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TAUBERIAN THEOREMS AND LIMIT DISTRIBUTIONS FOR UPPER ORDER STATISTICS

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Abstract. Starting with the Tauberian theorems of Karamata for regular variation we prove a slight extension of a Tauberian theorem by Trautner and the second author and use this to characterize limit relations for upper order statistics if we are in the domain of attraction of a max-stable distribution. Furthermore, we discuss the speed of convergence therein.

1. Introduction

One of the great contributions to Mathematics of Jovan Karamata is the notion of regular variation and its use in Tauberian theory (1930/31), see e.g., [**13, 14, 15, 16**]. Many authors have extended and applied these investigations in asymptotic Analysis and its applications. Another Mathematician from Yugoslavia, William Feller, introduced these concepts in his books [**11**] into Probability Theory where they had and still have many fruitful applications. The best modern book dealing with regular variation and its extensions together with various applications is the work of Bingham, Goldie and Teugels [**2**].

It is the aim of this paper to extend a Tauberian result of [**29**] slightly in order to get a characterization of the distributions for which certain distributional limit theorems for upper order statistics hold. Thereby we simplify the proof of a result by Smid and Stam [**25**] and extend their result.

Let us begin with some facts which are the basics since Karamata. Consider a measurable function $f : \mathbb{R}_+ \to \mathbb{R}_+$; it is called regularly varying (at infinity) if

$$(1) \qquad \lim_{\lambda\to\infty} \frac{f(\lambda t)}{f(\lambda)} \text{ exists on a set of positive measure in } (0,\infty).$$

Then the limit exists for all $t \in \mathbb{R}_+$ and the limit function is t^ρ with some $\rho \in \mathbb{R}$ called the index of regular variation and we write $f \in \mathrm{RV}_\rho$. Furthermore we can represent such a function f as $f(t) = t^\rho L(t)$ with a slowly varying function $L(.)$, i.e., a regularly varying function of order zero. We know that the limit relation in

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(1) is locally uniform in $(0, \infty)$ and even uniform in $[b, \infty)$ for any $b > 0$ if $\rho < 0$. One specific example is $f(t) = ct^\rho$, $\rho \geqslant 0$, which has the Laplace transform

$$\hat{f}(x) := x \int_0^\infty e^{-xt} f(t) dt = c\Gamma(\rho + 1) x^{-\rho}, \; x > 0.$$

This extends to the following Abelian type theorem

$$\text{If } f \in \mathrm{RV}_\rho \quad \text{then} \quad \hat{f}(1/x) \in \mathrm{RV}_\rho .$$

The well-known Tauberian theorem of Karamata says that for nondecreasing functions the converse is true as well (see also [**31, 32, 33**]). Whereas Karamata's proof uses approximation of a rectangular function by polynomials, another approach which is simple and effective is the use of sequential compactness arguments (see e.g., [**17, 11, 26, 27, 28, 29**]). Many subsequence principles have been shown in addition to the classical ones, consult e.g., the papers [**1, 18, 23, 24, 30**]. In our section 2 we alter a result from [**29**] slightly and use this in section 3 to prove the converse part in the characterization of certain distributional limit theorems for upper order statistics for iid random variables being in the domain of attraction of max-stable distributions. Besides Tauberian arguments the notion of regular variation and its extensions are essential in this context.

2. Some more Tauberian results

We shall present some ratio-Tauberian theorems following the arguments in [**27, 29**]. Therefore we consider the following classes of functions:

$$\mathcal{F}_\nearrow := \{f : \mathbb{R}_+ \to \mathbb{R}_+, f \text{ is nondecreasing and right continuous}\} \text{ and}$$

$$\mathcal{F}_D := \left\{ f \in \mathcal{F}_\nearrow, M_f(t) := \limsup_{\lambda \to \infty} \frac{f(\lambda t)}{f(\lambda)} < \infty \text{ for some (all) } t > 1 \right\}$$

the class of dominatedly varying functions. Analogously to $\mathcal{F}_\nearrow$ we define $\mathcal{F}_\searrow$. Then the following result was shown in [**27**].

Theorem A. *Let be given some function $f \in \mathcal{F}_D$. Then we have for any $g \in \mathcal{F}_\nearrow$ that*

$$\frac{\hat{g}(1/x)}{\hat{f}(1/x)} \to 1, \; x \to \infty, \qquad \textit{implies} \;\; \frac{g(t)}{f(t)} \to 1, \; t \to \infty$$

iff

$$M_f(1+) = 1 \tag{2}$$

Remarks. (i) Note that $M_f(1) = 1$, this means that condition (2) is equivalent to the continuity of M_f at $t = 1$. Functions satisfying (2) are also called intermediate regularly varying, see e.g., [**6**].

(ii) If we use $f \in \mathrm{RV}_\rho$, $\rho \geqslant 0$, then $M_f(t) = t^\rho$, $t > 0$, so (2) is obviously satisfied and we get back Karamata's Tauberian theorem.

(iii) If $f \in \mathcal{F}_D$, then there exist some constants c, $\beta > 0$ such that

$$\frac{f(\lambda t)}{f(\lambda)} \leqslant \begin{cases} ct^\beta, & t \geqslant 1 \\ 1, & 0. \end{cases} \tag{3}$$

The Laplace transform uses the exponential function as its kernel. A related result, however, is true for kernels which have similar properties as the exponential function, such as positivity or fast decay. If positivity is given up two-sided Tauberian conditions are usually needed. But in some recent results Drasin-Shea-Jordan theorems (see [**2**]), ratio Mercerian theorems and Tauberian theorems using one-sided Tauberian conditions are extended to kernels with sign changes, see the papers [**3, 4, 5**]. The typical Tauberian condition in Wiener's theory (see e.g., [**34, 20**]) for general kernels is, however, two-sided.

From now on we consider integral transforms of type

$$\widetilde{f}(x) := x \int_0^\infty K(xt) f(t) dt$$

with kernels satisfying

$$K \in L(\mathbb{R}_+),\ K \geqslant 0,\ \int_0^\infty K(v) dv = 1; \tag{4}$$

$$\begin{gathered}\text{The map } T_K : f \to \widetilde{f} \text{ is injective on} \\ \Big\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}_+),\ \text{with } \int_0^\infty K(xt)|f(t)|dt < \infty,\ \text{for all } x > 0 \Big\};\end{gathered} \tag{5}$$

$$\int_0^\infty K(t) t^\rho dt < \infty \text{ for all } \rho > 0. \tag{6}$$

Condition (5) is true if the Mellin transform $\int_0^\infty t^{ix} K(t) dt$ of the kernel does not vanish on the real line (see e.g., [**2**, Thm. 4.8.4]).

For two sets of functions $\mathcal{A}, \mathcal{B}$ on $\mathbb{R}_+$ we say that $f \in \mathcal{A} \cdot \mathcal{B}$ if f can be split into two functions $h \in \mathcal{A}$ and $g \in \mathcal{B}$ such that $f = h \cdot g$. Then we are interested in the following slight variation of the sufficiency part of Theorem 5 in [**29**] which is suitable for the investigations below.

THEOREM 1. *Let be given a kernel K satisfying conditions* (4)–(6) *and some function $f_1 \in \mathcal{F}_D$ satisfying* (2). *Then we have for an arbitrary function $f_2 \in \mathcal{F}_\searrow \cdot \mathrm{RV}_\varphi$, $\varphi \geqslant 0$ being locally bounded in* $[0, \infty)$ *that*

$$\frac{\widetilde{f}_2(1/x)}{\widetilde{f}_1(1/x)} \to 1,\ x \to \infty, \quad \textit{implies} \quad \frac{f_2(t)}{f_1(t)} \to 1,\ t \to \infty.$$

PROOF. We follow the arguments in the sufficiency part in the proof of Theorem 5 in [**29**] and give a short outline of the proof only. Assume that as $x \to \infty$ we have $\widetilde{f}_1(1/x) \sim \widetilde{f}_2(1/x)$ but with some sequence $(\mu_n) \to \infty$ we have $f_1(\mu_n)/f_2(\mu_n) \to c \neq 1$. By our assumption we obtain for any $x > 0$ as $\lambda \to \infty$

$$\widetilde{f}_1(x/\lambda) \sim \widetilde{f}_2(x/\lambda). \tag{7}$$

Consider now the family of functions $h_\lambda(t) = f_2(\lambda t)/f_1(\lambda)$, $\lambda > 0$. Note that we have $\limsup_{\lambda \to \infty} h_\lambda(1) < \infty$. Otherwise there exists a sequence $\lambda_n \to \infty$ such that with $f_2(t) = t^\varphi L(t) g_2(t)$ and $g_2 \searrow$

$$\frac{f_2(\lambda_n)}{f_1(\lambda_n)} \to \infty \text{ and on } \left[\frac{1}{2}, 1\right]: \ \frac{f_2(\lambda_n t)}{f_1(\lambda_n)} \geqslant \frac{(\lambda_n t)^\varphi L(\lambda_n t)}{\lambda_n^\varphi L(\lambda_n)} \frac{\lambda_n^\varphi L(\lambda_n) g_2(\lambda_n)}{f_1(\lambda_n)} \to \infty,$$

and then we find by (7) and the fact that $f_1 \in \mathcal{F}_D$ the following contradiction

$$O(1) = \int_0^\infty K(xt) \frac{f_1(\lambda_n t)}{f_1(\lambda_n)} dt \sim \int_0^\infty K(xt) h_{\lambda_n}(t) dt \geqslant \int_{1/2}^1 K(xt) h_{\lambda_n}(t) dt$$

since the right side is unbounded for suitable $x > 0$.

Furthermore since $f_1 \in \mathcal{F}_D$ and f_2 is locally bounded there exist constants $c, \rho > 0$ with

$$h_\lambda(t) \leqslant \begin{cases} c, & 0 \\ ct^\rho, & t \geqslant 1. \end{cases}$$

Now by the subsequence principle for monotone functions we find using a suitable subsequence μ_{n_k} which we denote by μ_n again that $f_1(\mu_n t)/f_1(\mu_n) \to f^*$ and $h_{\mu_n}(t) \to h^*$ and by dominated convergence we get that $\widetilde{f^*} = \widetilde{h^*}$. Now (5) yields that $f^* = h^*$ a.e. in particular we have that $t = 1$ is a continuity point of f^* and so $1 = f^*(1) = h^*(1)$ (note that h^* is a power times a monotone function) which is a contradiction to our choice of (μ_n). □

Corollary. *The statement of Theorem 1 holds if for given $\varphi \geqslant 0$ the assumption* (6) *on the kernel is reduced to $\int_0^\infty K(t) t^\rho dt < \infty$ for $0 \leqslant \rho < \rho_0$ with some $\rho_0 > \varphi$ and if f_1 satisfies* (3) *with $\beta < \rho_0$.*

Proof. Note that we have with arbitrary small $\delta > 0$ and the notions from the proof above

$$h_\lambda(t) \leqslant \frac{(\lambda t)^\varphi L(\lambda t)}{\lambda^\varphi L(\lambda)} \frac{\lambda^\varphi L(\lambda) g_2(\lambda)}{f_1(\lambda)} \leqslant c_\delta t^{\varphi + \delta}$$

for $t \geqslant 1$ and the proof works as before. □

3. Limit theorems for upper order statistics

From now on let $X_1, X_2, \ldots$ be a sequence of iid random variables with a continuous distribution F. For any $n \in \mathbb{N}$ we denote by $(X_{(1:n)}, X_{(2:n)}, \ldots, X_{(n:n)})$ the n-th order statistics of $(X_1, \ldots, X_n)$. By $\overline{F}(x)$ we mean the tail $1 - F(x)$ and by $x_\infty = \sup\{x : F(x) < 1\}$ we denote the right endpoint of the support of F.

It is well known (cf. e.g., [**22**]) that if one finds centering and norming sequences (a_n) and (b_n) (with $b_n \in \mathbb{R}$ und $a_n > 0$) such that $(X_{(n:n)} - b_n)/a_n$ converges in law to some nondegenerate distribution G, then G is of one of the following types:

(i) $$\Phi_\alpha(x) = \begin{cases} 0, & \text{if } x \leqslant 0 \\ \exp\{-x^{-\alpha}\}, & \text{if } x > 0 \end{cases} \quad \alpha > 0,$$

(ii) $$\Psi_\alpha(x) = \begin{cases} \exp\{-(-x)^\alpha\}, & \text{if } x < 0 \\ 1, & \text{if } x \geqslant 0 \end{cases} \quad \alpha > 0,$$

(iii) $\Lambda(x) = \exp\{-e^{-x}\}$.

The respective domains of attraction (denoted by $D(\Phi_\alpha(x))$, $D(\Psi_\alpha(x))$ and $D(\Lambda(x))$) are also well known, namely:

(i) $F \in D(\Phi_\alpha(x))$ iff $\overline{F} \in \mathrm{RV}_{-\alpha}$.

(ii) $F \in D(\Psi_\alpha(x))$ iff $\overline{F}$ is regularly varying with index $-\alpha$ at some point $x_\infty \in (0,\infty)$.

Recall that a measurable function $f : (0, x_\infty) \to \mathbb{R}_+$; is regularly varying at x_∞ with index ρ if we have for λ large enough that $f(x_\infty - \lambda^{-1}) = \lambda^\rho L(\lambda)$ with a slowly varying function L.

(iii) $F \in D(\Lambda(x))$ iff $1/\overline{F}$ is Γ-varying at some point $x_\infty \in (-\infty, \infty]$.

Here a nondecreasing right-continuous function $f : (x_0, x_\infty) \to \mathbb{R}_+$ (with some $x_0 < x_\infty$) is Γ-varying at x_∞ if there exists a measurable function $\alpha : (x_0, x_\infty) \to \mathbb{R}_+$, called the auxiliary function of f, such that for all $u \in \mathbb{R}$:

$$\frac{f(\lambda + u\alpha(\lambda))}{f(\lambda)} \to e^u \quad (\lambda \to x_\infty).$$

It turns out that for all of these three classes of distributions certain limit relations for upper order statistics hold. We start with the class of distributions with a regularly varying tail at infinity and state a simplified version of a result on the limit distributions of quotients of upper order statistics given by Smid and Stam [**25**]. For earlier results concerning the Abelian part of the Theorem, see e.g., [**9, 10, 19**].

THEOREM B. *The following statements are equivalent for some $\alpha > 0$:*

(i) $\overline{F} \in \mathrm{RV}_{-\alpha}$

(ii) $\lim_{n\to\infty} P\Big(\dfrac{X_{(n-j:n)}}{X_{(n-j+1:n)}} \leqslant x\Big) = x^{j\alpha}$ *for all $x \in (0,1)$ and some (all) $j \in \mathbb{N}$.*

For completeness we give a simplified proof using Theorem 1 instead of Wiener's Tauberian theorem, which was employed by Smid and Stam.

PROOF. '$(i) \Longrightarrow (ii)$': The joint distribution of order statistics is well-known and therefrom we get

$$P(X_{(n-j+1:n)} \geqslant 1/x | X_{(n-j:n)}) \overset{a.s.}{=} \big(\overline{F}(\max\{1/x, X_{(n-j:n)}\})/\overline{F}(X_{(n-j:n)})\big)^j .$$

W.l.o.g. we put $j = 1$ and then we obtain for any $x \in (0,1)$

$$P\Big(\frac{X_{(n:n)}}{X_{(n-1:n)}} \geqslant \frac{1}{x}\Big) = \int_\Omega \frac{\overline{F}(X_{(n-1:n)}/x)}{\overline{F}(X_{(n-1:n)})} dP \to x^\alpha,\ n \to \infty,$$

since by our assumptions $X_{(n-1:n)} \xrightarrow{a.s} \infty$, $\overline{F} \in \mathrm{RV}_{-\alpha}$ and the dominated convergence theorem applies.

'$(ii) \Longrightarrow (i)$': We have for arbitrary $x \in (0,1)$

$$P(X_{(n-1:n)} > xX_{(n:n)}) = n(n-1)\int_0^\infty (1 - \overline{F}(u))^{n-2}\overline{F}(u/x)dF(u)$$

with $\beta(t) := (1/\overline{F}(t))^{\leftarrow}$ and the substitution $u = \beta(t)$ we find (note that $\beta(1/\overline{F}(t)) = t$, F–a.e. and $\beta(\overline{F}(t)) = 1/t$, $t \geqslant 1$, since F is continuous)

$$\begin{aligned} &= n(n-1)\int_1^\infty (1-1/t)^{n-2}\overline{F}(\beta(t)/x)\frac{dt}{t^2} \\ &= \int_1^\infty \Big(\frac{n(n-1)}{t^3}\Big(1-\frac{n}{nt}\Big)^{n-2} - \frac{n^2}{t^3}e^{-n/t}\Big)t\overline{F}(\beta(t)/x)dt \\ &\quad + \int_1^\infty \frac{n^2}{t^3}e^{-n/t}t\overline{F}(\beta(t)/x)dt = I_n + II_n \ \text{ say.}\end{aligned}$$

Using Scheffé's Lemma and the fact that $|t\overline{F}(\beta(t)/x)| \leqslant 1$ we find that

$$|I_n| \leqslant \int_0^\infty \left|\frac{1}{v^3}\Big(1-\frac{1}{nv}\Big)^{n-2}\mathbf{1}_{[1/n,\infty)}(v) - \frac{1}{v^3}e^{-1/v}\right| dv \to 0, \ n\to\infty.$$

Hence we get by our assumptions that with $K(t) = e^{-1/t}t^{-3}\mathbf{1}_{(0,\infty)}(t)$

$$II_n = o(1) + \frac{1}{n}\int_0^\infty K(t/n)t\overline{F}(x^{-1}\beta(t))dt \to x^\alpha, \ n\to\infty.$$

and then it is easy to see that

$$\frac{1}{\lambda}\int_0^\infty K(t/\lambda)t\overline{F}(x^{-1}\beta(t))dt \to x^\alpha, \quad \text{as } \lambda\to\infty.$$

Setting $f_2(t) = t\overline{F}(x^{-1}\beta(t))$ and $f_1(t) \equiv x^\alpha$ the assumptions of the Corollary are satisfied (note that the Mellin transform of K is $\Gamma(2-ix) \neq 0$ on $\mathbb{R}$ and hence (5) is satisfied, see e.g., [**2, 34**]) and thus we obtain

$$t\frac{\overline{F}(x^{-1}\beta(t))}{x^\alpha} \to 1, \ t\to\infty$$

which means that for $x \in (0,1)$

$$\frac{\overline{F}(x^{-1}\beta(t))}{\overline{F}(\beta(t))} \to x^\alpha.$$

Since for an arbitrary sequence $(\lambda_n) \nearrow \infty$ we find a sequence (t_n) such that $\beta(t_n-) \leqslant \lambda_n \leqslant \beta(t_n)$ and $\overline{F}(\beta(t_n-))/\overline{F}(\beta(t_n)) \to 1$ we obtain that $\overline{F} \in \mathrm{RV}_{-\alpha}$. □

We continue by showing that for the domains of attraction of the other two types of max-limit laws we have analogues of Theorem B. The result about distributions with a tail that is regularly varying at some finite point is

THEOREM 2. *The following statements are equivalent for some $\alpha > 0$:*

(i) *$\overline{F}$ is regularly varying at x_∞ with index $-\alpha$*

(ii) $\lim_{n\to\infty} P\Big(\frac{x_\infty - X_{(n-j+1:n)}}{x_\infty - X_{(n-j:n)}} \leqslant x\Big) = x^{j\alpha}$ *for all $x \in (0,1)$ and some (all) $j \in \mathbb{N}$.*

PROOF. This is an easy consequence of the fact that the random variables $Y_n = 1/(x_\infty - X_n)$ have a distribution whose tail is in $\mathrm{RV}_{-\alpha}$ and that obviously $Y_{(j:n)} = 1/(x_\infty - X_{(j:n)})$. □

We come to the class Γ. Here it is well known that $\dfrac{X_{(n-j+1:n)}}{X_{(n-j:n)}} \xrightarrow{p} 1$ $(n \to \infty)$, but the analogue of the results above is as follows.

THEOREM 3. *The following statements are equivalent:*

(i) $1/\overline{F} \in \Gamma$ *with auxiliary function* α,

(ii) $\lim\limits_{n\to\infty} P\Big(\dfrac{X_{(n-j+1:n)} - X_{(n-j:n)}}{\alpha(X_{(n-j:n)})} > x\Big) = e^{-xj}$ *for all* $x > 0$ *and some (all)* $j \in \mathbb{N}$, *where we can assume w.l.o.g. that the auxiliary function satisfies for any* $u > 0$: $\lambda + u\alpha(\lambda)$ *is eventually nondecreasing.*

PROOF. The proof proceeds along the same lines as the proof of Theorem B.

'$(i) \Longrightarrow (ii)$': W.l.o.g. we put $j = 1$. As in Theorem B we have for any $x > 0$

$$P\left(X_{(n:n)} > X_{(n-1:n)} + x\alpha(X_{(n-1:n)})\right) = \int_\Omega \frac{\overline{F}(X_{(n-1:n)} + x\alpha(X_{(n-1:n)}))}{\overline{F}(X_{(n-1:n)})} dP \to e^{-x},$$

$n \to \infty$, by the dominated convergence theorem. By Theorem 3.10.8 in [**2**] we can replace the auxiliary function by one which absolutely continuous with derivative tending to zero, so the additional requirement in (ii) is satisfied.

'$(ii) \Longrightarrow (i)$': For arbitrary $x > 0$ we get by the substitution $u = \beta(t)$

$$\begin{aligned} P\big(X_{(n:n)} &> X_{(n-1:n)} + x\alpha(X_{(n-1:n)})\big) \\ &= n(n-1)\int_1^\infty \overline{F}(\beta(t) + x\alpha(\beta(t)))F(\beta(t))^{n-2}\frac{dt}{t^2} \\ &= n(n-1)\int_1^\infty \frac{\overline{F}(\beta(t) + x\alpha(\beta(t)))}{\overline{F}(\beta(t))}F(\beta(t))^{n-2}\frac{dt}{t^3} \end{aligned}$$

The same sort of application of Scheffé's Lemma as before yields

$$\frac{1}{n}\int_0^\infty K(t/n)t\overline{F}(\beta(t) + x\alpha(\beta(t)))dt \to e^{-x},\ n \to \infty.$$

Setting $f_2(t) = t\overline{F}(\beta(t) + x\alpha(\beta(t)))$ and $f_1(t) \equiv e^{-x}$ the assumptions of the Corollary are again satisfied (with the same kernel as before) and thus we obtain

$$t\frac{\overline{F}(\beta(t) + x\alpha(\beta(t)))}{e^{-x}} \to 1,\ t \to \infty$$

which means that

$$\frac{\overline{F}(\beta(t) + x\alpha(\beta(t)))}{\overline{F}(\beta(t))} \to e^{-x},\ t \to \infty$$

and hence $\overline{F}$ is Γ-varying. □

REMARKS. If one considers the joint limit distribution of the expressions above for $1 \leqslant j \leqslant k$ then one finds as in the proof of Theorem 1 in [**25**] that its components are independent, that is, e.g., in case $1/\overline{F} \in \Gamma$ we have

$$\lim_{n\to\infty} P\Big(\frac{X_{(n:n)} - X_{(n-1:n)}}{\alpha(X_{(n-1:n)})} > x_1, \dots, \frac{X_{(n-k+1:n)} - X_{(n-k:n)}}{\alpha(X_{(n-k:n)})} > x_k\Big) = e^{-x_1\cdot 1 - \dots - x_k \cdot k}$$

for any fixed $k \in \mathbb{N}$ and any tupel $(x_1, \ldots, x_k) \in \mathbb{R}^k$. In the special case of standard exponential random variables it is well known that $\alpha(.) \equiv 1$ and that

$$nX_{(1:n)},\ (n-1)(X_{(2:n)} - X_{(1:n)}),\ \ldots,\ 2(X_{(n-1:n)} - X_{(n-2:n)}),\ (X_{(n:n)} - X_{(n-1:n)})$$

are independent and exponentially distributed.

In case $1/\overline{F} \in \mathrm{RV}_{-\alpha}$ we have

$$\lim_{n\to\infty} P\Big(\frac{X_{(n:n)}}{X_{(n-1:n)}} > \frac{1}{x_1}, \ldots, \frac{X_{(n-k+1:n)}}{X_{(n-k:n)}} > \frac{1}{x_k}\Big) = x_1^{1\alpha} \cdots x_k^{k\alpha}$$

for any fixed $k \in \mathbb{N}$ and any tupel $(x_1, \ldots, x_k) \in (0,1)^k$. From this we get e.g., that

$$\lim_{n\to\infty} P\Big(\frac{X_{(n:n)}}{X_{(n-2:n)}} > \frac{1}{x}\Big) = 2x^\alpha(1 - x^\alpha/2),\ 0 < x < 1.$$

If one wants to gain some information on the speed of convergence in Theorem B it is clear that some more knowledge about the underlying distribution function is needed. A reasonable possibility is to look at second-order regular variation.

Recall that a measurable function is called second-order regularly varying with first order index γ if there exist a function a with ultimately constant sign such that $a(\lambda) \to 0$ as $\lambda \to \infty$ and

$$\lim_{\lambda\to\infty} \frac{\frac{f(\lambda t)}{f(\lambda)} - t^\gamma}{a(\lambda)} \to H_{\gamma,\kappa}(t) := ct^\gamma \int_1^t u^{\kappa-1} du \quad (c \neq 0) \tag{8}$$

for all $t > 0$. We write this as $f \in 2\,\mathrm{RV}(\gamma, \kappa, a)$.

It is well known that if (8) holds then $H_{\gamma,\kappa}$ is of the above form and convergence is locally uniform in $(0, \infty)$. Moreover, $a \in \mathrm{RV}(\kappa)$ and thus $\kappa \leqslant 0$. For these and other results on second order regular variation the reader may consult [**12, 8**].

Now the second-order result corresponding to Theorem B stated in its simplest form is:

Theorem 4. *Let $\overline{F} \in 2\,\mathrm{RV}(-\alpha, -\rho, a)$ with $\alpha > 0, \rho \geqslant 0$. Set again $\beta(t) := (1/\overline{F}(t))^{\leftarrow}$. Then we have for all $x \in (0,1)$:*

$$\frac{1}{a(\beta(n))}\Big(P\Big(\frac{X_{(n-1:n)}}{X_{(n:n)}} \leqslant x\Big) - x^\alpha\Big) \longrightarrow H_{-\alpha,-\rho}(x) \cdot \Gamma\Big(\frac{\alpha+\rho}{\alpha} + 1\Big) \tag{9}$$

Proof. We may assume w.l.o.g. that $c > 0$ in (8) and we write $H := H_{-\alpha,-\rho}$. Fix $x \in (0,1)$ throughout the whole proof. Note that $\beta \in \mathrm{RV}_{1/\alpha}$ (e.g., Thm. 1.5.12 in [**2**]).

Now choose $\epsilon, \zeta \in (0,1)$ arbitrarily small but fixed and then $x_0 = x_0(\epsilon, \zeta)$ and $n_0 = n_0(x_0, \epsilon, \zeta)$ such that the following conditions are satisfied

$$F(x_0) \geqslant 1/2, \tag{10}$$

$$\frac{\overline{F}(\lambda/x)/\overline{F}(\lambda) - x^\alpha}{a(\lambda)} \in [(1-\epsilon)H(x), (1+\epsilon)H(x)] \text{ for all } \lambda \geqslant x_0, \tag{11}$$

$$P(X_{(n-1:n)} \leqslant x_0) \leqslant \frac{\epsilon}{2} a(\beta(n)) \text{ for all } n \geqslant n_0, \tag{12}$$

$$\frac{a(y)}{a(x)} \leqslant 2 \max\left\{(y/x)^{-\rho+\zeta}, (y/x)^{-\rho-\zeta}\right\} \text{ for all } x, y \geqslant x_0, \tag{13}$$

$$\frac{\beta(y)}{\beta(x)} \leqslant 2 \max\left\{(y/x)^{3/(2\alpha)}, (y/x)^{1/(2\alpha)}\right\} \text{ for all } x, y \geqslant 1/\overline{F}(x_0). \tag{14}$$

Conditions (13) and (14) are plain applications of the so called "Potter bounds" (cf. [**2**, Thm. 1.5.6]). To see that (12) is true note that

$$P(X_{(n-1:n)} \leqslant x_0) = \sum_{\nu=0}^{1} \binom{n}{\nu} \overline{F}(x_0)^\nu \cdot F(x_0)^{n-\nu} \leqslant nF(x_0)^{n-1} = o(a(\beta(n))),$$

since $a \circ \beta \in \mathrm{RV}_{-\rho/\alpha}$.

Proceeding as in the proof of Theorem B we obtain

$$\begin{aligned} &\frac{1}{a(\beta(n))}\Big(P\Big(\frac{X_{(n-1:n)}}{X_{(n:n)}} \leqslant x\Big) - x^\alpha\Big) \\ &= \frac{1}{a(\beta(n))}\Big(\int_{\{X_{(n-1:n)} \leqslant x_0\}} + \int_{\{X_{(n-1:n)} > x_0\}}\Big)\Big(\frac{\overline{F}(X_{(n-1:n)}/x)}{\overline{F}(X_{(n-1:n)})} - x^\alpha\Big) dP \end{aligned} \tag{15}$$

Using (12) the first integral in (15) can be estimated from above by

$$\frac{1}{a(\beta(n))} \int_{\{X_{(n-1:n)} \leqslant x_0\}} \Big(\frac{\overline{F}(X_{(n-1:n)}/x)}{\overline{F}(X_{(n-1:n)})} + x^\alpha\Big) dP \leqslant \frac{2P(X_{(n-1:n)} \leqslant x_0)}{a(\beta(n))} \leqslant \epsilon. \tag{16}$$

For the second integral of (15) we get from (11):

$$\begin{aligned} &\frac{1}{a(\beta(n))} \int\limits_{\{X_{(n-1:n)} > x_0\}} \Big(\frac{\overline{F}(X_{(n-1:n)}/x)}{\overline{F}(X_{(n-1:n)})} - x^\alpha\Big) \cdot \frac{a(X_{(n-1:n)})}{a(X_{(n-1:n)})} dP \\ &= \tilde{c} \cdot H(x) \cdot \int_{\{X_{(n-1:n)} > x_0\}} \frac{a(X_{(n-1:n)})}{a(\beta(n))} dP. \end{aligned} \tag{17}$$

with some $\tilde{c} \in [1-\epsilon, 1+\epsilon]$.

Now assume for the moment that

$$\int_{\{X_{(n-1:n)} > x_0\}} \frac{a(X_{(n-1:n)})}{a(\beta(n))} dP \to \Gamma\Big(\frac{\alpha+\rho}{\alpha} + 1\Big) \tag{18}$$

then combining (16) and (17) we obtain for

$$\frac{1}{a(\beta(n))}\Big(P\Big(\frac{X_{(n-1:n)}}{X_{(n:n)}}\Big) - x^\alpha\Big)$$

that the upper limit is at most

$$(1+\epsilon)H(x)\cdot\Gamma\Big(\frac{\alpha+\rho}{\alpha}+1\Big)+\epsilon$$

and the lower limit is at least

$$(1-\epsilon)H(x)\cdot\Gamma\Big(\frac{\alpha+\rho}{\alpha}+1\Big)-\epsilon.$$

Since ϵ was arbitrary this proves (9).

So all that remains to be shown is (18). To this end fix $\delta \in (0,1)$. Then $\delta\beta(n) \geqslant x_0$ for all sufficiently large n. Now we split

$$\begin{aligned}&\int_{\{X_{(n-1:n)}>x_0\}}\frac{a(X_{(n-1:n)})}{a(\beta(n))}dP\\ =&\Big(\int_{\{X_{(n-1:n)}\geqslant\delta\beta(n)\}}+\int_{\{x_0<X_{(n-1:n)}<\delta\beta(n)\}}\Big)\frac{a(X_{(n-1:n)})}{a(\beta(n))}dP=I_n+II_n.\end{aligned}$$

We consider I_n first. It is well known (cf. e.g., [**21**]) that $X_{(n-1:n)}/\beta(n) \xrightarrow{d} Y$ where $P(Y\leqslant x)=e^{-1/x^{\alpha}}(1+x^{-\alpha})$ for all positive x. Thus on a suitable probability space there exists a sequence of random variables (Y_n) such that $Y_n \stackrel{d}{=} X_{(n-1:n)}$ and $Y_n/\beta(n)\to Y$ a.s. Then in particular $Y_n\to\infty$ a.s. and by the uniform convergence in (1) on $[1/m,\infty)$ for each m we find that

$$\frac{a(Y_n)}{a(\beta(n))}=\frac{a((Y_n/\beta(n))\cdot\beta(n))}{a(\beta(n))}\to Y^{-\rho}\text{ a.s. }(n\to\infty).$$

If $Y_n\geqslant\delta\beta(n)$ we can also conclude from (13) that

$$\frac{a(Y_n)}{a(\beta(n))}\leqslant 2\max\big\{(Y_n/\beta(n))^{-\rho+\zeta},(Y_n/\beta(n))^{-\rho-\zeta}\big\}\leqslant\delta^{-\rho-\zeta}.$$

Using dominated convergence we have

$$I_n=\int_{\Omega}\frac{a(Y_n)}{a(\beta(n))}1_{\{Y_n/\beta(n)\geqslant\delta\}}dP\to\int_{\Omega}Y^{-\rho}1_{\{Y\geqslant\delta\}}dP\quad(n\to\infty).$$

The latter integral equals

$$\int_{\delta}^{\infty}t^{-\rho}d\Big(e^{-1/t^{\alpha}}(1+t^{-\alpha})\Big)=\alpha\int_{\delta}^{\infty}t^{-\rho-2\alpha-1}e^{-1/t^{\alpha}}dt$$

and substituting $u=t^{-\alpha}$ yields

$$I_n\to\int_0^{\delta^{-\alpha}}u^{(\rho+\alpha)/\alpha}e^{-u}dt\quad(n\to\infty).\tag{19}$$

Now we give an upper bound for II_n. Note that for $x_0<X_{(n-1:n)}<\delta\beta(n)$ we find using (13)

$$\frac{a(X_{(n-1:n)})}{a(\beta(n))}\leqslant 2\max\Big\{\Big(\frac{X_{(n-1:n)}}{\beta(n)}\Big)^{-\rho+\zeta},\Big(\frac{X_{(n-1:n)}}{\beta(n)}\Big)^{-\rho-\zeta}\Big\}\leqslant 2\Big(\frac{\beta(n)}{X_{(n-1:n)}}\Big)^{\rho+\zeta}.$$

Hence

$$\begin{aligned}
I\!I_n &\leqslant 2\int_{\{x_0<X_{(n-1:n)}<\delta\beta(n)\}}\Big(\frac{\beta(n)}{X_{(n-1:n)}}\Big)^{\rho+\zeta}dP \\
&= 2n(n-1)\int_{x_0}^{\delta\beta(n)}\Big(\frac{\beta(n)}{y}\Big)^{\rho+\zeta}F(y)^{n-2}\overline{F}(y)dF(y) \\
&= 2n(n-1)\int_{F(x_0)}^{F(\delta\beta(n))}\Big(\frac{\beta(n)}{F^{\leftarrow}(y)}\Big)^{\rho+\zeta}y^{n-2}(1-y)dy \\
&\leqslant 2n^2\int_{F(x_0)}^{F(\delta\beta(n))}\Big(\frac{\beta(n)}{\beta(1/(1-y))}\Big)^{\rho+\zeta}y^{n-2}(1-y)dy.
\end{aligned}$$

Note that by (14) we have

$$\frac{\beta(n)}{\beta(1/(1-y))}\leqslant 2\max\left\{(n(1-y))^{3/(2\alpha)},(n(1-y))^{1/(2\alpha)}\right\}.$$

Since $\delta<1$ it follows that

$$n(1-y)\geqslant n\overline{F}(\delta\beta(n))=\frac{\overline{F}(\delta\beta(n))}{\overline{F}(\beta(n))}\to\delta^{-\alpha}>1$$

such that for sufficiently large n:

$$\frac{\beta(n)}{\beta(1/(1-y))}\leqslant 2(n(1-y))^{3/(2\alpha)}.$$

Therefore

$$I\!I_n\leqslant 4n^{(3\rho+3\zeta)/(2\alpha)+2}\int_{F(x_0)}^{F(\delta\beta(n))}y^{n-2}(1-y)^{(3\rho+3\zeta)/(2\alpha)+1}dy.$$

Substituting $y=e^{-u}$ leads to

$$I\!I_n\leqslant 4n^{(3\rho+3\zeta)/(2\alpha)+2}\int_{-\log F(\delta\beta(n))}^{-\log F(x_0)}e^{(1-n)u}(1-e^{-u})^{(3\rho+3\zeta)/(2\alpha)+1}du.$$

From (10) we obtain $0\leqslant-\log F(\delta\beta(n))\leqslant u\leqslant-\log F(x_0)\leqslant\log 2$ and therefore $0\leqslant(1-e^{-u})/u\leqslant c_1$ for some positive constant c_1. Thus we obtain with some further constant $c_2>0$:

$$\begin{aligned}
I\!I_n &\leqslant 4c_2n^{(3\rho+3\zeta)/(2\alpha)+2}\int_{-\log F(\delta\beta(n))}^{-\log F(x_0)}e^{(1-n)u}u^{(3\rho+3\zeta)/(2\alpha)+1}du \\
&\leqslant \frac{4c_2}{F(x_0)}\int_{-\log F(\delta\beta(n))}^{-\log F(x_0)}e^{-nu}(nu)^{(3\rho+3\zeta)/(2\alpha)+1}ndu \\
&\leqslant \frac{4c_2}{F(x_0)}\int_{-n\log F(\delta\beta(n))}^{-n\log F(x_0)}e^{-u}u^{(3\rho+3\zeta)/(2\alpha)+1}du.
\end{aligned}$$

As $n\to\infty$ we have

$$-n\log F(\delta\beta(n))\sim\frac{\overline{F}(\delta\beta(n))}{\overline{F}(\beta(n))}\to\delta^{-\alpha}$$

and we find with some constant $c_3 > 0$

$$\limsup_{n\to\infty} II_n \leqslant c_3 \int_{\delta^{-\alpha}}^{\infty} e^{-u} u^{(3\rho+3\zeta)/(2\alpha)+1} du. \tag{20}$$

Now (18) follows from (20) and (19) since by choosing δ small enough we can make I_n arbitrarily close to $\Gamma((\alpha+\rho)/\alpha+1)$ and II_n arbitrarily small. □

REMARK. If (9) holds with some auxiliary function a tending to zero, then we know by Theorem B that $\overline{F} \in \mathrm{RV}_{-\alpha}$. However, we do not have a simple Tauberian condition on $\overline{F}$ in order that we can obtain from (9) that $\overline{F}$ is in $2\,\mathrm{RV}$.

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