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UNIVALENT HARMONIC MAPPINGS BETWEEN JORDAN DOMAINS

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ABSTRACT. We give a classification of univalent harmonic functions of a Jordan domain onto a convex Jordan domain with boundary which does not contain linear segments. It is interesting that the boundary function must be continuous but not necessarily a univalent function, contrary to the case of conformal mappings.

1. Introduction and notation

The complex twice-differentiable function $w = f(z) = u + iv$ is called harmonic if u and v are real harmonic functions. Let be f a harmonic diffeomorphism. If there is $k < 1$ such that $|f_{\overline{z}}| \le k|f_z|$, then we say that f is a quasiconformal function (q.c.). We denote by QCH the class of harmonic quasiconformal functions.

The following formula is the Poisson integration formula and it plays a very important role in harmonic function theory. For every bounded harmonic function f defined on the unit disc U there is a bounded L^1 function g defined on the unit circle S^1 such that:

$$f(z) = P[g](z) = \int_0^{2\pi} P(z,\theta) g(e^{i\theta})\, d\theta$$

where

$$P(z,\theta) = \frac{1 - |z|^2}{2\pi |z - e^{i\theta}|^2}$$

is the Poisson kernel.

LEMMA 1.1. [**1**] *If g is a continuous function then f has a continuous extension on $\overline{U}$.*

LEMMA 1.2. [**3**] *Let $\{\varphi_n, n \in \mathbb{N}\}$ be a sequence of non-decreasing functions from $[0, 2\pi]$ to $[-2\pi, 4\pi]$. Then there is a subsequence $(\varphi_{n_k}) \subset (\varphi_n)$ and a function φ such that: $\varphi_{n_k}(x) \to \varphi(x)$ for $x \in [0, 2\pi]$ and φ is a non-decreasing function.*

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LEMMA 1.3. *Let $f : U \to V$ be a harmonic mapping of the unit disk U into the Jordan domain V. If $f = P[f(e^{i\theta})]$ and if there exists a $\theta_0 \in [0, 2\pi]$ such that $\lim_{\theta \uparrow \theta_0} f(e^{i\theta}) = A_0$ and $\lim_{\theta \downarrow \theta_0} f(e^{i\theta}) = B_0$, then for $\lambda \in [-1, 1]$ we have:*

$$\lim_{R\to\infty} f\left(e^{i\theta_0}\left(\frac{Re^{i\lambda\pi/2}-1}{Re^{i\lambda\pi/2}+1}\right)\right) = \frac{1}{2}(1-\lambda)A_0 + \frac{1}{2}(1+\lambda)B_0$$

The proof of this lemma follows from definition, but it has some technical difficulties. For details see [9].

PROPOSITION 1.1 (Choquet). *Let $f(z) = P[e^{i\varphi(\theta)}](z)$, such that φ is a non-decreasing* 1-1 *function and $\varphi(0) = 0$ and $\varphi(2\pi) = 2\pi$. Then f is a univalent harmonic function of the unit disk onto itself.*

Note that, this proposition is valid in a more general form. Indeed, only the convexity of the co-domain is important.

PROPOSITION 1.2. *Let $f_n : \Omega \to U$ be a sequence of k-q.c. mappings of a Jordan domain Ω into the unit disc U. Also let $f = \lim_{n\to\infty} f_n$; then either:*

(1) *f is constant* **or**
(2) *f is function with two points value* **or**
(3) *f is k-q.c.*

This proposition is due to Lehto and Virtanen [**5**].

PROPOSITION 1.3. *Let $f_n : U \to V$, be a sequence of harmonic diffeomorphisms of the unit disc U onto a Jordan domain V such that $f_n \xrightarrow{K} f$. Then:*

(1) *f is univalent of U into V* **or**
(2) *$f(z) = c + e^{i\varphi} \cdot R(z)$ where R is real harmonic function $\neq 0$* **or**
(3) $f \equiv$ const.

PROOF. Without loss of generality we my assume that f_n are sense preserving mappings. Let $a_n = \overline{f_{n\overline{z}}}/f_{nz}$. Then $a_n : U \to U$ is an analytic function. Since $f_n \xrightarrow{K} f$ one gets $f_{nz} \xrightarrow{K} f_z$ and $\overline{f_{n\overline{z}}} \xrightarrow{K} \overline{f_{\overline{z}}}$. Because $|f_{n\overline{z}}| \le |f_{nz}|$ it follows that $|f_{\overline{z}}| < |f_z|$. If $f_z \equiv 0$, then $f \equiv$ const which gives (1).

Otherwise $a = \overline{f_{\overline{z}}}/f_z$ is an analytic function except for some points. Since $|a| \le 1$, these points are admissible singularities of a. Hence a is analytic on U. If $|a(z)| = 1$ at some point, then there exists a φ such that $a(z) \equiv e^{i2\varphi}$. Hence

$$f(z) = c + e^{i\varphi}\sum_1^\infty (a_n e^{-i\varphi} z^n + \overline{a_n} e^{i\varphi}\overline{z}^n) = c + e^{i\varphi} \cdot R(z)$$

In this case (2) is true.

Let us now suppose $|a(z)| < 1$. We are going to prove that (1) holds. Let α, β be distinct points in U such that $|\alpha| < r$, $|\beta| < r$, and $r < 1$. Let $D_r = \{z : |z| < r\}$. Since a is an analytic function one gets that there is a $k < 1$ such that $|a(z)| < k$ for $|z| \le r$. Next, since $a_n \xrightarrow{K} a$, we have that there is a $k' < 1$ such that $|a_n| < k'$ on $\overline{D_r}$, for n small enough. Then the functions $F_n = f_n|_{D_r} - k'$ are q.c. Since $F_n \to F = f|_{D_r}$, from Proposition 1.2 it follows that $F : D_r \to F(D_r)$ is k' q.c. Consequently $f(\alpha) = F(\alpha) \neq F(\beta) = f(\beta)$. It follows that f is 1-1. This completes the proof. □

2. The main results

LEMMA 2.1. *Let $f : U \to V$ be a harmonic sense preserving diffeomorphism of the unit disk U into a Jordan domain V. Then there exists a function $\varphi : S \to \partial V$ with at most countably many points of discontinuity, all of them of the first type, such that: $f = P[\varphi]$.*

PROOF. Let $g : V \to U$ be a biholomorphism, which exists by Riemann mapping theorem. Then the function $F = g \circ f : U \to U$ is a sense preserving diffeomorphism. Let $U_n = \{z : |z| < \frac{n-1}{n}\}$, $\Delta_n = F^{-1}(U_n)$ and let g_n be a biholomorphism of the Jordan domain U onto the domain Δ_n such that $g_n(0) = 0$, and $g_n'(0) > 0$. Without loss of generality we can suppose $0 \in \Delta_n$ because the last relation holds for n large enough. Then the function:

$$F_n = \frac{n}{n-1} F \circ g_n = \frac{n}{n-1} g \circ f \circ g_n : \overline{U} \to \overline{U}$$

is a sense preserving homeomorphism. Let $\varphi_n = F_n|_S$ and let (φ_{n_k}) be a convergent subsequence of (φ_n) which exists because of Lemma 1.1. Then $\varphi_n(e^{i\theta}) = e^{i\phi_n(\theta)}$ where $\phi_n(\theta)$ is a monotone non-decreasing function. Let $\varphi_0 = \lim \varphi_{n_k}$. Then φ_0 is a monotone non-decreasing function. Hence

$$\frac{n_k}{n_k - 1} g \circ f \circ g_{n_k}|_S \to \varphi_0 \ \text{ if } \ k \to \infty.$$

And consequently

$$\lim_{k\to\infty} f \circ g_{n_k}(e^{i\theta}) = g^{-1}(\varphi_0(e^{i\theta})) \text{ for all } \theta$$

because g is a homeomorphism from $\overline{V}$ onto $\overline{U}$. Since $\phi_k = f \circ g_{n_k}|_S$ is continuous and $f \circ g_{n_k}$ is a harmonic function then from Lebesgue's Dominated Convergence Theorem, (because the function $g^{-1} \circ \varphi_0$ is bounded), we obtain: $f \circ g_{n_k} = P[\phi_k] \to P[g^{-1} \circ \varphi_0]$, as $k \to \infty$. It follows that the sequence g_{n_k} is convergent. Let $g_0(z) = \lim_{k\to\infty} g_{n_k}(z)$. Since g_0 is a conformal mapping from the unit disk onto itself which satisfies $g_0(0) = 0$, and $g_0'(0) > 0$ it follows that $g_0 = id$. Hence $f = P[g^{-1} \circ \varphi_0] = P[\phi]$, where g^{-1} is continuous and φ_0 is a monotone non-decreasing function. Hence, it has no more than countably many points of discontinuity, which are of the first type. The lemma is proved. □

THEOREM 2.1. *Let $f : U \to \Omega$ be a harmonic diffeomorphism of the unit disk U onto a Jordan domain Ω with boundary which contains no linear segments. Then the function f has a continuous extension from $\overline{U}$ onto $\overline{\Omega}$.*

PROOF. Follows from Lemma 1.2, Lemma 1.3 and homeomorphic properties of diffeomorphisms. □

REMARK 2.1. If a homeomorphism f reverses sense, then the homeomorphism $\overline{f}$ preserves sense.

COROLLARY 2.1. *Let $f : \Omega \to V$ be a harmonic diffeomorphism of a Jordan domain Ω onto a strict convex bounded domain V. Then f has a continuous extension of $\overline{\Omega}$ onto $\overline{V}$.*

PROOF. Let $\varphi : U \to \Omega$ be a conformal diffeomorphism of the unit disk U onto the Jordan domain Ω. Then $F = f \circ \varphi : U \to \Omega$ is a harmonic diffeomorphism. Hence, the corollary follows from Caratheodory's theorem and Theorem 2.1. $\square$

REMARK 2.2. It is a natural question, whether the extension of this harmonic diffeomorphism is a homeomorphism. The answer to this question, in the general case, is negative.

Indeed, the next theorem holds.

THEOREM 2.2. *Let g_n be a convergent sequence of homeorphisms between the unit circle and the convex Jordan domain $\gamma = \operatorname{int}\Omega$. Let $g = \lim_{n\to\infty} g_n$ be a non constant and a non two valued function and let $\operatorname{conv}(g(S^1)) = \Omega$. Then $f(z) = P[g](z)$ is a harmonic diffeomorphism of the unit disk onto Ω.*

PROOF. By Choquet's theorem it follows that the functions $f_n = P[g_n]$ is a univalent function from the unit disk onto Ω. On the other hand, because the family f_n is normal it has a convergent subsequence f_{n_k}. Let $f = \lim_{n\to\infty} f_{n_k} = P[g]$. Because g is not constant and it is not a two-valued function, according to Proposition 1.3 it follows that it is univalent. (The function $\theta \to e^{i\varphi(\theta)}$ has at last three value points.) On the other hand, because $\operatorname{conv}(f(S^1)) = \Omega$, it follows that $f(U) = \Omega$. $\square$

COROLLARY 2.2. *The harmonic function $f : U \to U$ is a sense preserving diffeomorphism of the unit disk U onto itself iff $f = P[e^{i\varphi(\theta)}]$ where φ is a continuous non-decreasing function such that $\varphi(0) = a$ and $\varphi(2\pi) = 2\pi + a$, $(a \in (-2\pi, 2\pi))$.*

The proof follows from Theorem 2.1 and Theorem 2.2

EXAMPLE 2.1. Let $\varphi(\theta) = \theta + k\sin\theta$, $\theta \in [0, 2\pi]$, $0 < k \le 1$. Then the function $f = P[e^{i\varphi(\theta)}]$ is a harmonic diffeomorphism of the unit disk onto itself such that if $0 < k < 1$ it is quasiconformal.

For the proof of the last assertion in the example, see [**8**].

REMARK 2.3. In a private conversation I learned that A. Lyzzaik and W. Hengartner have similar unpublished results.

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