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REGULAR VARIATION AND THE FUNCTIONAL CENTRAL LIMIT THEOREM FOR HEAVY TAILED RANDOM VECTORS

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ABSTRACT. Multivariable regular variation is used, along with the martingale central limit theorem, to give a very simple proof that the partial sum process for a sequence of independent, identically distributed random vectors converges to a Brownian motion whenever the summands belong to the generalized domain of attraction of a normal law. This includes the heavy tailed case, where the covariance matrix is undefined because some of the marginals have infinite variance.

1. Introduction

Regular variation is a powerful tool in probability theory, and in many other areas of mathematics. Bingham, Goldie, and Teugels [**2**] catalog numerous applications of the one variable theory. Feller [**4**] gives an elegant proof of the central limit theorem using regular variation, including the infinite variance case. In fact, the centered and normalized partial sums of independent and identically distributed random variables are asymptotically normal if and only if the truncated variance function is slowly varying, so that even the statement of the central limit theorem properly includes the idea of regular variation. Meerschaert and Scheffler [**16**, Theorem 8.1.7] provides an extension of this result to random vectors, using multivariable regular variation. Multivariable regular variation can be traced back to the works of Stam [**25**], Stadtmüller and Trautner [**24**], Jakimiv [**10**], de Haan et al. [**5, 6**], and Ostrogorski [**17, 18, 19**]. The theory was extended in Meerschaert [**13**] and Meerschaert and Scheffler [**15**] to the generality needed for operator-normed limit theorems in probability. The book [**16**] contains a detailed introduction to the general theory of multivariable regular variation, independent of probabilistic considerations.

In this paper, multivariable regular variation is used to prove the functional central limit theorem for independent and identically distributed random vectors

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on a finite dimensional real vector space. It is assumed that the sequence of partial sums can be centered (by subtracting the mean) and normalized (by applying an appropriate linear operator) so that the resulting sequence converges in distribution to a Gaussian limit. In this case, the distribution of these random vectors is said to belong to the generalized domain of attraction of a multivariate normal law. This includes the classical situation, in which the covariance matrix of the summands exists, but also allows that the covariance matrix is undefined. This can occur when some of the marginals have an infinite variance, as long as the multivariable regular variation condition specified in [**16**] holds. The main idea of the proof is to verify the conditions for the martingale central limit theorem, using multivariable regular variation.

2. Result

Let $\xi, \xi_1, \xi_2, \ldots$ be independent and identically distributed random vectors in $\mathbb{R}^d$. We say that the probability distribution $\mathcal{L}(\xi)$ belongs to the generalized domain of attraction (GDOA) of a d-variate normal law if there exist a sequence of invertible linear operators B_n on $\mathbb{R}^d$ and a sequence of nonrandom vectors b_n such that

$$\mathcal{L}\left(B_n \sum_{i=1}^{n} \xi_i - b_n\right) \Rightarrow N(0, I). \tag{2.1}$$

Here $\Rightarrow$ denotes weak convergence, and I is the identity matrix. Then $\mathcal{L}(\xi)$ is full (e.g., see Lemma 3.3.3 in [**16**]). That is, $\mathcal{L}(\xi)$ is not supported on any $d-1$ dimensional hyperplane. Equivalently, $\langle \xi, \theta \rangle$ is nondegenerate $\forall \theta$. Here $\langle \cdot, \cdot \rangle$ denotes the usual inner product on $\mathbb{R}^d$. Throughout this article all distributions will be assumed to be full. Moreover, if $\mathcal{L}(\xi)$ is in the GDOA of the multivariate normal law then Theorem 3 in Hudson, Veeh, and Weiner [**9**] shows that $E\|\xi\| < \infty$. Therefore, without loss of generality, we will assume that $E\xi = 0$, and in this case one may take $b_n = 0$, see Hahn and Klass [**8**]. According to Meerschaert [**14**], we may assume that $B_n = n^{-1/2} L_n$ where L_n is slowly varying, which means that $L_{[\lambda n]} L_n^{-1} \to I$ for all $\lambda > 0$. Consequently, $\|B_n\| \to 0$. Define the partial sum process $M_n(t) = 0$ for $t \in [0, 1/n)$, and for $t \geqslant 1/n$, let

$$M_n = B_n \sum_{i=1}^{[nt]} \xi_i, \tag{2.2}$$

where $[\,\cdot\,]$ is the greatest integer function. Then M_n has sample paths in the space $D([0,\infty), \mathbb{R}^d) = \{f : [0,\infty) \to \mathbb{R}^d : \forall t\ f(t+) = f(t) \text{ and } f(t-) \text{ exists}\}$. We endow this space with the usual topology, and we let W be a realization of standard Brownian motion on this space.

THEOREM 2.1. *Suppose that $\xi_1, \xi_2, \ldots$ are independent and identically distributed mean zero random vectors in $\mathbb{R}^d$ whose distribution belongs to the generalized domain of attraction of a d-variate normal law, so that* (2.1) *holds. Then $M_n \Rightarrow W$ in $D([0,\infty), \mathbb{R}^d)$ where M_n is the partial sum process defined in* (2.2) *above.*

PROOF. The proof of Theorem 2.1 is an application of the martingale central limit theorem that appears in Ethier and Kurtz [**3**, Theorem 7.1.4a, p. 339]. In the statement below, which is included for convenience, M_n^i is the i^{th} coordinate of M_n, and $[M_n^i, M_n^j]$ is the cross variation process. (See [**3**, p. 79].) $\square$

THEOREM 2.2. (Martingale Central Limit Theorem) *For $n = 1, 2 \ldots$, let $\{\mathcal{F}_t^n\}$ be a filtration and let M_n be an $\{\mathcal{F}_t^n\}$ local martingale with $M_n(0) = 0$ and sample paths in $D([0,\infty), \mathbb{R}^d)$. Let $A_n = (A_n^{ij})$ be symmetric $d \times d$ matrix valued processes such that $A_n^{ij} = [M_n^i, M_n^j]$ has sample paths in $D([0,\infty), \mathbb{R})$ and $A_n(t) - A_n(s)$ is nonnegative definite for $t > s \geqslant 0$. If:*

(i) *for each $T > 0$, $\lim_{n\to\infty} E \sup_{t \leqslant T} \|M_n(t) - M_n(t-)\| = 0$; and*
(ii) *$A_n(t) \to tI$ in probability;*

then $M_n \Rightarrow W$, a standard Brownian motion on $D([0,\infty), \mathbb{R}^d)$.

Define the empirical covariance operator

$$C_n = \sum_{i=1}^{n} \xi_i \xi_i^T \tag{2.3}$$

where ^{T} denotes the transpose, and let $A_n(t) = 0$ for $t < 1/n$,

$$A_n(t) = B_n C_{[nt]} B_n^T = \sum_{i=1}^{[nt]} B_n \xi_i \xi_i^T B_n^T$$

for $t \geqslant 1/n$. It is obvious that M_n is a martingale adapted to the natural filtration, $\mathcal{F}_t^n = \sigma(M_n(s) : s \leqslant t) = \sigma(\xi_i : i \leqslant nt)$ with sample paths in $D([0,\infty), \mathbb{R}^d)$.

Clearly, A_n^{ij} has sample paths in $D([0,\infty), \mathbb{R})$. Also, if $t > s \geqslant 0$, and $\theta \in \mathbb{R}^d$, then

$$\begin{aligned}
\langle A_n(t)\theta, \theta\rangle &= \sum_{i=1}^{[nt]} \langle B_n \xi_i \xi_i^T B_n^T \theta, \theta\rangle = \sum_{i=1}^{[nt]} \langle \xi_i \xi_i^T B_n^T \theta, B_n^T \theta\rangle \\
&= \sum_{i=1}^{[nt]} (B_n^T\theta)^T \xi_i \xi_i^T (B_n^T \theta) = \sum_{i=1}^{[nt]} \langle \xi_i, B_n^T \theta\rangle^2 \\
&\geqslant \sum_{i=1}^{[ns]} \langle \xi_i, B_n^T \theta\rangle^2 = \langle A_n(s)\theta, \theta\rangle
\end{aligned}$$

so that $A_n(t) - A_n(s)$ is nonnegative definite. For $\theta, \phi \in \mathbb{R}^d$ arbitrary

$$\begin{aligned}
\langle \theta, A_n(t)\phi\rangle &= \theta^T A_n(t)\phi = \theta^T B_n C_{[nt]} B_n^T \phi = (B_n^T\theta)^T C_{[nt]} (B_n^T \phi) \in \mathbb{R} \\
&= \left((B_n^T\theta)^T C_{[nt]} (B_n^T\phi)\right)^T = (B_n^T\phi)^T C_{[nt]}^T (B_n^T\theta) = (B_n^T\phi)^T C_{[nt]} (B_n^T\theta) \\
&= \phi^T B_n C_{[nt]} B_n^T \theta = \phi^T A_n(t)\theta = \langle A_n(t)\theta, \phi\rangle
\end{aligned}$$

so $A_n(t)$ is also symmetric.

We now verify condition (i). First, fix $T > 0$. Without loss of generality, we may assume that T is an integer. Furthermore, assume that $n > T$. Now $M_n(t) =$

$M_n(t-)$ except possibly at $t = j/n$, $j = 1, 2, \ldots$. In which case, $M_n(t) = \sum_{i=1}^{j} B_n\xi_i$ and $M_n(t-) = \sum_{i=1}^{j-1} B_n\xi_i$. Therefore, $M_n(t) - M_n(t-) = B_n\xi_j$. Now,

$$\begin{aligned}
E\left[\sup_{t\leqslant T}\|M_n(t) - M_n(t-)\|\right] &= E\left[\max_{j\leqslant nT}\|B_n\xi_j\|\right] \\
&= E\left[\max\left\{\max_{(k-1)n<j\leqslant kn}\|B_n\xi_j\| : k = 1, \ldots, T\right\}\right] \\
&\leqslant E\left[\sum_{k=1}^{T}\max_{(k-1)n<j\leqslant kn}\|B_n\xi_j\|\right] \\
&= TE\left[\max_{j\leqslant n}\|B_n\xi_j\|\right]
\end{aligned}$$

Therefore, it suffices to show that $E\max_{j\leqslant n}\|B_n\xi_j\| \to 0$. The following lemma is a significant strengthening of results in Sepanski [**20**] and Maller [**12**], where the convergence was shown in probability. The proof is an application of multivariable regular variation.

LEMMA 2.1. *If $\xi, \xi_1, \xi_2, \ldots$ are independent and identically distributed random vectors in $\mathbb{R}^d$ with distribution in the generalized domain of attraction of a d-variate normal law, then $E\max_{j\leqslant n}\|B_n\xi_j\| \to 0$.*

PROOF. Define $T_n = \max_{j\leqslant n}\|B_n\xi_j\|$. Write

$$\begin{aligned}
ET_n &= \int_0^\infty \Pr[T_n > t]dt \\
&= \int_0^\delta \Pr[T_n > t]dt + \int_\delta^\infty \Pr[T_n > t]dt \\
&\leqslant \delta + \int_\delta^\infty (1 - \Pr[T_n \leqslant t])dt \\
&= \delta + \int_\delta^\infty (1 - \Pr[\|B_n\xi\| \leqslant t]^n)dt \\
&= \delta + \int_\delta^\infty (1 - (1 - \varepsilon_n(t))^n)dt \\
&= \delta + I_n
\end{aligned}$$

where $\delta > 0$ is small and $\varepsilon_n(t) = \Pr[\|B_n\xi\| > t] \leqslant \varepsilon_n(\delta) \to 0$ uniformly in $t > \delta$ as $n \to \infty$ since $n\Pr[\|B_n\xi\| > \delta] \to 0$ by the standard convergence criteria for triangular arrays (Theorem 3.2.2 in [**16**]). Note that $1-(1-x)^n \leqslant nx$ for all $x > 0$ small, and so for all n large

$$I_n \leqslant n\int_\delta^\infty \varepsilon_n(t)dt = nE\|B_n\xi\|I(\|B_n\xi\| > \delta) \to 0$$

by the following regular variation argument. Recall that we are assuming that $E\xi = 0$. Note that

$$\|B_n\xi\| \leqslant \sum_{j=1}^{d} |\langle B_n\xi, e_j\rangle|$$

where $e_1 \cdots e_k$ is the standard basis for $\mathbb{R}^d$. Then

$$I_n \leqslant nE\sum_{j=1}^{d} |\langle B_n\xi, e_j\rangle| I(\|B_n\xi\| > \delta)$$

where

$$nE|\langle B_n\xi, e_j\rangle| I(\|B_n\xi\| > \delta) = J_1 + J_2$$

with

$$\begin{aligned} J_1 &= nE|\langle B_n\xi, e_j\rangle| I(|\langle B_n\xi, e_j\rangle| > \delta) \\ J_2 &= nE|\langle B_n\xi, e_j\rangle| I(\|B_n\xi\| > \delta \text{ and } |\langle B_n\xi, e_j\rangle| \leqslant \delta) \end{aligned}$$

Define the truncated moments U_b, V_c as in Feller [**4**] i.e., let

$$\begin{aligned} U_b(r,x) &= E|\langle \xi, x\rangle|^b I[|\langle \xi, x\rangle| \leqslant r], \\ V_c(r,x) &= E|\langle \xi, x\rangle|^c I[|\langle \xi, x\rangle| > r]. \end{aligned}$$

Since every full multivariable normal law on $\mathbb{R}^d$ is of the same type, we can assume without loss of generality that the limit is standard normal, with characteristic function $e^{-\frac{1}{2}Q(t)}$ where $Q(t) = \|t\|^2$. Corollary 8.1.4 in [**16**] shows that ξ belongs to the generalized domain of attraction of this multivariate normal law if and only if the truncated second moment function $M(t) = E\big(\langle \xi, t\rangle^2 I(|\langle \xi, t\rangle| \leqslant 1)\big)$ satisfies the regular variation condition

$$nM(B_n^T t_n) \to Q(t) \text{ whenever } t_n \to t \neq 0 \text{ in } \mathbb{R}^d. \tag{2.4}$$

This condition is equivalent to regular variation of $M(t)$ at zero with exponent $-\frac{1}{2}I$, and then Theorem 5.3.4 in [**16**] shows that $U_2(r,x)$ is slowly varying uniformly in $\|x\| = 1$ as $r \to \infty$. Write $B_n^T e_j = r_n^{-1}\theta_n$ where $\|\theta_n\| = 1$ and $r_n \to \infty$ so that $|\langle B_n\xi, e_j\rangle| = |\langle \xi, B_n^T e_j\rangle| = r_n^{-1}|\langle \xi, \theta_n\rangle|$. Then

$$\begin{aligned} J_1 &= nE|\langle B_n\xi, e_j\rangle| I(|\langle B_n\xi, e_j\rangle| > \delta) \\ &= nr_n^{-1} E|\langle \xi, \theta_n\rangle| I(|\langle \xi, \theta_n\rangle| > \delta r_n) \\ &= nr_n^{-1} V_1(\delta r_n, \theta_n) \\ &= \frac{r_n V_1(\delta r_n, \theta_n)}{U_2(\delta r_n, \theta_n)} \cdot \frac{U_2(\delta r_n, \theta_n)}{U_2(r_n, \theta_n)} \cdot nr_n^{-2} U_2(r_n, \theta_n) \to 0 \end{aligned}$$

since the first factor tends to zero by Theorem 5.3.11 in [**16**] (a uniform version of Feller [**4**, XVII.9, Theorem 2]), the second factor tends to 1 since U_2 is uniformly slowly varying, and the third factor

$$\begin{aligned} nr_n^{-2}U_2(r_n,\theta_n) &= nr_n^{-2}E|\langle\xi,\theta_n\rangle|^2 I(|\langle\xi,\theta_n\rangle| \leqslant r_n) \\ &= nE|\langle\xi,r_n^{-1}\theta_n\rangle|^2 I(|\langle\xi,r_n^{-1}\theta_n\rangle| \leqslant 1) \\ &= nE|\langle\xi,B_n^T e_j\rangle|^2 I(|\langle\xi,B_n^T e_j\rangle| \leqslant 1) \\ &= M(B_n^T e_j) \to Q(e_j) = 1 \end{aligned}$$

as $n \to \infty$ in view of (2.4).

Next write

$$\begin{aligned} J_2 &= nE|\langle B_n\xi, e_j\rangle| I(\|B_n\xi\| > \delta \text{ and } |\langle B_n\xi,e_j\rangle| \leqslant \delta) \\ &\leqslant \delta\cdot n\Pr[\|B_n\xi\| > \delta] \to 0 \end{aligned}$$

by the standard convergence criteria for triangular arrays (Theorem 3.2.2 in [**16**]). This concludes the proof of the lemma, and so condition (i) is established.

Finally we verify condition (ii). Recall that C_n is the empirical covariance operator defined by equation (2.3). Let H_n be the normalizing sequence constructed in Hahn and Klass [**7, 8**], and note that $H_n = H_n^T$. Let B_n be any other norming sequence for which the basic convergence in distribution (2.1) holds. As shown in Sepanski [**21**], $H_nC_nH_n \to I$ in probability. In fact, Sepanski [**22**] shows that $H_nC_n^{1/2} \to I$ in probability. However, since both sequences $B_n\sum_{i=1}^n \xi_i$ and $H_n\sum_{i=1}^n \xi_i$ converge in law to $N(0,I)$, Billingsley's convergence of types theorem [**1**] says that, $B_n = \delta_n R_n H_n$, where $\delta_n \to I$, and R_n are orthogonal transformations. But then we also have $B_nC_nB_n^T \to I$ in probability, since

$$\begin{aligned} &\|B_nC_nB_n^T - I\| \\ &\quad= \|\delta_nR_nH_nC_nH_nR_n^T\delta_n^T - I\| \\ &\quad= \|(\delta_nR_nH_nC_nH_nR_n^T\delta_n^T - \delta_n) + (\delta_n - I)\| \\ &\quad\leqslant \|\delta_nR_nH_nC_nH_nR_n^T\delta_n^T - \delta_n\| + \|\delta_n - I\| \\ &\quad\leqslant \|\delta_n\|\ \|R_nH_nC_nH_nR_n^T\delta_n^T - I\| + \|\delta_n - I\| \\ &\quad= \|\delta_n\|\ \|(R_nH_nC_nH_nR_n^T\delta_n^T - \delta_n^T) + (\delta_n^T - I)\| + \|\delta_n - I\| \\ &\quad\leqslant \|\delta_n\|\left\{\|R_nH_nC_nH_nR_n^T\delta_n^T - \delta_n^T\| + \|\delta_n^T - I\|\right\} + \|\delta_n - I\| \\ &\quad\leqslant \|\delta_n\|\left\{\|R_nH_nC_nH_nR_n^T - I\|\ \|\delta_n^T\| + \|\delta_n^T - I\|\right\} + \|\delta_n - I\| \\ &\quad= \|\delta_n\|\ \|\delta_n^T\|\ \|R_nH_nC_nH_nR_n^T - R_nR_n^T\| + \|\delta_n\|\ \|\delta_n^T - I\| + \|\delta_n - I\| \\ &\quad\leqslant \|\delta_n\|\ \|\delta_n^T\|\ \|R_n\|\ \|R_n^T\|\ \|H_nC_nH_n - I\| + \|\delta_n\|\ \|\delta_n^T - I\| + \|\delta_n - I\| \end{aligned} \tag{2.5}$$

tends to zero in probability. Next recall that $B_n = n^{-1/2}L_n$ where L_n is slowly varying, so that $L_{[nt]}L_n^{-1} \to I$. Then we have

$$\begin{aligned} B_nB_{[nt]}^{-1} &= n^{-1/2}L_n\,L_{[nt]}^{-1}[nt]^{1/2} = n^{-1/2}[nt]^{1/2}L_nL_{[nt]}^{-1} \\ &= n^{-1/2}[nt]^{1/2}(L_{[nt]}L_n^{-1})^{-1} \to t^{1/2}I^{-1} = t^{1/2}I \end{aligned} \tag{2.6}$$

while

$$
\begin{aligned}
(B_{[nt]}^{-1})^T B_n^T &= [nt]^{1/2}(L_{[nt]}^{-1})^T L_n^T n^{-1/2} \\
&= [nt]^{1/2} n^{-1/2}(L_{[nt]}^{-1})^T L_n^T \\
&= [nt]^{1/2} n^{-1/2}(L_n L_{[nt]}^{-1})^T \\
&= [nt]^{1/2} n^{-1/2}((L_{[nt]} L_n^{-1})^{-1})^T \\
&\to t^{1/2}(I^{-1})^T = t^{1/2} I.
\end{aligned}
\tag{2.7}
$$

Combining (2.5), (2.6), and (2.7) we see that

$$
\begin{aligned}
A_n(t) &= B_n C_{[nt]} B_n^T \\
&= B_n B_{[nt]}^{-1} \cdot B_{[nt]} C_{[nt]} B_{[nt]}^T \cdot (B_{[nt]}^{-1})^T B_n^T \\
&\to t^{1/2} I \cdot I \cdot t^{1/2} I = tI
\end{aligned}
$$

in probability, and this concludes the proof. □

3. Remarks

Theorem 2.1 was proven by Sepanski [**23**] using specialized Fourier analytic methods. The proof given in this paper is considerably simpler. It may also be possible to prove Theorem 2.1 using the more general result of Kuelbs [**11**], which shows that a functional limit theorem for Banach space valued random variables follows from the central limit theorem, under suitable conditions.

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