for $L(t)$ is $L(t) = (t+1)^a$, $a\in\mathcal{R}$, $a\neq 0$. We can apply the corollary proved above. Then

$$
L^{(1)}(t/r)L^{-1}(t/r) = a\frac{r}{t+r}.
$$

Hence $W(t,r) = \dfrac{ar}{t+r}$ and $\dfrac{ar}{t+r}\varphi(t)\to a\varphi(t)$, $r\to\infty$, in $\mathcal{S}_+$ and $\gamma = 0$.

By the theorem it follows now: If $\langle s(t), xe^{-xt}\rangle \to s_0$, $x\to\infty$, and if

$$
\Big(\frac{r}{t+r}\Big)^a\Big(H(u)*(u+1)^a\frac{d}{du}s(u)\Big)(t/r)\to 0,\quad r\to\infty,\ \text{in } \mathcal{S}'_+
$$

for an $a\in\mathcal{R}$, $a\neq 0$, then $s(t/r)\to s_0$, $r\to\infty$, in $\mathcal{S}'_+$.

If $a = 0$, then by (3) $s(x) = D(x)$ and the process is useless.

Another possibility is to take $L(t) = e^{-at}$, $a\in\mathcal{R}_+$. Then $L^{(1)}(t)L^{-1}(t) = -a$. Condition (6) is satisfied and the convergence condition (5) has the form

$$
e^{ax/r}\Big(H(u)*e^{-au}\frac{d}{du}s(u)\Big)(x/r)\to 0,\quad r\to\infty,\ \text{in } \mathcal{S}'_+.
$$

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