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# SPECTRAL RADIUS AND SPECTRUM OF THE COMPRESSION OF A SLANT TOEPLITZ OPERATOR

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Abstract. A slant Toeplitz operator $A_\varphi$ with symbol $\varphi$ in $L^\infty(T)$, where $T$ is the unit circle on the complex plane, is an operator whose representing matrix $M = (a_{ij})$ is given by $a_{ij} = \langle \varphi, z^{2i-j} \rangle$, where $\langle \cdot, \cdot \rangle$ is the usual inner product in $L^2(T)$. The operator $B_\varphi$ denotes the compression of $A_\varphi$ to $H^2(T)$ (Hardy space). In this paper, we prove that the spectral radius of $B_\varphi$ is greater than the spectral radius of $A_\varphi$, and if $\varphi$ and $\varphi^{-1}$ are in $H^\infty$, then the spectrum of $B_\varphi$ contains a closed disc and the interior of this disc consists of eigenvalues with infinite multiplicity.

## 1. Introduction

Let $\varphi \in L^\infty(T)$. Then $\varphi(z) \sim \sum_{i=-\infty}^{\infty} a_i z^i$, where $a_i = \langle \varphi, z^i \rangle$ is the $i$-th Fourier coefficient of $\varphi$ and $\{z^i : i \in Z\}$ is the usual basis, and $Z$ is the set of integers. The slant Toeplitz operator $A_\varphi$ is defined as follows: $A_\varphi(z^k) = \sum_{i=-\infty}^{\infty} a_{2i-k} z^i$. Furthermore, by [4, Proposition 1] $A_\varphi = WM_\varphi$, where $M_\varphi$ is a multiplication operator and $Wz^{2n} = z^n$, $Wz^{2n-1} = 0$, for $n \in Z$.

$B_\varphi$, the compression of $A_\varphi$ to $H^2(T)$, is by definition $B_\varphi = PA_\varphi|_{H^2}$. Equivalently, $B_\varphi P = PA_\varphi P$, where $P$ is the orthogonal projection from $L^2$ on to $H^2$. By [4, p. 846], $B_\varphi = WT_\varphi$, where $T_\varphi$ is the Toeplitz operator on $H^2(T)$.

## 2. Spectral radius

Our aim is to prove that the spectral radius of $B_\varphi$ is greater than the spectral radius of $A_\varphi$. To do this we need the following lemmas.

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LEMMA 2.1. *$(I-P)M_{z^n} \to 0$, as $n \to \infty$ in the strong operator topology, where $M_{z^n}$ is the multiplication by $z^n$ on $L^2(T)$.*

PROOF. Let $f \in L^2(T)$ and $f(z) \sim \sum_{i=-\infty}^{\infty} a_i z^i$ be its Fourier expansion. Then $(I-P)M_{z^n} f = (I-P)\Big(\sum_{i=-\infty}^{\infty} a_i z^{i+n}\Big) = \sum_{i=-\infty}^{-n-1} a_i z^{i+n}$. Since, $\Big\| \sum_{i=-\infty}^{-n-1} a_i z^{i+n} \Big\|^2 = \sum_{i=-\infty}^{-n-1} |a_i|^2 \to 0$, as $n \to \infty$, the assertion follows. $\square$

The proof of the following lemma is similar to that of [1, Theorem 5].

LEMMA 2.2. *$M_{\bar{z}^n} B_\varphi P M_{z^{2n}} \to A_\varphi$, as $n \to \infty$, in the strong operator topology.*

PROOF. From Lemma 2.1, we know $(I-P)M_{z^n} \to 0$. This implies that $M_{\bar{z}^n}(I-P)M_{z^n} \to 0$ which is equivalent to $M_{\bar{z}^n} P M_{z^n} \to I$. Consider

$$M_{\bar{z}^n} B_\varphi P M_{z^{2n}} = M_{\bar{z}^n} P A_\varphi P M_{z^{2n}} = (M_{\bar{z}^n} P M_{z^n})(M_{\bar{z}^n} A_\varphi M_{z^{2n}})(M_{\bar{z}^{2n}} P M_{z^{2n}}).$$

Since, for each $n = 1, 2, \ldots$, $M_{\bar{z}^n} A_\varphi P M_{z^{2n}} = A_\varphi$, [4, Proposition 3], and the first and the last factors converge to $I$, as $n \to \infty$, the assertion follows. $\square$

The following theorem is proved in [4, p. 851] but we give here a different proof.

THEOREM 2.3. *$\|A_\varphi\| = \|B_\varphi\|$.*

PROOF. For each $n = 1, 2 \ldots$, we have $\|M_{\bar{z}^n} B_\varphi P M_{z^{2n}}\| \le \|B_\varphi\|$. So from Lemma 2.2, we get $\|A_\varphi\| \le \|B_\varphi\|$. Since $B_\varphi$ is the compression of $A_\varphi$, we get $\|A_\varphi\| \ge \|B_\varphi\|$. The proof is complete. $\square$

We are now ready to prove our main result.

THEOREM 2.4. *The spectral radius of $B_\varphi$ is greater than the spectral radius of $A_\varphi$.*

PROOF. First, we prove the following claim by induction.

Claim: For $k = 1, 2, 3, \ldots$, $M_{\bar{z}^n} B_\varphi^k P M_{z^{2^k n}} \to A_\varphi^k$, as $n \to \infty$ in the strong operator topology.

For $k = 1$, the claim is true by Lemma 2.2. Let $m$ be any positive integer and assume that it is true for $k \le m$, and consider

$$\begin{aligned}
&\|M_{\bar{z}^n} B_\varphi^{m+1} P M_{z^{2^{m+1} n}} - A_\varphi^{m+1}\| \\
&\quad = \|M_{\bar{z}^n}(B_\varphi^{m+1} P M_{z^{2^{m+1} n}} - M_{z^n} A_\varphi^{m+1})\| = \|B_\varphi^{m+1} P M_{z^{2^{m+1} n}} - M_{z^n} A_\varphi^{m+1}\| \\
&\quad \le \|B_\varphi^{m+1} P M_{z^{2^{m+1} n}} - P M_{z^n} A_\varphi^{m+1}\| + \|P M_{z^n} A_\varphi^{m+1} - M_{z^n} A_\varphi^{m+1}\|
\end{aligned}$$

By Lemma 2.1, the second term tends to 0 as $n$ approaches infinity. As to the first term, we use $M_{z^n} A_\varphi = A_\varphi M_{z^{2n}}$ [4, Proposition 3] and get the following approximation

$$\begin{aligned}\|B_\varphi^{m+1}PM_{z^{2^{m+1}n}} - PM_{z^n}A_\varphi^{m+1}\| &= \|PA_\varphi B_\varphi^m PM_{z^{2^{m+1}n}} - PA_\varphi M_{z^{2n}}A_\varphi^m\| \\ &\le \|A_\varphi\|\,\|B_\varphi^m PM_{z^{2^{m+1}n}} - M_{z^{2n}}A_\varphi^m\| \\ &= \|A_\varphi\|\,\|(B_\varphi^m PM_{z^{2^m n}} - M_{z^n}A_\varphi^m)M_{z^{2^m n}}\| \\ &\le \|A_\varphi\|\,\|B_\varphi^m PM_{z^{2^m n}} - M_{z^n}A_\varphi^m\| \\ &= \|A_\varphi\|\,\|M_{z^n}B_\varphi^m PM_{z^{2^m n}} - A_\varphi^m\|\end{aligned}$$

By the induction assumption, the last expression tends to 0 as $n$ approaches infinity. Therefore, the claim is proved. The fact that $\|M_{z^n}B_\varphi^k PM_{z^{2^k n}}\| \le \|B_\varphi^k\|$, for $k = 1, 2, \ldots$, and the above claim imply that $\|B_\varphi^k\| \ge \|A_\varphi^k\|$. This in turn implies that $r(B_\varphi) \ge r(A_\varphi)$, where $r$ represents spectral radius. □

DEFINITION 2.5. $H^\infty = \{\varphi \in L^\infty(T) : \langle \varphi, z^n\rangle = 0 \text{ for } n < 0\}$. The elements of $H^\infty$ are called analytic and their conjugates are called coanalytic.

COROLLARY 2.6. *$r(A_\varphi) = r(B_\varphi)$, if $\varphi$ is analytic or coanalytic.*

PROOF. If $\varphi$ is analytic, then $B_\varphi = A_\varphi|_{H^2}$. Therefore, for each $k = 1, 2, 3, \ldots$, $\|B_\varphi^k\| \le \|A_\varphi^k\|$. This implies $r(B_\varphi) \le r(A_\varphi)$. This together with Theorem 2.4, gives the required assertion. If $\varphi$ is coanalytic, then $B_\varphi^* = A_\varphi^*|_{H^2}$. By the same argument, we get $r(A_\varphi^*) = r(B_\varphi^*)$, Consequently $r(A_\varphi) = r(B_\varphi)$. □

The following fact is indicated in [4, p. 856], but we give a different proof below.

THEOREM 2.7. *If $\varphi$ is invertible in $L^\infty(T)$, then $r(A_\varphi) \ge [r(A_{\varphi^{-1}})]^{-1}$.*

PROOF. First, we show that $\varphi(z)$ is invertible if and only if $\varphi(z^2)$ is invertible. Suppose $\varphi$ is invertible. Then $\varphi\varphi^{-1} = 1$. Therefore, by [4, p. 846] $W^*\varphi W^*\varphi^{-1} = 1$. Equivalently $\varphi(z^2)\varphi^{-1}(z^2) = 1$. Hence $\varphi(z^2)$ is invertible. Conversely, if $\varphi(z^2)$ is invertible, then $\varphi(z^2)\varphi^{-1}(z^2) = 1$. This and [4, p. 847] implies $W\varphi(z^2)W\varphi^{-1}(z^2) = 1$, which is equivalent to $\varphi\varphi^{-1} = 1$. Therefore $\varphi$ is invertible.

Let $h(z) = \varphi(z^2)$ be invertible. Then $hh^{-1} = 1$. This and [4, p. 847] implies that $(Wh)(Wh^{-1}) = 1$. Therefore $(Wh)^{-1} = W(h^{-1})$. This in turn implies that, for each $n = 1, 2, 3, \ldots$

$$(\varphi^{-1})_n = (\varphi_n)^{-1},$$

where

$$\varphi_n = \overbrace{W(W(\ldots(W}^{n\text{ times}}\underbrace{|h|^2)|h|^2\ldots)|h^2|)}_{n\text{ times}}.$$

From this and [4, p. 851], we get

$$\begin{aligned} r(A_{h^{-1}}) &= \lim_{n\to\infty}\|(\varphi^{-1})_n\|_\infty^{1/2n} = \lim_{n\to\infty}\|(\varphi_n)^{-1}\|_\infty^{1/2n} \\ &\ge \left[\lim_{n\to\infty}\|\varphi_n\|_\infty^{1/2n}\right]^{-1} = [r(A_h)]^{-1}\end{aligned}$$

Therefore, $r(A_h) \ge [r(A_{h^{-1}})]^{-1}$.

Since $\sigma(A_\varphi) = \sigma(A_{\varphi(z^2)})$, where $\sigma$ denotes the spectrum [4, Lemma 9], it follows that $r(A_\varphi) \geq [r(A_{\varphi^{-1}})]^{-1}$. □

Corollary 2.8. *$r(B_\varphi) \geq r(B_{\varphi^{-1}})^{-1}$, if $\varphi$ and $\varphi^{-1}$ are analytic or $\varphi$ and $\varphi^{-1}$ are coanalytic.*

Proof. This is an immediate consequence of Corollary 2.6 and Theorem 2.7. □

## 3. Spectrum

Ho [4] showed that, for invertible $\varphi$ in $L^\infty$, the spectrum of $A_\varphi$ contains a closed disk consisting of eigenvalues of $A_\varphi$. We also show, for $\varphi$ and $\varphi^{-1}$ in $H^\infty$, that the spectrum of $B_\varphi$ contains a closed disc and the interior of this disc consists of eigenvalues with infinite multiplicity, by using the idea of the proof of Proposition 10 in [4].

Theorem 3.1. *Let $\varphi$ and $\varphi^{-1}$ be in $H^\infty$. Then the spectrum of $B_\varphi$ contains a closed disc and the interior of this disc consists of eigenvalues with infinite multiplicity.*

Proof. Assume first that $\lambda \neq 0$. Suppose that $B^*_{\bar{\varphi}(z^2)} - \lambda$ is onto. Since $B_{\varphi(z^2)} = T_\varphi W$, we have, for $f$ in $H^2$,

$$(B^*_{\bar{\varphi}(z^2)} - \lambda)f = (W^*T_\varphi - \lambda)f = (W^*T_\varphi - \lambda P_e)f \oplus (-\lambda P_0 f)$$

where $P_e$ is the projection on the closed span of $\{z^{2n} : n = 0, 1, 2\ldots\}$ in $H^2(T)$ and $P_0 = I - P_e$. Now let $0 \neq g_0$ be in $P_0(H^2)$. Since $B^*_{\bar{\varphi}(z^2)} - \lambda$ is onto, there exists a nonzero vector $f$ in $H^2(T)$ such that $(B^*_{\bar{\varphi}(z^2)} - \lambda)f = g_0$. But then from the computations above, we have $(W^*T_\varphi - \lambda P_e)f = 0$, because $g_0 \neq 0$. Since $\lambda \neq 0$ and $T_\varphi$ is invertible [2, Theorem 7.1], it follows that $\lambda W^* T_\varphi(\lambda^{-1} - T_{\varphi^{-1}}W)f = 0$, and the fact that $W^*$ is an isometry implies that $(\lambda^{-1} - T_{\varphi^{-1}}W)f = 0$. This in turn implies $\lambda^{-1} \in \sigma_p(B_{\varphi^{-1}(z^2)})$, where $\sigma_p$ denotes the point spectrum. Since $\dim P_0(H^2) = \infty$, it follows that $\lambda^{-1}$ is of infinite multiplicity. Now, for $\lambda \in \rho(B^*_{\bar{\varphi}(z^2)})$, the resolvent of $B^*_{\bar{\varphi}(z^2)}$, the operator $B^*_{\bar{\varphi}(z^2)} - \lambda$ is invertible (hence onto), so we have

$$D = \{\lambda^{-1} : \lambda \in \rho(B^*_{\bar{\varphi}(z^2)})\} \subseteq \sigma_\rho(B_{\varphi^{-1}(z^2)}),$$

Since $B_{\varphi(z^2)} = T_\varphi W$, $B_\varphi = WT_\varphi$ and $T_\varphi$ is invertible, we have $\sigma_\rho(B_{\varphi(z^2)}) = \sigma_\rho(B_\varphi)$ [3, Problem 61]. Therefore $D \subseteq \sigma_\rho(B_{\varphi^{-1}})$. So by replacing $\varphi^{-1}$ with $\varphi$, we have shown that for any invertible $\varphi$ in $H^\infty$, the spectrum of $B_\varphi$ contains a disc consisting of eigenvalues with infinite multiplicity. Therefore, by the fact that the spectrum of any operator is compact, $\sigma(B_\varphi)$ contains a closed disc and the interior of this disc consists of eigenvalues with infinite multiplicity. □

Remark 3.2. If $\varphi$ and $\varphi^{-1}$ are coanalytic, then $T_\varphi T_\varphi^{-1} = T_\varphi^{-1} T_\varphi = I$ [2, Theorem 7.1]. Therefore, one can repeat the proof above and arrive at the same conclusion as Theorem 3.1, that is, if $\varphi$ and $\varphi^{-1}$ are coanalytic, then the spectrum of $B_\varphi$ contains a closed disc and the interior of this disc consists of eigenvalues of infinite multiplicity.

REMARK 3.3. The radius of the closed disc contained in $\sigma_p(B_\varphi)$ is equal to $(r(B_{\varphi^{-1}}))^{-1}$, because if $D_0 = \{0\} \cup \{\lambda^{-1} : |\lambda| > r(B_{\varphi^{-1}})\}$, then $D_0 \subseteq \{\lambda^{-1} : \lambda \in \rho(B^*_{\varphi^{-1}}\} \cup \{0\} \subseteq \sigma_p(B_\varphi)$ and the radius of the disc $D_0$ is equal to $(r(B_{\varphi^{-1}}))^{-1}$. Hence $r(B_\varphi) \geq [r(B_{\varphi^{-1}})]^{-1}$. This relation is also proved in Theorem 2.7.

REMARK 3.4. If $\varphi(z) = 1$, then $r(B_{\varphi^{-1}}) = r(B_\varphi) = 1$ by the spectral radius formula for $A_\varphi$ [4, p. 851] and Corollary 2.6. Hence, the spectrum of $B_\varphi$ is the closed unit disc by Theorem 3.1 and Remark 3.3. Since the eigenvalues are of infinite multiplicity, it follows that the essential spectrum of $B_\varphi$ is the same as the spectrum of $B_\varphi$.

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