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ON THE LAPLACE TRANSFORMS OF LOGCONCAVE DENSITIES

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Abstract. This paper describes for any given logconcave density f the set of all finite measures μ whose Laplace transforms are asymptotic to the Laplace transform of f. It is shown that the density of μ is asymptotic to f if it is logconcave. Thus logconcavity is a Tauberian condition for Laplace transforms of finite measures.

1. Introduction

Let $\mathcal{M}$ denote the space of all finite measures μ on $\mathbb{R}$ with the topology of vague convergence, and $\mathcal{LC}$ the subset of those measures μ with density $f = e^{-\varphi}$ with $\varphi \in \mathcal{C}$. Here $\mathcal{C}$ consists of all convex function $\varphi : \mathbb{R} \to (-\infty, \infty]$ with non-empty open domain $D = \{\varphi < \infty\}$, such that $e^{-\varphi}$ is integrable. Such densities $e^{-\varphi}$ are called *logconcave*. They may be regarded as natural generalizations of the Gaussian density and play an important role in statistics.

We shall approach the exponential transform of μ via the exponential family generated by μ. This approach was developed in studying the domains of attraction for the limit laws of exponential families. It was introduced in Feigin and Yashchin [1983]. It has the advantage that it does not distinguish between finite and infinite upper endpoint of the moment generating function (lower endpoint of the Laplace transform). Proofs are based on probabilistic arguments for weak convergence.

With $\mu \in \mathcal{M}$ associate the Radon measures $d\mu_\lambda(x) = e^{\lambda x} d\mu(x)$, $\lambda \in \mathbb{R}$. The *moment generating function* (mgf) K of μ

$$K(\lambda) = \int e^{\lambda x} d\mu(x) = \mu_\lambda(\mathbb{R}) \quad \text{with domain } \Lambda = \{K < \infty\}$$

is continuous on the vertical strip $\Lambda \times i\mathbb{R}$ in the complex plane and analytic on the interior of the strip. Assume $\mu \not\equiv 0$. Then for each $\lambda \in \Lambda$ one may introduce a random variable X_λ with probability distribution $\mu_\lambda / K(\lambda)$. The set of random

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variables X_λ, $\lambda \in \Lambda$, is the *natural exponential family* generated by μ. The function $\kappa = \log K : \mathbb{R} \to (-\infty, \infty]$ is the *cumulant generating function* (cgf) of μ. For interior points λ of Λ

$$\kappa'(\lambda) = EX_\lambda, \qquad \kappa''(\lambda) = \sigma^2(\lambda) = \mathrm{var}(X_\lambda).$$

So κ is strictly convex unless μ is degenerate.

We are interested in finite measures μ with a logconcave density. The subset $\mathcal{LC} \subset \mathcal{M}$ is invariant under exponential tilting: If $\mu \in \mathcal{LC}$ then $\mu_\lambda \in \mathcal{LC}$ provided $\mu_\lambda \in \mathcal{M}$. Convexity is preserved, but integrability may be lost by tilting. This invariance is one reason for being interested in logconcave densities. Another reason is that mgfs are logconvex. One might hope to establish some kind of duality between logconcave densities and their mgfs. Such a duality exists in the case of Gaussian densities and has been extended to the case of logconcave densities with Gaussian tails, see Balkema, Klüppelberg and Stadtmüller [1997]. In the present paper we turn our attention to the full class of logconcave densities.

Integrable logconcave functions are well behaved. See Barndorff-Nielsen [1978] for details. For an integrable logconcave density $e^{-\varphi}$ there exists a constant $c > 0$ and $\epsilon > 0$ so that

$$e^{-\varphi(x)} \leqslant ce^{-\epsilon|x|}, \qquad x \in \mathbb{R}. \tag{1.1}$$

Set $\varphi_\lambda(x) = \varphi(x) - \lambda x$. The set of λ for which $e^{-\varphi_\lambda}$ is integrable is open. This set is exactly the domain Λ of the mgf K of $e^{-\varphi}$. The function κ' is strictly increasing on Λ and maps Λ onto the domain $D = \{\varphi < \infty\}$. (To prove surjectivity assume $\kappa'(\lambda) < x_0$ for all $\lambda \in \Lambda$ for some $x_0 \in D$. By a change of coordinates we may assume that $x_0 = 0$. Let $x \in (0, x_\infty)$. Then $\mu_\lambda(-\infty, 0) \to 0$ and $\mu_\lambda(x, \infty) \to \infty$ for $\lambda \to \lambda_\infty$ while the expectation remains negative!) The map κ' is a homeomorphism. Hence we may define an analytic function $\alpha : D \to (0, \infty)$ by

$$\alpha(\kappa'(\lambda)) = \sigma(\lambda) = \sqrt{\kappa''(\lambda)}, \qquad \lambda \in \Lambda. \tag{1.2}$$

Observe that $\alpha'(x) = -(1/\sigma)'(\lambda)$ for $x = \kappa'(\lambda)$. For $x \in D$ one may describe $\alpha(x)$ as the standard deviation of the random variable X_λ where λ is chosen so that x is the expectation. The function α and its derivative will play an important role below.

It is well known that asymptotic equality of measures in their upper endpoint implies asymptotic equality of the Laplace transforms: For $\mu \in \mathcal{M}$ the *upper endpoint* is $x_\infty = \sup\{x \in \mathbb{R} \mid \mu[x, \infty) > 0\}$. If μ and $\tilde{\mu}$ have the same upper endpoint x_∞ and if $\tilde{\mu}[x, \infty) \sim \mu[x, \infty)$ for $x \uparrow x_\infty$ then the mgfs K and $\tilde{K}$ have the same *upper endpoint* $\lambda_\infty = \sup \Lambda = \sup \tilde{\Lambda}$ and $K(\lambda) \sim \tilde{K}(\lambda)$ for $\lambda \uparrow \lambda_\infty$. The converse implication need not hold. Karamata's celebrated theorem on the Laplace transform of measures μ whose distribution functions $M(x) = \mu(-\infty, x]$ vary regularly in infinity is amazing precisely because no extra Tauberian condition is needed for the converse.

2. Main results

Our first theorem states that logconcavity is a Tauberian condition for Laplace transforms: If the Laplace transforms of f and g are asymptotically equal, and if f and g each are asymptotic to a logconcave density then they are asymptotic to each other. The second theorem characterizes those finite measures $\tilde{\mu}$ whose Laplace transforms are asymptotic to the Laplace transform of a given measure μ with a logconcave density f. Such measures may be obtained by roughening the smooth measure $f d\lambda$ replacing Lebesgue measure by the rough approximation ρ.

THEOREM 1. *Let f and g be integrable logconcave functions on $\mathbb{R}$ with mgfs K_f and K_g. The following two statements are equivalent:*

1) *f and g have the same upper endpoint x_∞ and $f(x) \sim g(x)$ for $x \to x_\infty$;*

2) *K_f and K_g have the same upper endpoint λ_∞ and $K_f(\lambda) \sim K_g(\lambda)$ for $\lambda \to \lambda_\infty$.*

THEOREM 2. *Let $f = e^{-\varphi}$ be a logconcave integrable function on $\mathbb{R}$ with domain $D = \{\varphi < \infty\}$ and mgf K. Set $x_\infty = \sup D$. Let $\tilde{\mu}$ be a finite measure on $(-\infty, x_\infty)$ with mgf $\tilde{K}$. Let $R : D \to \mathbb{R}$ be a distribution function of the measure $d\rho = (1/f)d\tilde{\mu}$ on D. Let $\alpha : D \to (0, \infty)$ be the standard deviation associated with the exponential family generated by the density f, see (1.2), and let $\tilde{\alpha}$ be the function associated with $\tilde{\mu}$. Then $x + u\alpha(x) \in D$ if $x \in D$ and $|u| \leqslant a_0$ where a_0 is the universal constant defined in Section 5. The following two statements are equivalent:*

1) *For $u \in [0, a_0]$*

$$\frac{R(x + u\alpha(x)) - R(x)}{\alpha(x)} \to u, \qquad x \to x_\infty; \tag{2.1}$$

2) *The mgfs K and $\tilde{K}$ have the same upper endpoint λ_∞ and $\tilde{K} \sim K$ in λ_∞.*

If 1) or 2) holds then $\tilde{\alpha} \sim \alpha$ in x_∞.

3. An example

This section exhibits two measures whose mgfs are asymptotically equal.

Let $\tilde{\mu}$ be the discrete measure with mass $1/n!$ in the points $n = 0, 1, 2, \ldots$ The mgf is

$$\tilde{K}(\lambda) = \sum_{n=0}^{\infty} \frac{e^{\lambda n}}{n!} = e^{e^\lambda}, \qquad \lambda \in \Lambda = \mathbb{R}.$$

We claim that the mgf K of the density $f(x) = 1/\Gamma(1 + x)$, $x > -1$, is asymptotic to $\tilde{K}$ in ∞. This may be seen by noting that

$$f(x) \sim f_0(x) = \frac{1}{\sqrt{2\pi x}} e^{-\psi(x)}, \qquad x \to \infty,\ \psi(x) = x \log x - x,$$

and evaluating the integral of $e^{\lambda x} f_0(x)$ for $\lambda \to \lambda_\infty$ using Laplace's principle.

The measure $\tilde{\mu}$ may be regarded as a discretization of the measure μ with density f. See Nagaev [1998]. This does not explain the asymptotic equality of the

Laplace transforms. Indeed let us delete the mass of $\tilde{\mu}$ in the points p which are prime and compensate by doubling the mass in the points $p+1$. Does this affect the asymptotic behaviour of the mgf? One could also delete all composite numbers and compensate by multiplying the mass $1/p!$ in the prime p by the distance to the next prime!

In order to evaluate the effect of these changes on the asymptotic behaviour of the Laplace transform we use Theorem 2 and look at the asymptotic behaviour of the function $\tilde{\alpha}$ in infinity. We observe that ρ is the counting measure on the non-negative integers and that $\alpha(x) = \sqrt{x}$ since $\sigma^2 = \kappa'$. Since $\sqrt{x} \to \infty$ deleting the occasional prime does not affect the asymptotic behaviour of the mgf $\tilde{K}$. As for deleting the mass in the composite numbers, the known bounds on the distance between adjacent primes are too far apart at present to decide whether the asymptotic behaviour of the Laplace transform is affected. □

The example lies outside the scope of Karamata's theorem [1930]. It may be regarded as a limiting case though, see Balkema, Klüppelberg and Resnick [2002]. There it is shown that for a measure $\mu \in \mathcal{M}$ with mgf $K = e^\kappa$ the following statements are equivalent:

1) The upper endpoint λ_∞ of K is finite and $\xi \mapsto K(\lambda_\infty - \xi)$ varies regularly for $\xi \downarrow 0$ with exponent $-\gamma < 0$;

2) The distribution function

$$F(x) = \int_{(-\infty,x]} e^{\lambda_\infty t} d\mu(t)$$

varies regularly in infinity with exponent γ;

3) $(1/\sigma)'(\lambda) \to -1/\sqrt{\gamma}$ for $\lambda \uparrow \lambda_\infty \notin \Lambda$;

4) the variables X_λ, $\lambda \in \Lambda$, in the exponential family generated by μ may be normalized to converge in law to a variable $U \geqslant 0$ with a gamma distribution with parameter γ.

The equivalence between 1) and 2) is Karamata's theorem.

In the example above $(1/\sigma)' \to 0$. This corresponds to $\gamma = \infty$. For $\gamma \to \infty$ the gamma distribution properly normed converges to the standard normal distribution. The condition $(1/\sigma)' \to 0$ can be shown to be equivalent to asymptotic normality of the associated exponential family.

One can now ask what happens if one only assumes that the derivative $(1/\sigma)'$ is bounded. It will be shown below that this condition is satisfied if μ has a logconcave density. On the other hand if $(1/\sigma)'$ has a limit in $(-1,1)$ then there exists a logconcave density whose mgf is asymptotic to K. This raises two interesting questions, answer unknown: How does one characterize the asymptotic behaviour of Laplace transforms of logconcave functions? Does there exist a simple class of continuous densities which yield a Tauberian condition for Laplace transforms for which $(1/\sigma)'$ is bounded?

Alternative generalizations of Karamata's theory are treated in Bingham, Goldie and Teugels [1987], de Haan and Stadtmüller [1985] and Balkema, Geluk and de Haan [1979].

4. Convergence of convex functions

In this section we briefly review some results about convergence of convex functions. See Barndorff-Nielsen [1978] for details.

Given any sequence of convex functions $\varphi_n : \mathbb{R} \to (-\infty, \infty]$ and any countable dense set $A \subset \mathbb{R}$ one may use the diagonal argument to select a subsequence which converges in each $x \in A$ to an element $\varphi(x) \in [-\infty, \infty]$. Let μ_n have density $e^{-\varphi_n}$, and suppose μ_n converges vaguely to a finite measure μ. If $\varphi(x) = -\infty$ in two distinct points $x_1 < x_2$ of A then $\mu_{k_n}[x_1, x_2] \to \infty$ contradicting vague convergence. If $\varphi_n(x) \to \infty$ for all $x \in D$ (with the possible exception of one point) then the limit measure μ is degenerate. We conclude that there is an open interval D so that φ is finite and convex on $D \cap A$, and such that $\varphi \equiv \infty$ on A outside the closure of D. Convergence $\varphi_n \to \varphi$ holds uniformly on compact subsets of D and $\varphi_n' \to \varphi'$ weakly on D. To the right of the closure of D the sequence φ_n tends to ∞ and so does φ_n'. To the left $\varphi_n \to \infty$ and $\varphi_n' \to -\infty$. This proves

PROPOSITION 3. *Let $\mu_n \in \mathcal{LC}$ have density $e^{-\varphi_n}$. If $\mu_n \to \mu$ vaguely for some $\mu \in \mathcal{M}$, and μ is non-degenerate then $\mu \in \mathcal{LC}$ with density $e^{-\varphi}$ and $\varphi_n \to \varphi$ pointwise on a dense set $A \subset \mathbb{R}$. The densities $f_n = e^{-\varphi_n}$ converge in $\mathbf{L}^1$ to $f = e^{-\varphi}$.*

PROPOSITION 4. *Suppose $\mu_n \in \mathcal{LC}$ for $n = 0, 1, \ldots$ has density $e^{-\varphi_n}$ and cgf κ_n with domain $\Lambda_n = \{\kappa_n < \infty\}$. Suppose $\mu_n \to \mu_0$ vaguely. Then $\kappa_n \to \kappa_0$ uniformly on compact subsets of $\Lambda_0 \times i\mathbb{R}$, and the same holds for the derivatives $\kappa_n^{(m)}$ for $m \geqslant 1$. In particular $\lambda \in \Lambda_0$ implies $\lambda \in \Lambda_n$ eventually.*

Proof. If $\sup D_0$ is finite then $\sup \Lambda_0 = \infty$ and $\sup \Lambda_n \to \infty$ by the arguments above. If $\sup D_0 = \infty$ then $\sup \Lambda_0 = \sup \varphi_0'(x)$. If $\lambda \in \Lambda_0$ then $\varphi_0'(x - 0) > \lambda$ for some $x \in D$, hence $(\varphi_0(x) - \varphi_0(x - h))/h > \lambda$ for some $h > 0$, hence this also holds for $n \geqslant n_0$ which implies $\varphi_n'(x - 0) > \lambda$ for $n \geqslant n_0$. Hence $\lambda < \sup \Lambda_n$. By symmetry we also find $\lambda > \inf \Lambda_0$ implies $\lambda > \inf \Lambda_n$ eventually.

Let $\lambda_1 < \lambda_2$ lie in Λ_0. Then $\varphi_0(x) \geqslant \chi(x) = c + (\lambda_1 x \vee \lambda_2 x)$ uniformly on $\mathbb{R}$ for some $c \in \mathbb{R}$, and $\varphi_n \geqslant \chi - 1$ on $\mathbb{R}$ holds for $n \geqslant n_0$. (Indeed let $\lambda_1 < 0 < \lambda_2$ in Λ_0. Choose $x < y$ in D_0 so that $\varphi_0'(x + 0) < \lambda_1$ and $\varphi_0'(y - 0) > \lambda_2$. These inequalities then also hold for φ_n' for $n \geqslant n_0$. Convergence is uniform on $[x, y]$. The inequality for the derivatives yields the inequality $\varphi_n \geqslant \chi - 1$ outside the interval $[x, y]$.)

It follows that $K_n(\zeta)$, $n \geqslant n_0$, $\zeta \in [\lambda_1, \lambda_2]$, is uniformly bounded. This also holds for complex ζ in $[\lambda_1, \lambda_2] \times i\mathbb{R}$ since K_n are mgfs. So the sequence K_n is relatively compact. The condition $\mu_n \to \mu_0$ implies that it converges uniformly on compact subsets of the vertical strip $(\lambda_1, \lambda_2) \times i\mathbb{R}$, as does the sequence of derivatives $K_n^{(m)}$ for any $m \geqslant 1$. □

5. The space $\mathcal{LC}_{101}$ and the universal constant a_0

Let $\mathcal{M}_{101}$ denote the set of all finite measures μ on $\mathbb{R}$ which satisfy

$$\int d\mu(x) = 1, \qquad \int x d\mu(x) = 0, \qquad \int x^2 d\mu(x) = 1$$

and define $\mathcal{LC}_{101} = \mathcal{LC} \cap \mathcal{M}_{101}$. We claim that $\mathcal{LC}_{101}$ is a compact set in $\mathcal{M}$.

For $a > 0$, $b \in \mathbb{R}$ and $c > 0$ define $T_{abc}(\mu)$ as the image of the measure μ/c under the affine transformation $x \mapsto (x-b)/a$. The map $T : (a, b, c, \mu) \to T_{abc}(\mu)$ from $(0, \infty) \times \mathbb{R} \times (0, \infty) \times \mathcal{M}$ to $\mathcal{M}$ is continuous.

PROPOSITION 5. *The function α' is invariant under the transformations T_{abc}:*

$$\tilde{\alpha}'(ax+b) = \alpha'(x), \qquad x = \kappa'(\lambda), \ \tilde{\mu} = T_{abc}\mu.$$

Proof. By computation. Let $\tilde{X} = (X_\lambda - b)/a$. Then

$$\tilde{\kappa}(\xi) = \kappa(\lambda + \xi/a) - \kappa(\lambda) - b\xi/a$$

gives $\tilde{\sigma}^2(\xi) = a^2\sigma^2(\lambda + \xi/a)$. Now use that $\alpha'(x) = (1/\sigma)'(\lambda)$ for $x = \kappa'(\lambda)$. □

Define $\mathcal{LC}_1$ as the set of all μ with density $e^{-\varphi}$ such that

$$\inf \varphi = 0, \qquad \varphi(x) \leqslant 1 \text{ on } (-1,1), \qquad \varphi(x) \geqslant 1 \text{ off } [-1,1].$$

It is clear that $\mathcal{LC}_1$ is compact. Any sequence φ_n which satisfies the three conditions above has a convergent subsequence whose limit satisfies these three conditions. Moreover the mgf K of any measure $\mu \in \mathcal{LC}_1$ is finite on $(-1/2, 1/2)$. The functions $\mu \mapsto \mu(\mathbb{R})$, $\mu \mapsto \operatorname{var}(\mu/\mu(\mathbb{R}))$ are positive and continuous on $\mathcal{LC}$, hence bounded below. Hence $\mathcal{LC}_{101}$ is the continuous image of $\mathcal{LC}_1$. In fact these two spaces are homeomorphic. The map which associates with $\mu \in \mathcal{LC}$ with density $e^{-\varphi}$ the standardized probability measure $\mu^* \in \mathcal{LC}_{101}$ with density $e^{-\varphi^*}$ is continuous. This yields the following result:

THEOREM 6. *The space $\mathcal{LC} \subset \mathcal{M}$ is homeomorphic to the product $(0,\infty) \times \mathbb{R} \times (0,\infty) \times \mathcal{LC}_{101}$ and the space $\mathcal{LC}_{101} \subset \mathcal{M}$ is compact.*

As a consequence a number of bounds hold uniformly on $\mathcal{LC}_{101}$. Some of these are listed below.

PROPOSITION 7. *There exist constants $b_0 \in \mathbb{R}$, $c_0 > 0$, $\delta_0 > 0$ so that uniformly in $\mu \in \mathcal{LC}_{101}$*

$$\varphi(x) \geqslant b_0 + \delta_0|x|, \qquad x \in \mathbb{R}$$

$$\varphi(x) \geqslant c_0 \ \Rightarrow \ |\varphi'(x)| \geqslant \delta_0$$

and so that the functions

$$(\mu, \lambda) \mapsto \kappa_\mu^{(m)}(\lambda), \qquad \mu \in \mathcal{LC}_{101}, |\lambda| \leqslant \delta_0 \tag{5.1}$$

are continuous for $m = 0, 1, 2, \ldots$.

For any $\epsilon > 0$ there is a constant $C > 0$ so that

$$\int_{\{\varphi > C\}} e^{-\varphi(x)} dx < \epsilon, \qquad |\{\varphi < C\}| < \epsilon e^C, \qquad \mu \in \mathcal{LC}_{101}$$

where μ has density $e^{-\varphi}$ and $|J|$ denotes the length of the interval J.

Proof. The corresponding results are obvious for $\mathcal{LC}_1$. Take $b_0 = -1/2$, $\delta_0 = 1/2$, $c_0 = 1$, and use Proposition 4. Hence they carry over to $\mathcal{LC}_{101}$. □

Corollary 1. *There exists a constant* $a_0 \in (0,1]$ *so that* $|\alpha'(x)| \leqslant 2/a_0$ *for* $\mu \in \mathcal{LC}$, $x \in D_\mu$.

Proof. The result holds for $\mu \in \mathcal{LC}_{101}$ by compactness and (5.1). Now use the invariance of Proposition 5. □

Corollary 2. *Suppose* $\mu \in \mathcal{LC}$. *Then*

$$\begin{aligned} \lambda + a_0/\sigma(\lambda) &< \lambda_\infty, \qquad \lambda \in \Lambda \\ \lambda - a_0/\sigma(\lambda) &\to \lambda_\infty, \qquad \lambda \to \lambda_\infty. \end{aligned}$$

Similar results hold for the function α*:*

$$\begin{aligned} x + a_0\alpha(x) &< x_\infty, \qquad x \in D \\ x - a_0\alpha(x) &\to x_\infty, \qquad x \to x_\infty. \end{aligned}$$

Proof. We only prove the top set. The first relation is obvious if $\lambda_\infty = \infty$. So suppose λ_∞ is finite. Then $1/\sigma(\lambda)$ converges for $\lambda \to \lambda_\infty$ since the derivative of $1/\sigma$ is bounded (by $2/a_0$ see Corollary 1). The limit is zero since else σ has a finite limit in λ_∞ and then also κ and K which contradicts the fact that Λ is open. Now observe that on $[\lambda, \lambda + a_0/\sigma(\lambda)]$ the function $1/\sigma$ is bounded below by $1/2\sigma(\lambda)$ since the derivative $(1/\sigma)'$ is bounded by $1/2a_0$.

Convergence $\lambda - a_0/\sigma(\lambda) \to \lambda_\infty$ for $\lambda \to \lambda_\infty$ follows from $a_0/\sigma(\lambda) < \lambda_\infty - \lambda$, see above. If $\lambda_\infty = \infty$ then $1/\sigma(\lambda) \leqslant 3\lambda/a_0$ eventually and hence $\lambda - a_0/\sigma(\lambda) \geqslant \lambda/4$. □

6. Proof of Theorem 1

We need only prove the Tauberian half. So we may assume that $\lambda_\infty = \sup \Lambda = \sup \tilde{\Lambda}$ and $\tilde{\kappa}(\lambda) - \kappa(\lambda) \to 0$ for $\lambda \to \lambda_\infty$. We claim that $\tilde{x}_\infty = x_\infty$ and that $\tilde{f}(x) \sim f(x) \to 0$ for $x \to x_\infty$.

Let $b_n \to x_\infty$ where $b_n = \kappa'(\lambda_n)$ and $\lambda_n \to \lambda_\infty$, and let $a_n = \alpha(b_n) = \sqrt{\kappa''(\lambda_n)}$. Let $\mu^*_{\lambda_n} = T_{a_n b_n c_n}(\mu_{\lambda_n})$ with $c_n = K(\lambda_n)$ denote the standardized probability measure. So $\mu^*_{\lambda_n} \in \mathcal{LC}_{101}$. By compactness of $\mathcal{LC}_{101}$ we may assume that $\mu^*_{\lambda_n} \to \pi$ vaguely for some $\pi \in \mathcal{LC}_{101}$ with density $e^{-\psi}$ and that $\varphi^*_{\lambda_n} \to \psi$. It suffices to prove that $b_n \to \tilde{x}_\infty$ and $\tilde{f}(b_n) \sim f(b_n)$.

Note that the cgfs converge:

$$\gamma_n(\xi) = \kappa^*_{\lambda_n}(\xi) = \kappa(\lambda_n + \xi/a_n) - \kappa(\lambda_n) - b_n\xi/a_n \to \gamma(\xi), \qquad |\xi| < \delta_0 \tag{6.1}$$

where γ is cgf of π. Hence by Corollary 2 $\lambda_n + \xi/a_n \to \lambda_\infty$, and $\tilde{\kappa} - \kappa \to 0$ gives

$$\tilde{\gamma}^0_n = \tilde{\kappa}^*_{\lambda_n}(\xi) := \tilde{\kappa}(\lambda_n + \xi/a_n) - \tilde{\kappa}(\lambda_n) - b_n\xi/a_n \to \gamma(\xi), \qquad |\xi| < \delta_0 \tag{6.2}$$

which implies $\tilde{\varphi}^0_{\lambda_n} \to \psi$ and $\varphi^*_{\lambda_n} \to \psi$ weakly on $\mathbb{R}$. Hence $\tilde{\kappa}^0_{\lambda_n}{}''(0) \to \gamma''(0)$ by Proposition 4 which implies that the first two moments of the normalized random variable $(\tilde{X}_{\lambda_n} - b_n)/a_n$ converge to $\gamma'(0) = 0$ and $\gamma''(0) = 1$ respectively. Hence $\tilde{\kappa}''(\lambda_n) \sim \kappa''(\lambda_n) = a_n^2$ and $\tilde{\kappa}'(\lambda_n) - \kappa'(\lambda_n) = o(a_n)$ for $n \to \infty$. Since $\kappa'(\lambda_n) \pm a_0 a_n \to x_\infty$ by Corollary 2 it follows that $\tilde{x}_\infty = x_\infty$.

Write

$$e^{-\varphi^*_{\lambda_n}(u)} = a_n f_{\lambda_n}(b_n + a_n u)/K(\lambda_n) = a_n f(b_n + a_n u)e^{\lambda_n(b_n+a_n u)}/K(\lambda_n) \tag{6.3}$$

and similarly

$$e^{-\tilde{\varphi}^0_{\lambda_n}(u)} = a_n \tilde{f}_{\lambda_n}(b_n + a_n u)/\tilde{K}(\lambda_n) = a_n \tilde{f}(b_n + a_n u)e^{\lambda_n(b_n+a_n u)}/\tilde{K}(\lambda_n). \tag{6.4}$$

Now $\varphi^*_{\lambda_n}(0) \to \psi(0)$ and $\tilde{\varphi}^0_{\lambda_n}(0) \to \psi(0)$ and $\tilde{K}(\lambda_n) \sim K(\lambda_n)$ imply $\tilde{f}(b_n) \sim f(b_n)$. $\square$

7. Proof of Theorem 2

The basic argument is that replacing Lebesgue measure by the measure ρ does not impair weak convergence for the exponential families. We divide the proof into two parts. First we need a lemma.

Lemma 8. *Suppose μ and $\tilde{\mu}$ in $\mathcal{M}$ have mgfs K and $\tilde{K}$ which are asymptotic in their common upper endpoint. Then the upper endpoints x_∞ and $\tilde{x}_\infty$ of μ and $\tilde{\mu}$ are equal.*

Proof. Assume $x_\infty = 0 < \tilde{x}_\infty$ by a change of coordinates. Then $K(\lambda) \to \mu(\{0\})$ for $\lambda \to \lambda_\infty$ and $\tilde{K}(\lambda) \to \infty$. By symmetry we have $x_\infty = \tilde{x}_\infty$. $\square$

Proposition 9. *Suppose $\mu \in \mathcal{LC}$ has density $e^{-\varphi}$ and mgf K and $\tilde{\mu} \in \mathcal{M}$ has mgf $\tilde{K}$. Let $\lambda_\infty = \sup\{K < \infty\}$. If $\tilde{K}$ is finite on $[0, \lambda_\infty)$ and $\tilde{K}(\lambda) \sim K(\lambda)$ for $\lambda \to \lambda_\infty$ then the upper endpoint $\tilde{x}_\infty$ of $\tilde{\mu}$ equals $x_\infty = \sup\{\varphi < \infty\}$ and*

$$\rho[x, x + u\alpha(x)]/\alpha(x) \to u, \qquad 0 \leqslant u \leqslant a_0,\ x \to x_\infty \tag{7.1}$$

where $d\rho = e^{\varphi} d\tilde{\mu}$.

Proof. Let $\lambda_n \to \lambda_\infty$, $a_n > 0$, $b_n \in \mathbb{R}$, $\delta > 0$. Assume (6.1) holds with γ non-linear, $\lambda_n + \delta/a_n < \lambda_\infty$ eventually and $\lambda_n - \delta/a_n \to \lambda_\infty$. Then (6.2) holds. Hence $U_{\lambda_n} = (X_{\lambda_n} - b_n)/a_n \Rightarrow U$ where U has cgf γ and $\tilde{U}_{\lambda_n} \Rightarrow U$. Note that U is non-degenerate since γ is non-linear. Hence U has density $e^{-\psi}$ with $\psi \in \mathcal{C}$.

We may restrict $\tilde{\mu}$ to any interval (x_0, x_∞) with $x_0 < x_\infty$ without affecting the asymptotic behaviour of $\tilde{K}$. Hence assume $\tilde{\mu}$ lives on $D = \{\varphi < \infty\}$ and write

$$d\tilde{\mu} = e^{-\varphi} d\rho.$$

Then $d\tilde{\mu}_\lambda = e^{-\varphi_\lambda} d\rho$ with $\varphi_\lambda(x) = \varphi(x) - \lambda x$. Let ρ_n be the image of ρ/a_n under the affine map $x \mapsto (x - b_n)/a_n$ and $\tilde{\pi}^0_n$ the probability distribution of $\tilde{U}_{\lambda_n}$, $e^{-\varphi^*_n}$ the density of U_{λ_n}. Then

$$c_n d\tilde{\pi}^0_n = e^{-\varphi^*_n} d\rho_n, \qquad c_n = \tilde{K}(\lambda_n)/K(\lambda_n).$$

We claim that $\rho_n \to \lambda$ weakly, where λ is Lebesgue measure.

Let χ be continuous with compact support contained in $\{\psi < \infty\}$. Then $c_n e^{\varphi_n^*} \to e^{\psi}$ uniformly on the support of χ and χe^{ψ} is continuous. Hence $\tilde{\pi}_n^0 \to e^{-\psi} d\lambda$ weakly implies

$$\int \chi d\rho_n = \int c_n e^{\varphi_n^*} \chi d\tilde{\pi}_n^0 \to \int \chi d\lambda.$$

This implies $\rho_n[0,u] \to u$ for $0 < u < a_0$. Hence (7.1) holds for the sequence (b_n). By compactness of $\mathcal{LC}_{101}$ it holds for $x \to x_\infty$. □

Corollary. *$\tilde{\sigma}^2(\lambda) \sim \sigma^2(\lambda)$ for $\lambda \to \lambda_\infty$, and hence $\tilde{\alpha}(x) \sim \alpha(x)$ for $x \to x_\infty$.*

For the second part we need two lemmas.

Lemma 10. *Let ψ be convex with domain $D = \{\psi < \infty\}$. Let $x_\infty = \sup D$. Suppose $c \in D$ and $\psi'(c+0) \geqslant 0$. Let $\alpha : D \to (0,\infty)$ be continuous and let ρ be a measure on D. Suppose*

$$\rho[c,y] \leqslant 2(y-c) + 2q_0, \qquad y > c, y \in D.$$

Then

$$\int_{[c,x_\infty)} e^{-\psi} d\rho \leqslant 2\int_c^\infty e^{-\psi(x)} dx + 2q_0 e^{-\psi(c)}. \tag{7.2}$$

Proof. By partial integration

$$\int_{[c,\infty)} e^{-\psi} d\rho = \int_c^\infty \varphi'(y) e^{-\varphi(y)} \rho[c,y] dy.$$

Let σ be the measure on $[c, x_\infty)$ with density 2 on (c, x_∞) and point mass $2q_0$ in c. A similar equality holds for σ. Then $\rho[c,y] \leqslant \sigma[c,y]$ gives (7.2). □

Lemma 11. *Suppose $\mu \in \mathcal{LC}$. Then*

$$\rho[x,y] \leqslant |y-x| + 2a_0(\alpha(x) \wedge \alpha(y)), \qquad x,y \in D. \tag{7.3}$$

Proof. Define $x_0 = x$ and $x_{n+1} = x_n + a_0\alpha(x_n)$. Let $x_m \leqslant y < x_{m+1}$. Then $\alpha(y) \geqslant \alpha(x_m)/2$ gives the inequality with $\alpha(y)$. Now increase x_0 until $x_m = y$. Then $\alpha(x_0) \leqslant 2\alpha(x)$ gives the inequality with $\alpha(x)$. □

Proposition 12. *Let $\mu \in \mathcal{LC}$ have density $e^{-\varphi}$ and mgf K. Let ρ on $D = \{\varphi < \infty\}$ satisfy* (7.1). *Set $d\tilde{\mu} = e^{-\varphi} d\rho$. Then the mgf $\tilde{K}$ of $\tilde{\mu}$ is finite on $[0, \lambda_\infty)$ and $\tilde{K} \sim K$ in λ_∞.*

Proof. First note that by (7.1) we may assume that

$$\rho[x, x + a_0\alpha(x)] \leqslant 2a_0\alpha(x), \qquad x \in D \tag{7.4}$$

replacing D by a cofinal interval (x_0, x_∞) if necessary.

Relation (7.1) also holds for the affine transforms ρ_n, D_n and α_n corresponding to the functions φ_n^*. Now assume $\varphi_n^* \to \psi$. The normalization implies $\alpha_n(0) = 1$, and $|\alpha_n'| \leqslant 2/a_0$ gives $a_0\alpha_n(v) \leqslant 1 + 2|v|$.

Set $J_n = \int e^{-\varphi_n^*} d\rho_n$. Then as in (6.3) and (6.4)

$$J_n = \int e^{\lambda_n x} d\tilde{\mu}_n / K(\lambda_n) = \tilde{K}(\lambda_n)/K(\lambda_n).$$

We have to prove that $J_n \to 1$.

We first look at the tails of the integral. Let $\epsilon > 0$ be arbitrary. Choose $C > 0$ large and let (u_n, v_n) be the open interval where $\varphi_n^* < C$. Lemma 10 applied with $q_0 = a_0\alpha(v_n)$ gives

$$\int_{[v_n,\infty)} e^{-\varphi_n^*} d\rho_n \leqslant 2\int_{v_n}^{\infty} e^{-\varphi_n^*(x)} dx + 2(1 + v_n)e^{-C}.$$

A similar result holds for the left tail. Proposition 7 now shows that the tails of the measures $\tilde{\mu}_n^0$, μ_n^* and of the limit measure π may be neglected uniformly.

Now consider the central part

$$J_n(C) = \int_{(u_n,v_n)} e^{-\varphi_n^*} d\rho_n.$$

We first assume that $b_n + u_n a_n \to x_\infty$. Since $u_n \to u$ and $v_n \to v$ where $(u, v) = \{\psi < C\}$ the functions α_n are uniformly bounded on the intervals (u_n, v_n) and $1_{(u_n,v_n)} d\rho_n \to 1_{(u,v)} d\lambda$ weakly where λ is Lebesgue measure. Hence $J_n(C) \to \int_u^v e^{-\psi(t)} dt$.

It is possible that $b_n + u_n a_n$ does not converge to x_∞. If this occurs then $\varphi'(x)$ is bounded and $\varphi(x) - \lambda_\infty x$ has a finite limit for $x \to \infty$. In this case $e^{-\varphi} \sim f_0(x) = e^{-c-\lambda_\infty x}$. Our result now follows from Karamata's theory: The mgf K_0 of f_0 has the form $K_0(\lambda_\infty - \xi) = e^{-c}/\xi$. Since $\tilde{K} \sim K \sim K_0$ in λ_∞ Karamata's Theorem states that $d\tilde{\mu}(x) = e^{-\lambda_\infty x} d\rho_0(x)$ where

$$R_0(x) = \rho_0(-\infty, x] \sim K_0(\lambda_\infty - 1/x) = e^{-c}x \qquad x \to \infty. \tag{7.5}$$

In this case $\alpha_0(x) \equiv x$ (for the exponential distributions expectation and standard deviation agree), $\alpha(x) \sim x$ and hence the asymptotic relation (7.5) is equivalent to our relation (2.1). □

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