PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 70(84) (2001), 79–93

REMAINDER TERM IN CHAKALOV–POPOVICIU QUADRATURES OF RADAU AND LOBATTO TYPE AND INFLUENCE FUNCTION

Miodrag M. Spalević

Communicated by Gradimir Milovanović

Abstract. Let f be a given real function defined on $[a,b]$, $-\infty < a < b \leq \infty$. We develop estimates of the remainder term in the quadrature formulas with multiple nodes (Q) below, where $\sigma = \sigma_n = (s_1, s_2, \ldots, s_n)$ is a given sequence of nonnegative integers, $p, q \in \mathbb{N}_0$, and $w(t)$ is a given weight function on (a,b). Let $N = 2(\sum_{\nu=1}^{n} s_\nu + n) + p + q$, and denote by $AC^k[a,b]$, $B^k[a,b]$, $C^k[a,b]$ the classes of functions whose the k-th derivative is absolutely continuous, bounded or continuous on $[a,b]$, respectively. An influence function is introduced, its relevant properties are investigated, and in classes of functions $AC^{N-1}[a,b]$, $B^N[a,b]$, $C^N[a,b]$ the error estimates are given. A numerical example is included.

1. Introduction

Let $d\varphi(t)$ be a given nonnegative measure on the real line $\mathbb{R}$, with compact or unbounded support, for which all moments $\mu_k = \int_{\mathbb{R}} t^k \, d\varphi(t)$ $(k = 0, 1, \ldots)$ exist and are finite, and $\mu_0 > 0$.

A quadrature formula of the form

$$\int_{\mathbb{R}} f(t) \, d\varphi(t) = \sum_{\nu=1}^{n} \sum_{i=0}^{2s} A_{i,\nu} f^{(i)}(\tau_\nu) + R(f), \tag{1.1}$$

where $A_{i,\nu} = A_{i,\nu}^{(n,s)}$, $\tau_\nu = \tau_\nu^{(n,s)}$ $(i = 0, 1, \ldots, 2s;\ \nu = 1, \ldots, n)$, which is exact for all algebraic polynomials of degree at most $2(s+1)n - 1$, was considered firstly by P. Turán (see [24]), in the case when $d\varphi(t) = dt$ on $[-1, 1]$. The case with a weight

2000 *Mathematics Subject Classification.* Primary 41A55; Secondary 65D32.
Key words and phrases. s-orthogonality, σ-orthogonality, multiple nodes, influence function, remainder term, error estimate.
Partially supported by Serbian Scientific Foundation

function $d\varphi(t) = w(t)\,dt$ on $[a, b]$ has been investigated by Italian mathematicians Ossicini, Ghizzetti, Guerra, Rosati, and also by Chakalov, Stroud, Stancu, Ionescu, Pavel, etc. (see the survey paper [11] for references).

The nodes τ_ν in (1.1) must be zeros of a (monic) polynomial $\pi_n(t)$ which minimizes the integral

$$\Phi \equiv \Phi(a_0, a_1, \dots, a_{n-1}) = \int_{\mathbb{R}} \pi_n(t)^{2s+2}\, d\varphi(t),$$

where $\pi_n(t) = t^n + a_{n-1}t^{n-1} + \ \dots\ + a_1 t + a_0$. In order to minimize Φ we must have

$$\int_{\mathbb{R}} \pi_n(t)^{2s+1} t^k\, d\varphi(t) = 0, \quad k = 0, 1,\ \dots\ , n-1. \tag{1.2}$$

Such polynomials $\pi_n(t)$, which satisfy this new type of orthogonality so called "*power orthogonality*" are known as s-orthogonal (or s-self associated) polynomials with respect to the measure $d\varphi(t)$.

For $s = 0$ we have the standard case of orthogonal polynomials.

Take now a sequence of nonnegative integers $\sigma = (s_1, s_2, \dots)$. Consider a generalization of Gauss-Turán quadrature formula (1.1) to rules having nodes with arbitrary multiplicities

$$\int_{\mathbb{R}} f(t)\, d\varphi(t) = \sum_{\nu=1}^{n} \sum_{i=0}^{2s_\nu} A_{i,\nu} f^{(i)}(\tau_\nu) + R(f), \tag{1.3}$$

where $A_{i,\nu} = A_{i,\nu}^{(n,\sigma)}$, $\tau_\nu = \tau_\nu^{(n,\sigma)}$ $(i = 0, 1, \dots, 2s_\nu;\ \nu = 1, \dots, n)$. Such formulas were derived independently by Chakalov [2, 3] and Popoviciu [18]. A deep theoretical progress in this subject was made by Stancu (see [23] and references in it).

In this case, it is important to assume that the nodes τ_ν $(= \tau_\nu^{(n,\sigma)})$ are ordered, say

$$\tau_1 < \tau_2 < \cdots < \tau_n, \quad \tau_\nu \in \operatorname{supp}(d\lambda), \tag{1.4}$$

with odd multiplicities $2s_1 + 1,\ 2s_2 + 1,\ \dots,\ 2s_n + 1$, respectively, in order to have uniqueness of Chakalov-Popoviciu quadrature formula (1.3) (cf. Karlin and Pinkus [8]). Then this quadrature formula has the maximum degree of exactness

$$d_{\max} = 2 \sum_{\nu=1}^{n} s_\nu + 2n - 1$$

if and only if

$$\int_{\mathbb{R}} \prod_{\nu=1}^{n} (t - \tau_\nu)^{2s_\nu+1} t^k\, d\varphi(t) = 0, \quad k = 0, 1, \dots, n-1. \tag{1.5}$$

The last *orthogonality conditions* correspond to (1.2). The existence of such quadrature rules was proved by Chakalov [2], Popoviciu [18], Morelli and Verna [15], and existence and uniqueness subject to (1.4) by Ghizzetti and Ossicini [6].

The conditions (1.5) define a sequence of polynomials $\{\pi_{n,\sigma}\}_{n\in\mathbb{N}_0}$,

$$\pi_{n,\sigma}(t) = \prod_{\nu=1}^{n}\left(t-\tau_\nu^{(n,\sigma)}\right), \quad \tau_1^{(n,\sigma)} < \tau_2^{(n,\sigma)} < \cdots < \tau_n^{(n,\sigma)}, \quad \tau_\nu^{(n,\sigma)} \in \operatorname{supp}(d\lambda),$$

such that

$$\int_{\mathbb{R}} \pi_{k,\sigma}(t)\prod_{\nu=1}^{n}\left(t-\tau_\nu^{(n,\sigma)}\right)^{2s_\nu+1} d\varphi(t) = 0, \quad k=0,1,\dots,n-1.$$

These polynomials are called σ-orthogonal polynomials and they correspond to the sequence $\sigma = (s_1, s_2, \dots)$. If we have $\sigma = (s, s, \dots)$, the above polynomials reduce to the s-orthogonal polynomials.

In this paper we consider the generalized Chakalov-Popoviciu quadrature formulae

$$\text{(Q)} \qquad \begin{aligned}\int_a^b w(t)f(t)\,dt &= \sum_{i=0}^{p-1} A_{i,0}f^{(i)}(a) + \sum_{\nu=1}^{n}\sum_{i=0}^{2s_\nu} A_{i,\nu}f^{(i)}(\tau_\nu)\\ &\quad + \sum_{i=0}^{q-1} A_{i,n+1}f^{(i)}(b) + R(f),\end{aligned}$$

with arbitrary, $\tau_\nu (\nu = 1, \dots, n)$, and fixed, a and (or) b, multiple nodes. A such quadrature formula has maximum degree of exactness $N-1$ if and only if

$$\int_a^b w(t)(1+t)^p(1-t)^q\prod_{\nu=1}^{n}(t-\tau_\nu)^{2s_\nu+1}t^k\,dt = 0, \quad k=0,1,\dots,n-1.$$

Recent proofs of the existence and the uniqueness of such quadrature rules have been obtained in [19], [22]. Proofs of convergence of such formulas can be found in [17], [20].

In Section 2 an influence function is considered, its relevant properties are investigated, and in the classes of functions $AC^{N-1}[a,b]$, $B^N[a,b]$, $C^N[a,b]$ the error estimates are given. In order to illustrate the possibility of use these error estimates we give a numerical example.

2. Error estimates for quadrature formulae of Radau and Lobatto type connected to σ-orthogonal polynomials

In [16] (see also [5]) Ossicini, for the Gauss-Turán quadratures (formula (Q) with $p = q = 0$, $s_1 = \cdots = s_n = s$, or (1.1) with $d\varphi(t) = w(t)\,dt$), and the Gauss-Turán quadratures of Lobatto type (formula (Q) with $s_1 = \cdots = s_n = s$

and $p = q = 2s + 1$), constructed an influence function, investigated its relevant properties, and in some classes of functions gave error estimates. Recently, we have generalized those results to the formula (Q) with $p = q = 0$ (or (1.3) with $d\varphi(t) = w(t)\,dt$) (see [14]). In this section we will consider the quadrature formula (Q), the general case.

For all undefined notions and notations we refer to [5].

Radau formula. Let

$$\int_a^b w(t)f(t)\,dt = \sum_{i=0}^{p-1} A_{i,0}^R f^{(i)}(a) + \sum_{\nu=1}^{n}\sum_{i=0}^{2s_\nu} A_{i,\nu}^R f^{(i)}(\tau_\nu) + R^R(f), \tag{2.1}$$

$-\infty < a < \infty$, $p \in \mathbb{N}$, with $R^R(f) = 0$ for $f \in \mathcal{P}_{2(\sum_{\nu=1}^n s_\nu + n)+p-1}$, be the generalized Chakalov-Popoviciu quadrature formula of Radau type. With $\mathcal{P}_k$ we denoted the set of all polynomials of degree at most k, $k \in \mathbb{N}_0$. Denote $N = 2(\sum_{\nu=1}^n s_\nu + n) + p$. Concerning the assumptions on $w(t), f(t)$ for the validity of (2.1) we have the following theorem:

THEOREM 2.1. *Formula* (2.1) *is valid under the following hypotheses:*

$$\begin{aligned}
&w(t) \in L[a,b], \quad f(t) \in AC^{N-1}[a,b], \qquad \textit{if } b \textit{ is finite},\\
&t^N w(t) \in L[a,\infty), \quad f(t) \in AC_{loc}^{N-1}[a,\infty),\\
&f^{(N)}(t)\int_t^\infty \xi^{N-1} w(\xi)\,d\xi \in L[0,\infty).
\end{aligned}$$

The proof is the same as the one of Theorem 4.13.I. in [5, pp. 132–133] and will be omitted.

Consider, for example, the case:

$$p - 1 < 2s_1 < 2s_2 < \cdots < 2s_n, \tag{2.2}$$

i. e., $N - p - 1 > N - 2s_1 - 2 > \cdots > N - 2s_n - 2$. Let p be odd, without loss of generality. Then, we have that N is odd. Assuming already computed the nodes τ_ν and the coefficients $A_{i,\nu}^R$ for the remainder in (2.1) we have (see [5]):

$$R^R(f) \equiv R(f) = \int_a^b \Phi(t) f^{(N)}(t)\,dt, \tag{2.3}$$

where the influence-function $\Phi(t)$ is expressed by

$$\Phi(t) = \varphi_{\nu+1}(t) \qquad \text{for} \quad \tau_\nu < t < \tau_{\nu+1}, \quad \nu = 0,1,\ldots,n;\ \tau_0 = a, \tau_{n+1} = b, \tag{2.4}$$

and the functions $\varphi_\nu(t)$, integrals of the differential equation $\varphi^{(N)}(t) = -w(t)$ (since N is odd), are given by the formulae

$$\begin{aligned}
\varphi_\nu(t) = -\int_a^t w(\xi)\frac{(t-\xi)^{N-1}}{(N-1)\,!}\,d\xi &+ \sum_{j=1}^{\nu-1}\sum_{i=0}^{2s_j}(-1)^i A_{i,j}\frac{(t-\tau_j)^{N-i-1}}{(N-i-1)\,!}\\
&+ \sum_{i=0}^{p-1}(-1)^i A_{i,0}\frac{(t-a)^{N-i-1}}{(N-i-1)\,!}\,,
\end{aligned} \tag{2.5}$$

where $\nu = 1, \dots, n+1$, and $A_{h,j} = A^R_{h,j}$.

For $\varphi_{n+1}(t)$ we have

$$\varphi_{n+1}(t) = \int_t^b w(\xi) \frac{(t-\xi)^{N-1}}{(N-1)\,!}\, d\xi. \tag{2.6}$$

From (2.4), (2.5) it follows, differentiating k times (with $0 \le k \le N-1$):

$$\Phi^{(k)}(t) = \varphi_\nu^{(k)}(t) \qquad \text{for} \quad t \in (\tau_{\nu-1}, \tau_\nu), \quad \nu = 1, \dots, n+1, \tag{2.7}$$

where for $l = \nu, \nu-1, \dots, 1$,

$$\begin{aligned}\varphi_\nu^{(k)}(t) = -\int_a^t w(\xi) \frac{(t-\xi)^{N-k-1}}{(N-k-1)\,!}\, d\xi + \sum_{j=1}^{l-1} \sum_{i=0}^{2s_j} (-1)^i A_{i,j} \frac{(t-\tau_j)^{N-i-k-1}}{(N-i-k-1)\,!} \\ + \sum_{j=l}^{\nu-1} \sum_{i=0}^{N-k-1} (-1)^i A_{i,j} \frac{(t-\tau_j)^{N-i-k-1}}{(N-i-k-1)\,!} + \sum_{i=0}^{p-1} (-1)^i A_{i,0} \frac{(t-a)^{N-i-k-1}}{(N-i-k-1)\,!}\,,\end{aligned}$$

with: (i) $0 \le k \le N - 2s_{\nu-1} - 2$, for $l = \nu$, (ii) $N - 2s_l - 1 \le k \le N - 2s_{l-1} - 2$, for $l = \nu-1, \nu-2, \dots, 1$, where we put $s_0 = (p-1)/2$, and

$$\begin{aligned}\varphi_\nu^{(k)}(t) = -\int_a^t w(\xi) \frac{(t-\xi)^{N-k-1}}{(N-k-1)\,!}\, d\xi + \sum_{j=1}^{\nu-1} \sum_{i=0}^{N-k-1} (-1)^i A_{i,j} \frac{(t-\tau_j)^{N-i-k-1}}{(N-i-k-1)\,!} \\ + \sum_{i=0}^{N-k-1} (-1)^i A_{i,0} \frac{(t-a)^{N-i-k-1}}{(N-i-k-1)\,!}\,,\end{aligned} \tag{2.8}$$

for $N - p \le k \le N-1$. (We used the convention $\sum_i^j \cdot \equiv 0$ for $j < i$.)

For the derivatives of $\varphi_{n+1}(t)$ $(t \in (\tau_n, b))$ we can use the following formulas:

$$\varphi_{n+1}^{(k)}(t) = \int_t^b w(\xi) \frac{(t-\xi)^{N-k-1}}{(N-k-1)\,!}\, d\xi, \quad t \in (\tau_n, b). \tag{2.9}$$

Now, we can conclude that

$$\begin{aligned}&\Phi^{(k)}(a) = 0, \;\; k = 0, 1, \dots, N-p-1, \\ &\Phi^{(k)}(b) = 0, \;\; k = 0, 1, \dots, N-1,\end{aligned} \tag{2.10}$$

and that the functions $\Phi(t), \Phi'(t), \dots, \Phi^{(n-2_n-2)}(t)$ are continuous in $[a, b]$, since $N - 2s_n - 2 = \min_{1 \le \nu \le n}(N - 2s_\nu - 2)$, while $\Phi^{(N-2s_n-1)}(t), \dots, \Phi^{(N-1)}(t)$ have discontinuities of the first kind at the points $\tau_1, \tau_2, \dots, \tau_n$. From (2.9) we conclude

$$(-1)^k \Phi^{(k)}(t) > 0 \quad \text{for} \quad t \in (\tau_n, b), \;\; k = 0, 1, \dots, N-1, \tag{2.11}$$

and, particularly, $\Phi(t) > 0$ on (τ_n, b).

Let the weight function $w(t)$ be not identically zero in any interval contained in $[a, b]$. We will prove that the influence function $\Phi(t)$ is positive inside $[a, b]$. We give the proof for the case (2.2). (The general case can be considered in analogous way, see the Lobatto case.) If we identify the function $\Phi(t)$ as a monospline, then the property $\Phi(t) > 0$ on (a, b) is a corollary from the Micchelli estimate [9] of the number of zeros for monosplines with multiplicities (cf. Braess [p. 241, 1]). Our proof is direct and use only Rolle theorem.

We show that $\Phi^{(N-2s_l-2)}(t)$ $(l = 1, 2, \ldots, n)$ has at most $2s_l + 2$ zeros in each interval $[\tau_{\nu-1}, \tau_\nu]$, $\nu = 1, 2, \ldots, n$. In fact, should it have $2s_l + 3$ of them, for the Rolle theorem, $\Phi^{(N-2s_l-1)}(t)$ would have at least $2s_l + 2$ zeros inside $[\tau_{\nu-1}, \tau_\nu]$, $\Phi^{(N-2s_l)}(t)$ would have at least $2s_l + 1$ zeros and so on, until we may conclude that $\Phi^{(N-1)}(t)$ would have at least two zeros inside $[\tau_{\nu-1}, \tau_\nu]$. But this is absurd since from (2.8) there follows that, for $t \in (\tau_{\nu-1}, \tau_\nu)$, we have

$$\Phi^{(N-1)}(t) = \varphi_\nu^{(N-1)}(t) = -\int_a^t w(\xi)\, d\xi + \sum_{j=0}^{\nu-1} A_{0,j}$$

and this function is decreasing (for the hypothesis on $w(t)$).

$\Phi^{(N-p)}(t)$ is continuous in $[a, \tau_1]$ and, we can prove as for $\Phi^{(N-2s_l-2)}(t)$, it has at most p zeros in $[a, \tau_1]$. $\Phi^{(N-2s_\nu-2)}(t)$ $(\nu = 1, 2, \ldots, n)$ is continuous in $[a, \tau_1]$ and let it have α zeros in $[a, \tau_1]$. Applying Rolle theorem (using (2.10) for a) we conclude that $\Phi^{(N-p)}(t)$ has at least α zeros in (a, τ_1). Since $\alpha \le p$, we have that $\Phi^{(N-2s_\nu-2)}(t)$ $(\nu = 1, 2, \ldots, n)$ has at most p zeros in $[a, \tau_1]$.

$\Phi^{(N-2s_1-2)}(t)$ is continuous in $[a, \tau_2]$ and has at most $p + (2s_1 + 2)$ zeros in $[a, \tau_2]$. $\Phi^{(N-2s_\nu-2)}(t)$ $(\nu = 2, \ldots, n)$ is continuous in $[a, \tau_2]$ and let it have α_1 zeros in $[a, \tau_2]$. Applying Rolle theorem (using (2.10) for a) we conclude that $\Phi^{(N-2s_\nu-1)}(t)$ has at least α_1 zeros in (a, τ_2), etc., $\Phi^{(N-2s_1-2)}(t)$ has at least α_1 zeros in (a, τ_2). Since $\alpha_1 \le p + (2s_1 + 2)$, we have that $\Phi^{(N-2s_\nu-2)}(t)$ $(\nu = 2, \ldots, n)$ has at most $p + (2s_1 + 2)$ zeros in $[a, \tau_2]$.

$\Phi^{(N-2s_2-2)}(t)$ is continuous in $[a, \tau_3]$ and has at most $p + (2s_1 + 2) + (2s_2 + 2)$ zeros in $[a, \tau_3]$. $\Phi^{(N-2s_\nu-2)}(t)$ $(\nu = 3, \ldots, n)$ is continuous in $[a, \tau_3]$ and let it have α_2 zeros in $[a, \tau_3]$. Applying Rolle theorem (using (2.10) for a) we conclude that $\Phi^{(N-2s_\nu-1)}(t)$ has at least α_2 zeros in (a, τ_3), etc., $\Phi^{(N-2s_2-2)}(t)$ has at least α_2 zeros in (a, τ_3). Since $\alpha_2 \le p + (2s_1 + 2) + (2s_2 + 2)$, we have that $\Phi^{(N-2s_\nu-2)}(t)$ $(\nu = 3, \ldots, n)$ has at most $p + (2s_1 + 2) + (2s_2 + 2)$ zeros in $[a, \tau_3]$.

In analogous way, we conclude that $\Phi^{(N-2s_n-2)}(t)$ is continuous in $[a, \tau_n]$ and has at most

$$p + \sum_{\nu=1}^{n-1} (2s_\nu + 2) = N - 2s_n - 2$$

zeros in $[a, \tau_n]$, and also in (a, b), because of (2.11).

We may then show that $\Phi(t)$ does not vanish inside $[a, b]$ and therefore it is positive, because it is such on (x_n, b). In fact, if $\Phi(t)$ should vanish at one point in

(a,b), using (2.2) and (2.10) and applying Rolle theorem, we find that $\Phi'(t)$ would vanish at least two times, etc., $\Phi^{(N-2s_n-2)}(t)$ would vanish at least $N-2s_n-1$ times, in contraposition with the preceding deduction, because $N-2s_n-1 \le N-2s_n-2$ gives $1 \le 0$.

So, we proved the theorem:

THEOREM 2.2. *Under the hypothesis that the weight function $w(t)$ is not identically zero in any interval contained in $[a,b]$, the influence function $\Phi(t)$ defined by* (2.4) (*together with* (2.5) *and* (2.6)) *belongs to the class $C^{N-2s_k-2}[a,b]$, where $N-2s_k-2 = \min_{1\le\nu\le n}(N-2s_\nu-2)$, and it is positive inside $[a,b]$.*

Now, we can estimate the remainder in the formulas of the type (2.1), by using (2.3).

1^0 If $f(t) \in AC^{N-1}[a,b]$ and $a,b \in R$ we have

$$|R(f)| \le \max_{a\le t\le b} \Phi(t)V_{N-1} = \Phi(x_0)V_{N-1},$$

where V_{N-1} denotes the total variation of the function $f^{(N-1)}(t)$ absolutely continuous on the interval $[a,b]$. Because $\Phi'(t)$ vanish in exact one point of the interval (a,b) it holds $(\exists x_0 \in (a,b))\ \max\limits_{a\le t\le b} \Phi(t) = \Phi(x_0)$.

2^0 If $f^{(N)}(t)$ is bounded in $[a,b]$, i. e., $M_N = \sup\limits_{a\le t<b} \left|f^{(N)}(t)\right|$, $b \le \infty$, then we have

$$|R(f)| \le M_N \int_a^b \Phi(t)\,dt.$$

3^0 If $f \in C^N[a,b]$, $b < \infty$, because $\Phi(t) > 0$ on (a,b) we may apply the mean value theorem and write

$$R(f) = f^{(N)}(\xi) \int_a^b \Phi(t)\,dt, \quad \xi \in (a,b).$$

Lobatto formula. Let

$$\begin{aligned}\int_a^b w(t)f(t)\,dt = &\sum_{i=0}^{p-1} A_{i,0}^L f^{(i)}(a) + \sum_{\nu=1}^{n}\sum_{i=0}^{2s_\nu} A_{i,\nu}^L f^{(i)}(\tau_\nu) \\ &+ \sum_{i=0}^{q-1} A_{i,n+1}^L f^{(i)}(b) + R^L(f),\end{aligned} \tag{2.12}$$

$-\infty < a < b < \infty$, $p,q \in \mathbb{N}$, with $R^L(f) = 0$ for $f \in \mathcal{P}_{2(\sum_{\nu=1}^n s_\nu + n)+p+q-1}$, be the generalized Chakalov-Popoviciu quadrature formula of Lobatto type. Denote $N = 2(\sum_{\nu=1}^n s_\nu + n) + p + q$. Concerning the assumptions on $w(t), f(t)$ for the validity of (2.12) we have the following theorem:

THEOREM 2.3. *Formula* (2.12) *is valid under the following hypothesis:*

$$w(t) \in L[a,b], \quad f(t) \in AC^{N-1}[a,b].$$

The proof is the same as one of the theorem 4.13.I. in [5, pp. 132–133] and will be omitted.

Let, for simplicity,

$$p-1 < 2s_1 < 2s_2 < \cdots < 2s_n < q-1, \tag{2.13}$$

i. e.,

$$N-p-1 > N-2s_1-2 > \cdots > N-2s_n-2 > N-q-1,$$

and, let $p+q$ – be even, without loss of generality. Then, N is even.

Assuming already computed the nodes τ_ν and the coefficients $A_{i,\nu}^L$ for the remainder in (2.12) we have (see [5]):

$$R^L(f) \equiv R(f) = \int_a^b \Phi(t) f^{(N)}(t)\,dt, \tag{2.14}$$

where the influence-function $\Phi(t)$ is expressed by

$$\Phi(t) = \varphi_{\nu+1}(t) \qquad \text{for} \quad \tau_\nu < t < \tau_{\nu+1}, \quad \nu = 0,1,\ldots,n;\ \tau_0 = a, \tau_{n+1} = b, \tag{2.15}$$

and the functions $\varphi_\nu(t)$, integrals of the differential equation $\varphi^{(N)}(t) = w(t)$ (since N is even), are given by the formulae

$$\begin{aligned}\varphi_\nu(t) = \int_a^t w(\xi)\frac{(t-\xi)^{N-1}}{(N-1)\,!}\,d\xi - \sum_{j=1}^{\nu-1}\sum_{i=0}^{2s_j}(-1)^i A_{i,j}\frac{(t-\tau_j)^{N-i-1}}{(N-i-1)\,!}\\ -\sum_{i=0}^{p-1}(-1)^i A_{i,0}\frac{(t-a)^{N-i-1}}{(N-i-1)\,!}\,,\end{aligned} \tag{2.16}$$

where $\nu = 1,\ldots,n+1$, and $A_{h,j} = A_{h,j}^L$.

For $\varphi_{n+1}(t)$ we have

$$\varphi_{n+1}(t) = -\int_t^b w(\xi)\frac{(t-\xi)^{N-1}}{(N-1)\,!}\,d\xi + \sum_{i=0}^{q-1}(-1)^i A_{i,n+1}\frac{(t-b)^{N-i-1}}{(N-i-1)\,!}\,. \tag{2.17}$$

From (2.15), (2.16) it follows, differentiating k times (with $0 \le k \le N-1$):

$$\Phi^{(k)}(t) = \varphi_\nu^{(k)}(t) \qquad \text{for} \quad t \in (\tau_{\nu-1},\tau_\nu), \quad \nu = 1,\ldots,n+1,$$

where for $l = \nu, \nu-1, \dots, 1$,

$$\varphi_\nu^{(k)}(t) = \int_a^t w(\xi)\frac{(t-\xi)^{N-k-1}}{(N-k-1)!}\,d\xi - \sum_{j=1}^{l-1}\sum_{i=0}^{2s_j}(-1)^i A_{i,j}\frac{(t-\tau_j)^{N-i-k-1}}{(N-i-k-1)!}$$
$$-\sum_{j=l}^{\nu-1}\sum_{i=0}^{N-k-1}(-1)^i A_{i,j}\frac{(t-\tau_j)^{N-i-k-1}}{(N-i-k-1)!} - \sum_{i=0}^{p-1}(-1)^i A_{i,0}\frac{(t-a)^{N-i-k-1}}{(N-i-k-1)!}\,,$$

with
(i) $0 \le k \le N-2s_{\nu-1}-2$, for $l=\nu$,
(ii) $N-2s_l-1 \le k \le N-2s_{l-1}-2$, for $l=\nu-1, \nu-2, \dots, 1$, where we put $s_0=(p-1)/2$,
and

$$\varphi_\nu^{(k)}(t) = \int_a^t w(\xi)\frac{(t-\xi)^{N-k-1}}{(N-k-1)!}\,d\xi - \sum_{j=1}^{\nu-1}\sum_{i=0}^{N-k-1}(-1)^i A_{i,j}\frac{(t-\tau_j)^{N-i-k-1}}{(N-i-k-1)!}$$
$$-\sum_{i=0}^{N-k-1}(-1)^i A_{i,0}\frac{(t-a)^{N-i-k-1}}{(N-i-k-1)!}\,,$$

for $N-p \le k \le N-1$.

For the derivatives of $\varphi_{n+1}(t)$ $(t \in (\tau_n, b))$ we can use the following formulas:

$$\varphi_{n+1}^{(k)}(t) = -\int_t^b w(\xi)\frac{(t-\xi)^{N-k-1}}{(n-k-1)!}\,d\xi + \sum_{h=0}^{I}(-1)^i A_{i,n+1}\frac{(t-b)^{N-i-k-1}}{(N-i-k-1)!}\,,$$

with $I=q-1$, for $0 \le k \le N-q-1$; $I=N-k-1$, for $N-q \le k \le N-1$.

Now, we can conclude that

$$\begin{aligned} &\Phi^{(k)}(a)=0,\ k=0,1,\dots,N-p-1,\\ &\Phi^{(k)}(b)=0,\ k=0,1,\dots,N-q-1, \end{aligned} \tag{2.18}$$

and that the functions $\Phi(t), \Phi'(t), \dots, \Phi^{(N-2s_n-2)}(t)$ are continuous in $[a,b]$, since $N-2s_n-2 = \min_{1\le\nu\le n}(N-2s_\nu-2)$, while $\Phi^{(n-2s_n-1)}(t), \dots, \Phi^{(n-1)}(t)$ have discontinuities of first kind at the points $\tau_1, \tau_2, \dots, \tau_n$.

The same conclusions can be derived for an arbitrary case $\sigma=(s_1, s_2, \dots, s_n)$, $p, q \in \mathbb{N}$, $s_\nu \in \mathbb{N}_0$ $(\nu=1,2,\dots,n)$. So, we have just proved that the influence function $\Phi(t)$ defined by (2.15) (together with (2.16) and (2.17)) belongs to the class $C^{N-2s_k-2}[a,b]$, where $N-2s_k-2 = \min\limits_{1\le\nu\le n}(N-2s_\nu-2)$.

If we put $f(t)=(t-a)^p(b-t)^q\prod_{\nu=1}^n(t-\tau_\nu)^{2s_\nu+2}$ in (2.14), then

$$(-1)^q N!\int_a^b \Phi(t)\,dt = \int_a^b w(t)(t-a)^p(b-t)^q\prod_{\nu=1}^n(t-\tau_\nu)^{2s_\nu+2}\,dt\,.$$

So, we obtain that

$$\int_a^b \Phi(t)\,dt \begin{cases} <0, & \text{if } q \text{ is odd,} \\ >0, & \text{if } q \text{ is even.} \end{cases}$$

Therefore, if $\Phi(t)$ does not vanish in (a,b) it holds a sign on this interval.

Let the weight function $w(t)$ be not identically zero in any interval contained in $[a,b]$. We will prove that the influence function $\Phi(t)$ holds a sign in (a,b).

We will give a proof for a sufficient general case. Then, proceed in analogous way, a proof for any other case can be performed. The consideration will be given in detail.

Let $n=9$ and

$$q-1>2s_1>2s_5>2s_4>2s_8>p-1>2s_9>2s_6>2s_3>2s_7>2s_2,$$

i. e.,

$$N-q-1<N-2s_1-2<\cdots<N-2s_2-2.$$

A point τ_ν, $\nu=2,3,\ldots,n-1$, we will call a *point of partition* of $[a,b]$ if for it holds $s_{\nu-1}\le s_\nu>s_{\nu+1}$, i. e., $N-2s_{\nu-1}-2\ge N-2s_\nu-2<N-2s_{\nu+1}-2$. τ_1 is the *point of partition* of $[a,b]$ if for it holds $p-1\le s_1>s_2$. τ_n is the *point of partition* of $[a,b]$ if for it holds $s_{n-1}\le s_n>q-1$. Denote by I the *index set* whose elements are the indices of the points of partition of $[a,b]$. It is clearly that $I\subset\{1,2,\ldots,n\}$. Therefore, in our case, the points of partition of $[a,b]$ are τ_1,τ_5,τ_8, and $I=\{1,5,8\}$. $[a,b]$ by the points of partition we divide into the *intervals of partition*, in our case $[a,\tau_1],[\tau_1,\tau_5],[\tau_5,\tau_8],[\tau_8,b]$, on which we consider the functions $\Phi^{(N-2s_\nu-2)}(t)$, $\nu\in I$. It is clearly that $[a,b]$ can be represented as the union of the intervals of partition.

For $\nu\in I$, order in the decreasing sequence the values $N-2s_\nu-2$, and consider the functions $\Phi^{(N-2s_\nu-2)}(t)$, respectively . Therefore, in our case we consider $\Phi^{(N-2s_8-2)}(t)$, $\Phi^{(N-2s_5-2)}(t)$, $\Phi^{(N-2s_1-2)}(t)$, respectively.

a) Firstly, consider $\Phi^{(N-2s_8-2)}(t)$, which is continuous in $[\tau_5,b]=[\tau_5,\tau_8]\cup[\tau_8,b]$.

a.1) Firstly, consider $[\tau_5,\tau_8]$. For $\nu\in\{5,6,7,8\}$ (the indices of the nodes belong to $[\tau_5,\tau_8]$), order in the decreasing sequence the values $N-2s_\nu-2$ so that the last be one which correspond to the point of partition τ_8, and then consider the functions $\Phi^{(N-2s_\nu-2)}(t)$, respectively. Therefore, in our case we consider $\Phi^{(N-2s_7-2)}(t)$, $\Phi^{(N-2s_6-2)}(t)$,$\Phi^{(N-2s_8-2)}(t)$, respectively.

$\Phi^{(N-2s_7-2)}(t)$ is continuous in $[\tau_6,\tau_8]=[\tau_6,\tau_7]\cup[\tau_7,\tau_8]$ and has at most $(2s_7+2)+(2s_7+2)$ zeros in it.

$\Phi^{(N-2s_6-2)}(t)$ is continuous in $[\tau_5,\tau_8]=[\tau_5,\tau_6]\cup[\tau_6,\tau_8]$. Let $\Phi^{(N-2s_6-2)}(t)$ have β_6 zeros in $[\tau_6,\tau_8]$. Then, using (2.18) and applying Rolle theorem, we conclude that $\Phi^{(N-2s_6-1)}(t)$ has at least β_6-1 zeros in (τ_6,τ_8), etc., $\Phi^{(N-2s_7-2)}(t)$ has at least $\beta_6-(2s_6-2s_7)$ zeros in (τ_6,τ_8). Therefore, we have $\beta_6-(2s_6-2s_7)\le(2s_7+2)+(2s_7+2)$, i. e., $\beta_6\le(2s_6+2)+(2s_7+2)$.

Therefore, $\Phi^{(N-2s_6-2)}(t)$ has at most $(2s_6+2)+(2s_6+2)+(2s_7+2)$ zeros in $[\tau_5,\tau_8]$.

Let $\Phi^{(N-2s_8-2)}(t)$ have α_8 zeros in $[\tau_5, \tau_8]$. Then, using (2.18) and applying Rolle theorem, we conclude that $\Phi^{(N-2s_8-1)}(t)$ has at least $\alpha_8 - 1$ zeros in (τ_5, τ_8), etc., $\Phi^{(N-2s_6-2)}(t)$ has at least $\alpha_8 - (2s_8 - 2s_6)$ zeros in (τ_5, τ_8). Therefore, we have $\alpha_8 - (2s_8 - 2s_6) \le (2s_6 + 2) + (2s_6 + 2) + (2s_7 + 2)$, i. e., $\alpha_8 \le (2s_6 + 2) + (2s_7 + 2) + (2s_8 + 2)$.

a.2) Consider $[\tau_8, b]$. For $\nu \in \{8, 9\}$ (the indices of the nodes belong to $[\tau_8, b]$), order in the decreasing sequence the values $N - 2s_\nu - 2$ so that the last be one which correspond to the point of partition τ_8, and then consider the functions $\Phi^{(N-2s_\nu-2)}(t)$, respectively. Therefore, in our case we consider $\Phi^{(N-2s_9-2)}(t)$, $\Phi^{(N-2s_8-2)}(t)$, respectively.

$\Phi^{(N-2s_9-2)}(t)$ is continuous in $[\tau_8, b] = [\tau_8, \tau_9] \cup [\tau_9, b]$ and has at most $(2s_9 + 2) + (2s_9 + 2)$ zeros in it.

Let $\Phi^{(N-2s_8-2)}(t)$ have β_8 zeros in $[\tau_8, b]$. Then, using (2.18) and applying Rolle theorem, we conclude that $\Phi^{(N-2s_8-1)}(t)$ has at least $\beta_8 - 1$ zeros in (τ_8, b), etc., $\Phi^{(N-2s_9-2)}(t)$ has at least $\beta_8 - (2s_8 - 2s_9)$ zeros in (τ_8, b). Therefore, we have $\beta_8 - (2s_8 - 2s_9) \le (2s_9 + 2) + (2s_9 + 2)$, i. e., $\beta_8 \le (2s_8 + 2) + (2s_9 + 2)$.

Therefore, using a.1), a.2), we conclude that $\Phi^{(N-2s_8-2)}(t)$ has at most $(2s_8 + 2) + \sum_{\nu=6}^{9}(2s_\nu + 2)$ zeros in $[\tau_5, b]$.

b) Now, consider $\Phi^{(N-2s_5-2)}(t)$, which is continuous in $[\tau_1, b] = [\tau_1, \tau_5] \cup [\tau_5, b]$.

b.1) Consider $[\tau_1, \tau_5]$. For $\nu \in \{1, 2, 3, 4, 5\}$ (the indices of the nodes belong to $[\tau_1, \tau_5]$), order in the decreasing sequence the values $N - 2s_\nu - 2$ so that the last be one which correspond to the point of partition τ_5, and then consider the functions $\Phi^{(N-2s_\nu-2)}(t)$, respectively. Therefore, in our case we consider $\Phi^{(N-2s_2-2)}(t)$, $\Phi^{(N-2s_3-2)}(t)$, $\Phi^{(N-2s_4-2)}(t)$, $\Phi^{(N-2s_5-2)}(t)$, respectively.

$\Phi^{(N-2s_2-2)}(t)$ is continuous in $[\tau_1, \tau_3] = [\tau_1, \tau_2] \cup [\tau_2, \tau_3]$ and has at most $(2s_2 + 2) + (2s_2 + 2)$ zeros in it.

$\Phi^{(N-2s_3-2)}(t)$ is continuous in $[\tau_1, \tau_4] = [\tau_1, \tau_3] \cup [\tau_3, \tau_4]$.

Let $\Phi^{(N-2s_3-2)}(t)$ have α_3 zeros in $[\tau_1, \tau_3]$. Then, using (2.18) and applying Rolle theorem, we conclude that $\Phi^{(N-2s_3-1)}(t)$ has at least $\alpha_3 - 1$ zeros in (τ_1, τ_3), etc., $\Phi^{(N-2s_2-2)}(t)$ has at least $\alpha_3 - (2s_3 - 2s_2)$ zeros in (τ_1, τ_3). Therefore, we have $\alpha_3 - (2s_3 - 2s_2) \le (2s_2 + 2) + (2s_2 + 2)$, i. e., $\alpha_3 \le (2s_2 + 2) + (2s_3 + 2)$.

Therefore, $\Phi^{(N-2s_3-2)}(t)$ has at most $(2s_2 + 2) + (2s_3 + 2) + (2s_3 + 2)$ zeros in $[\tau_1, \tau_4]$.

$\Phi^{(N-2s_4-2)}(t)$ is continuous in $[\tau_1, \tau_5] = [\tau_1, \tau_4] \cup [\tau_4, \tau_5]$.

Let $\Phi^{(N-2s_4-2)}(t)$ have α_4 zeros in $[\tau_1, \tau_4]$. Then, using (2.18) and applying Rolle theorem, we conclude that $\Phi^{(N-2s_4-1)}(t)$ has at least $\alpha_4 - 1$ zeros in (τ_1, τ_4), etc., $\Phi^{(N-2s_3-2)}(t)$ has at least $\alpha_4 - (2s_4 - 2s_3)$ zeros in (τ_1, τ_4). Therefore, we have $\alpha_4 - (2s_4 - 2s_3) \le (2s_2 + 2) + (2s_3 + 2) + (2s_3 + 2)$, i. e., $\alpha_4 \le (2s_2 + 2) + (2s_3 + 2) + (2s_4 + 2)$.

Therefore, $\Phi^{(N-2s_4-2)}(t)$ has at most $\sum_{\nu=2}^{4}(2s_\nu + 2) + (2s_4 + 2)$ zeros in $[\tau_1, \tau_5]$.

Finally, let $\Phi^{(N-2s_5-2)}(t)$ has α_5 zeros in $[\tau_1, \tau_5]$. As above, we conclude that $\alpha_5 \le \sum_{\nu=2}^{5}(2s_\nu + 2)$.

b.2) Let $\Phi^{(N-2s_5-2)}(t)$ have β_5 zeros in $[\tau_5, b]$. In analogous way as above, by using the conclusions from a), we conclude that $\beta_5 \le \sum_{\nu=5}^{9}(2s_\nu + 2)$.

Therefore, $\Phi^{(N-2s_5-2)}(t)$ has at most $\sum_{\nu=2}^{9}(2s_\nu+2)+(2s_5+2)$ zeros in $[\tau_1, b]$.

c) Finally, consider $\Phi^{(N-2s_1-2)}(t)$, which is continuous in $[a,b]=[a,\tau_1]\cup[\tau_1,b]$.

c.1) Consider $[a,\tau_1]$. $\Phi^{(N-p)}(t)$ has at most p zeros in $[a,\tau_1]$. Let $\Phi^{(N-2s_1-2)}(t)$ have α_1 zeros in $[a,\tau_1]$. Then, using (2.18) (for the point a) and applying Rolle theorem, we conclude that $\Phi^{(N-2s_1-1)}(t)$ has at least α_1 zeros in (a,τ_1), etc., $\Phi^{(N-p)}(t)$ has at least α_1 zeros in (a,τ_1). Therefore, $\alpha_1 \le p$.

c.2) By using the conclusions for $\Phi^{(N-2s_5-2)}(t)$, from b), in analogous way as above, we conclude that $\Phi^{(N-2s_1-2)}(t)$ has at most $\sum_{\nu=1}^{9}(2s_\nu+2)$ zeros in $[\tau_1,b]$.

Therefore, on the basis of c.1), c.2), we conclude that $\Phi^{(N-2s_1-2)}(t)$ has at most $p+\sum_{\nu=1}^{9}(2s_\nu+2)=N-q$ zeros in $[a,b]$.

We may then show that $\Phi(t)$ does not vanish inside $[a,b]$ and therefore holds a sign in it. In fact, if $\Phi(t)$ should vanish at one point in (a,b), using (2.18) and applying Rolle theorem, we find that $\Phi'(t)$ would vanish at least two times, $\Phi^{(N-q-1)}(t)$ would vanish at least $N-q$ times, $\Phi^{(N-q)}(t)$ would vanish at least $N-q+1$ times, etc., $\Phi^{(N-2s_1-2)}(t)$ would vanish at least $N-q+1$ times, in contraposition with the preceding deduction, because $N-q+1 \le N-q$ gives $1 \le 0$.

On the basis of the upper considerations we have just proved the following statement:

THEOREM 2.4. *Under the hypothesis that the weight function $w(t)$ is not identically zero in any interval contained in $[a,b]$, the influence function $\Phi(t)$ defined by* (2.15) (*together with* (2.16) *and* (2.17)) *belongs to the class $C^{n-2s_k-2}[a,b]$, where $N-2s_k-2=\min_{1\le\nu\le n}(N-2s_\nu-2)$, and one holds a sign inside $[a,b]$.*

Now, we can estimate the remainder in the formulas of the type (2.12), by using (2.14).

1^0 If $f(t)\in AC^{N-1}[a,b]$ we have

$$|R(f)| \le \max_{a\le t\le b}|\Phi(t)|\,V_{N-1} = |\Phi(x_0)|\,V_{N-1},$$

where V_{N-1} denotes the total variation of the function $f^{(N-1)}(t)$ absolutely continuous on the interval $[a,b]$. Because $\Phi'(t)$ vanish in exact one point of the interval (a,b) it holds $(\exists x_0\in(a,b))\ \max\limits_{a\le t\le b}\Phi(t)=\Phi(x_0)$.

2^0 If $f^{(N)}(t)$ is bounded in $[a,b]$, i.e., $M_N=\sup\limits_{a\le t\le b}\left|f^{(N)}(t)\right|$, then we have

$$|R(f)| \le M_N\int_a^b|\Phi(t)|\,dt.$$

3^0 If $f\in C^N[a,b]$, because $\Phi(t)$ holds a sign on (a,b) we may apply the mean value theorem and write

$$R(f)=f^{(N)}(\xi)\int_a^b\Phi(t)\,dt,\quad \xi\in(a,b).$$

Example. An iterative process for computing the coefficients of s-orthogonal polynomials in a special case, when the interval $[a, b]$ is symmetric with respect to the origin and the weight function w is an even function, was proposed by Vincenti [25]. He applied his process to the Legendre case. When n and s increase, the process becomes numerically unstable.

In [10] (see also [4]) a stable procedure for numerical construction of s-orthogonal polynomials with respect to $d\varphi(t)$ on $\mathbb{R}$ is given.

Recently, a simple and numerically stable procedure for construction of σ-orthogonal polynomials is proposed by Milovanović and Spalević [14].

A stable numerical procedure for calculating the coefficients $A_{i,\nu}$ in (1.1) was given by Gautschi and Milovanović [4]. Some alternative methods were proposed by Stroud and Stancu [23], Golub and Kautsky [7], and Milovanović and Spalević [12] (see also [21]). A generalization of methods, for the weights, from [4, 12] to the general case when $s_\nu \in \mathbb{N}_0$, $\nu = 1, \ldots, n$, was derived recently by Milovanović and Spalević [13].

Finally, a method for calculating the nodes and the coefficients in the generalized Chakalov-Popoviciu quadrature formulae of Radau and Lobatto type, by using the results from [13, 14], has been proposed in [22]. We use that method, for calculating in this example, in order to tabulate the corresponding influence function. Consider the Legendre case with $w(t) = 1$ on $[-1, 1]$. Let $p = q = 1$ and $\sigma = (1, 0, 1)$ in (2.12). Therefore, we have a symmetric task. The results show that the nodes of the corresponding quadrature are symmetrically distributed with respect to the origin, namely,

$$\tau_1 = -\tau_3 = -0.66772435790692, \quad \tau_2 = 0.$$

The coefficients of the corresponding Chakalov-Popoviciu quadrature formula of Lobatto type are:

$$\begin{aligned} &A_{0,1} = A_{0,3} = 6.68946557387391(-01), \quad A_{1,1} = -A_{1,3} = 2.90757109134605(-02), \\ &A_{2,1} = A_{2,3} = 8.27917955975223(-03), \quad A_{0,2} = 5.47406124470793(-01), \\ &A_{0,0} = A_{0,4} = 5.7350380377213(-02). \end{aligned}$$

The results show that the influence function is even (see Table), and

$$\max_{-1 \le t \le 1} |\Phi(t)| = -\Phi(0) = 7.06(-12).$$

TABLE

t	∓ 1	$\pm\tau_1$	∓ 0.5	∓ 0.4	∓ 0.3	∓ 0.2	∓ 0.1
$\Phi(t)$	0	$-4.05(-15)$	$-1.94(-13)$	$-8.23(-13)$	$-2.23(-12)$	$-4.31(-12)$	$-6.25(-12)$

Therefore, these results can be use in estimations of the remainder given above for the corresponding Chakalov-Popoviciu quadrature formula of Lobatto type.

All computations were done using FORTRAN in double precision arithmetics. The numbers in parentheses denoted the decimal exponents.

References

1. D. Braess, *Nonlinear Approximation Theory*, Springer Verlag, Berlin, 1986.
2. L. Chakalov, *General quadrature formulae of Gaussian type*, Bulgar. Akad. Nauk Izv. Mat. Inst. **1** (1954), 67–84. (Bulgarian)
3. L. Chakalov, *Formules générales de quadrature mécanique du type de Gauss*, Colloq. Math. **5** (1957), 69–73.
4. W. Gautschi and G.V. Milovanović, *S-orthogonality and construction of Gauss–Turán type quadrature formulae*, J. Comput. Appl. Math. **86** (1997), 205–218.
5. A. Ghizzetti and A. Ossicini, *Quadraturae formulae*, Academie Verlag, Berlin, 1970.
6. A. Ghizzetti and A. Ossicini, *Sull' esistenza e unicità delle formule di quadratura gaussiane*, Rend. Mat. **(6) 8** (1975), 1–15.
7. G.H. Golub and J. Kautsky, *Calculation of Gauss quadratures with multiple free and fixed knots*, Numer. Math. **41** (1983), 147–163.
8. S. Karlin and A. Pinkus, *Gaussian quadrature with multiple nodes*, in: Studies in Spline Functions and Approximation Theory (S. Karlin, C.A. Micchelli, A. Pinkus and I.J. Schoenberg, eds.), Academic Press, New York, 1976, pp. 113–141.
9. C.A. Micchelli, *The fundamental theorem of algebra for monosplines with multiplicities*, in: P. Butzer, J.P. Kahane, B.Sz. Nagy (eds.), Linear Operators and Approximation, ISNM vol. 20, Birkhäuser, Basel, 1972, pp. 372–379.
10. G.V. Milovanović, *Construction of s-orthogonal polynomials and Turán quadrature formulae*, in: Numerical Methods and Approximation Theory III, Niš, 1987 (G.V. Milovanović, ed.), Univ. Niš, Niš, 1988., pp. 311–328.
11. G.V. Milovanović, *Quadratures with Multiple Nodes, Power Orthogonality, and Moment-preserving Spline Approximation*, in Numerical Analysis in 20$^{\text{th}}$ Century, Vol. 5 (W. Gautschi, F. Marcellan, L. Reichel, eds.), J. Comput. Appl. Math. **127** (2001) 267–286.
12. G.V. Milovanović and M.M. Spalević, *A numerical procedure for coefficients in generalized Gauss–Turán quadratures*, Filomat (Niš) **9:1** (1995), 1–8.
13. G.V. Milovanović and M.M. Spalević, *Construction of Chakalov–Popoviciu's type quadrature formulae*, Rend. Circ. Mat. Palermo **52** (1998), 625–636.
14. G.V. Milovanović and M.M. Spalević, *Quadrature formulae connected to σ-orthogonal polynomials*, J. Comput. Appl. Math. **140** (2002), 619–637.
15. A. Morelli and I. Verna, *Formula di quadratura in cui compaiono i valori della funzione e delle derivate con ordine massimo variabile da nodo a nodo*, Rend. Circ. Mat. Palermo **(2) 18** (1969), 91–98.
16. A. Ossicini, *Le funzioni di influenza nel problema di Gauss sulle formule di quadratura*, Matematiche (Catania) **23** (1968), 7–30.
17. A. Ossicini and F. Rosati, *Sulla convergenza dei funzionali ipergaussiani*, Rend. Mat. **11** (1978), 97–108.
18. T. Popoviciu, *Sur une généralisation de la formule d'integration numérique de Gauss*, Acad. R. P. Romîne Fil. Iaşi Stud. Cerc. Şti. **6** (1955), 29–57.
19. Y.G. Shi, *Generalized Gaussian quadrature formulas for Tchebycheff systems*, Far East J. Appl. Math. **3** (1999), 153–170.
20. Y.G. Shi, *Convergence of Gaussian quadrature formulas*, J. Approx. Theory **105** (2000), 279–291.
21. M.M. Spalević, *Product of Turán quadratures for Cube, Simplex, Surface of the Sphere, $\bar{E}_n^r$, $E_n^{r^2}$*, J. Comput. Appl. Math. **106** (1999), 99–115.
22. M.M. Spalević, *Calculation of Chakalov–Popoviciu's quadratures of Radau and Lobatto type*, ANZIAM J. (3) **43** (2002), 429–447 .
23. A.H. Stroud, D.D. Stancu, *Quadrature formulas with multiple gaussian nodes*, J. SIAM Numer. Anal. Ser. B **2** (1965), 129–143.
24. P. Turán, *On the theory of mecchanical quadrature*, Acta Sci. Math. Szeged **12** (1950), 30–37.

25. G. Vincenti, *On the computation of the coefficients of s-orthogonal polynomials*, SIAM J. Numer. Anal **23** (1986), 1290–1294.

Prirodno-matematički fakultet
34000 Kragujevac, p.p. 60
Yugoslavia
spale@knez.uis.kg.ac.yu

(Received 01 09 2001)