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ON CLASSICAL SOLUTIONS OF MIXED BOUNDARY PROBLEMS FOR ONE-DIMENSIONAL PARABOLIC EQUATION OF SECOND ORDER

Nebojša L. Lažetić

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Abstract. We prove the existence and uniqueness of classical solutions to mixed boundary problems for the equation

$$\frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) + q(x)u(x,t) = f(x,t)$$

on a closed rectangle, with arbitrary self-adjoint boundary conditions. The initial function, the potential $q(x)$ and $f(x,t)$ belong to some subclasses of $W_p^{(k)}(\cdot)$ $(1 < p < \infty,\ k \in \mathbb{N})$ that are defined by monotonicity conditions of "Dirichlet–Jordan type". We also give a priori estimates of the solutions.

Introduction

1. On the problem. Let $G = (a,b)$ be a finite interval of the real axis $\mathbb{R}$, and let $T > 0$ be an arbitrary number. In this paper we consider the problem of existence of a real-valued function $u = u(x,t)$ defined on the closed rectangle $\overline{\Omega} = [a,b] \times [0,T]$ and satisfying the partial differential equation

$$\frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) + q(x)u(x,t) = f(x,t), \quad (x,t) \in \Omega, \tag{1}$$

the initial condition

$$u(x,0) = \varphi(x), \qquad x \in \overline{G}, \tag{2}$$

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and the boundary conditions

$$\begin{aligned} &\alpha_{10}u(a,t)+\alpha_{11}u'_x(a,t)+\beta_{10}u(b,t)+\beta_{11}u'_x(b,t)=0,\\ &\alpha_{20}u(a,t)+\alpha_{21}u'_x(a,t)+\beta_{20}u(b,t)+\beta_{21}u'_x(b,t)=0, \quad t\in[0,T], \end{aligned} \tag{3}$$

where $(\alpha_{i0},\alpha_{i1},\beta_{i0},\beta_{i1})\in\mathbb{R}^4$ $(i=1,2)$ are linearly independent vectors, and $q(x)$, $\varphi(x)$, $f(x,t)$ are given real-valued functions. We suppose that conditions (3) are such that the formal Schrödinger operator

$$\mathcal{L}(v)(x)=-v''(x)+q(x)v(x), \qquad x\in G, \tag{4}$$

and the boundary conditions

$$\begin{aligned} &\alpha_{10}v(a)+\alpha_{11}v'(a)+\beta_{10}v(b)+\beta_{11}v'(b)=0,\\ &\alpha_{20}v(a)+\alpha_{21}v'(a)+\beta_{20}v(b)+\beta_{21}v'(b)=0, \end{aligned} \tag{5}$$

generate an arbitrary self-adjoint operator L, with discrete spectrum. (By this we mean a self-adjoint extension L of the corresponding symmetric operator L_0; we suppose that the potential $q(x)$ allows such an extension (see [**3**, §18]).)

A real-valued function $u=u(x,t)$is called a *s-classical solution* of the mixed boundary problem (1)–(3) if it has the following properties:

1_u) $u(x,t),u'_t(x,t),u''_{x^2}(x,t)\in C(\overline{\Omega})$;

2_u) $u(x,t)$ satisfies equation (1) for all $(x,t)\in\Omega$;

3_u) $u(x,t)$ satisfies conditions (2)–(3) in the ordinary sense.

Using the Fourier method, we first prove the existence (and uniqueness) of a s-classical solution to the problem (1)–(3) (Theorem 1), under certain "reasonable" smoothness conditions imposed on functions $q(x),\varphi(x)$ and $f(x,t)$. Then we modify the above definition as to replace condition 1_u) by the following one:

1^*_u) $u(x,t)\in C(\overline{\Omega})$, $\quad u'_t(x,t),u''_{x^2}(x,t)\in C(\Omega)$.

For this class of solutions, which we call *classical solutions* (see [**1**]), two existence and uniqueness theorems are obtained: Theorem 2, proved by the Fourier method, and Theorem 3 which is established by a modification of the Fourier method.

In all the cases considered an a priori estimate of the solution is given.

2. Main theorems. Let $AC(\overline{G})$ be the class of (real-valued) absolutely continuous functions on the closed interval $\overline{G}=[a,b]$, and let $BV(\overline{G})$ be the class of functions having the bounded variation on this interval. By $\overset{\circ}{W}{}^{(k)}_p(G)$ we denote the set of functions $h(x)$ in the class $W^{(k)}_p(G)$ $(1\le p<+\infty,\ k\in\mathbb{N})$ such that $h(a)=h'(a)=\cdots=h^{(k-1)}(a)=0=h(b)=h'(b)=\cdots=h^{(k-1)}(b)$. (By definition, $h(x)\in W^{(k)}_p(G)$ if functions $h(x),h'(x),\dots,h^{(k-2)}(x)$ are continuously differentiable on $[a,b]$, $h^{(k-1)}(x)\in AC(\overline{G})$ and $h^{(k)}(x)$ is in $L_p(G)$.)

A function $h(x)$, defined on a set $A\subseteq[a,b]$, is said to be *piecewise monotone* on A if there exists a set $\{x_0,x_1,\dots,x_{n(h,A)}\}\subset[a,b]$ such that

$$a=x_0<x_1<\cdots<x_{n(h,A)}=b,$$

and $h(x)$ is monotone on the set $A \cap [x_{i-1}, x_i]$ for every $i \in \{1, \ldots, n(h, A)\}$. (In the class of piecewise monotone functions we include functions that are monotone (non-increasing or non-decreasing) on the set A.) Let $g(x,t)$ be a function defined on a set $\mathcal{D}(g) \subseteq \overline{\Omega}$, where

$$\mathcal{D}(g) \stackrel{\text{def}}{=} \bigcup_{t \in [0,T]} (A_t \times \{t\}).$$

This function is called *piecewise monotone uniformly with respect to* $t \in [0,T]$ if it is piecewise monotone on $A_t \subseteq \overline{G}$ for every $t \in [0,T]$, and the set

$$\{n(g, A_t) \in \mathbb{N} \mid t \in [0,T]\}$$

is bounded. In this case it will be said that $g(x,t)$ has *property* $(\mathcal{A})$.

We say that a function $g(x,t)$, defined on $\overline{\Omega}$, *belongs to the class* $BV(\overline{G})$ *uniformly with respect to* $t \in [0,T]$ if $g(x,t) \in BV(\overline{G})$ for every $t \in [0,T]$, and the set $\{V_a^b(g(\cdot,t)) \mid t \in [0,T]\}$ is bounded. This property is said to be *property* $(\mathcal{B})$.

Finally, let $g(x,t)$ be a function defined on the closed rectangle $\overline{\Omega}$. We say that this function *satisfies the Hölder condition on* $[0,T]$*, with an exponent* $\alpha \in (0,1]$*, uniformly with respect to* $x \in \overline{G}$ if

$$(\forall x \in \overline{G})(\forall\, t, t' \in [0,T])\ |g(x,t) - g(x,t')| \le B|t - t'|^{\alpha}, \tag{6}$$

where the constant $B > 0$ does not depend on x.

We can state now our results.

THEOREM 1. *Let us suppose: 1)* $q(x) \in AC(\overline{G})$.

2) $\varphi(x) \in W_1^{(3)}(G)$, *and* $\varphi(x)$ *satisfies the boundary conditions (5);* $\mathcal{L}(\varphi)(a) = 0 = \mathcal{L}(\varphi)(b)$*;* $\mathcal{L}(\varphi)'(x)$ *is a bounded, piecewise monotone function on its domain, or* $\mathcal{L}(\varphi)'(x) \in BV(\overline{G})$.

3) $f(x,t) \in C(\overline{\Omega})$*;* $f(x,t) \in \overset{\circ}{W}{}_1^{(1)}(G)$ *for every* $t \in [0,T]$*;* $f_x'(x,t)$ *has the property* $(\mathcal{A})$ *and it is bounded on* $\mathcal{D}(f_x')$*, or* $f_x'(x,t)$ *has the property* $(\mathcal{B})$.

Then, there exists a unique s-classical solution of the problem (1)–(3) which can be represented as a series converging absolutely and uniformly on $\overline{\Omega}$*. This series can be differentiated twice with respect to the variable* x*, and once with respect to the variable* t*. The obtained series for the derivatives of the solution converge absolutely and uniformly on* $\overline{\Omega}$.

Also, if $u(x,t)$ *is the s-classical solution, then the following a priori estimate holds:*

$$\|u\|_{C(\overline{\Omega})} \le D\left(\|\varphi\|_{L_1(G)} + \|\varphi''\|_{L_1(G)} + \|f\|_{C(\overline{\Omega})}\right),$$

where the constant $D > 0$ *does not depend on functions* φ *and* f.

In the case of classical solutions we can prove

THEOREM 2. *Let the following conditions be satisfied: 1) $q(x) \in C(\overline{G})$.*

2) $\varphi(x) \in \overset{\circ}{W}{}_1^{(1)}(G)$, and $\varphi(x)$ satisfies the boundary conditions (5); $\varphi'(x)$ is a bounded, piecewise monotone function on its domain $\mathcal{D}(\varphi')$, or $\varphi'(x) \in BV(\overline{G})$.

3) $f(x,t) \in C(\overline{\Omega})$; $f(x,t) \in \overset{\circ}{W}{}_1^{(1)}(G)$ for every $t \in [0,T]$; $f'_x(x,t)$ has the property $(\mathcal{A})$ and it is bounded on $\mathcal{D}(f'_x)$, or $f'_x(x,t)$ has the property $(\mathcal{B})$.

Then, there exists a unique classical solution of the problem (1)–(3) which can be represented as a series converging absolutely and uniformly on $\overline{\Omega}$. This series can be differentiated twice with respect to x, and once with respect to t on every closed rectangle $\overline{\Omega}_\epsilon = \overline{G} \times [\epsilon, T]$ ($\epsilon \in (0,T)$). The obtained series for the derivatives of the solution converge absolutely and uniformly on $\overline{\Omega}_\epsilon$.

Further, if $u(x,t)$ is the classical solution, then the following a priory estimate holds:

$$(*) \qquad \|u\|_{C(\overline{\Omega})} \le D\left(\|\varphi\|_{L_1(G)} + M(\varphi') + \|f\|_{C(\overline{\Omega})}\right),$$

where $M(\varphi') = n(\varphi', \mathcal{D}(\varphi')) \cdot \sup_{x \in \mathcal{D}(\varphi')} |\varphi'(x)|$ if $\varphi'(x)$ is bounded and piecewise monotone on $\mathcal{D}(\varphi')$, or $M(\varphi') = V_a^b(\varphi')$ if $\varphi'(x) \in BV(\overline{G})$. The constant $D > 0$ does not depend on $\varphi(x)$ and $f(x,t)$.

THEOREM 3. *Let the following conditions be satisfied: 1) $q(x) \in C(\overline{G})$.*

2) $\varphi(x) \in \overset{\circ}{W}{}_1^{(1)}(G)$, and $\varphi(x)$ satisfies the boundary condition (5); $\varphi'(x)$ is a bounded, piecewise monotone function on its domain, or $\varphi'(x) \in BV(\overline{G})$.

3) $f(x,t) \in C(\overline{\Omega})$, and $f(x,t)$ satisfies the Hölder condition on $[0,T]$, with an exponent $\alpha \in (1/2, 1]$, uniformly with respect to $x \in \overline{G}$.

Then, there exists a unique classical solution of the problem (1)–(3) which can be represented as a series converging absolutely and uniformly on $\overline{\Omega}$. This series can be differentiated once with respect to x or t on every closed rectangle $\overline{\Omega}_\epsilon$. The obtained series for the first derivatives of the solution converge absolutely and uniformly on $\overline{\Omega}_\epsilon$.

Also, if $u(x,t)$ is the classical solution, then the a priori estimate $(*)$ holds.

Remark 1. If we change the definition of classical solution as to replace $t \in [0,T]$ in (3) by $t \in (0,T]$, then assumption "$\varphi(x)$ satisfies the boundary conditions (5)", appearing in Theorems 2 and 3, can be dropped.

Remark 2. Let us suppose that the coefficients of linear forms (5) satisfy condition $\alpha_{11}\beta_{21} - \alpha_{21}\beta_{11} \ne 0$. Then,

1) assumption $\mathcal{L}(\varphi)(a) = 0 = \mathcal{L}(\varphi)(b)$ (Theorem 1) can be dropped;
2) assumption $\varphi(x) \in \overset{\circ}{W}{}_1^{(1)}(G)$ (Theorems 2 and 3) can be replaced by $\varphi(x) \in W_1^{(1)}(G)$:
3) assumption $f(x,t) \in \overset{\circ}{W}{}_1^{(1)}(G)$ (Theorems 1 and 2) can be replaced by $f(x,t) \in W_1^{(1)}(G)$.

It seems that the problem of existence of classical solutions to mixed boundary problems with general self-adjoint boundary conditions, for the equation considered, has not been thoroughly studied yet (see also [**5**]). In this paper some contributions in that direction are given. The technique used in proofs of the theorems is essentially different from the techniques known so far for justification of the Fourier method in the case of classical mixed problems (for instance, with the zero boundary conditions). It is based only on uniform and exact, with respect to order, estimates for eigenfunctions (and their derivatives) of the operator (4)–(5). Theorems 1 and 2 are proved by using only differentiation of the formal series representing the solution. But in the proof of Theorem 3 we use a general extension of a method developed by Chernyatin [**7**]. This method contains only one differentiation of the series mentioned, and gives us a possibility to decrease the smoothness condition imposed on function $f(x,t)$. In his paper Chernyatin supposed conditions 1) and 3) from Theorem 3 were satisfied, and proved the existence and uniqueness of the classical solution to equation (1) satisfying the conditions

$$u(x,0)=0, \quad x\in[0,\pi]; \qquad u(0,t)=u(\pi,t)=0, \quad t\in[0,T].$$

We prove the uniqueness of the solutions by a technique used in [**5**].

Note that the corresponding mixed boundary problem for an one-dimensional hyperbolic equation of second order was investigated in [**9**]–[**10**].

3. Auxiliary propositions. Our approach to justification of the Fourier method is based on a set of results obtained by several authors.

Consider an arbitrary *non-negative* self-adjoint extension L of the operator (4) with the potential $q(x)\in L_1(G)$. (This extension is defined by the corresponding *self-adjoint* boundary conditions (5); its spectrum is discrete. Recall that the operator L is defined in the following way. Let $\mathcal{D}(L)$ be the set of functions $g(x)\in L_2(G)$ such that functions $g(x)$, $g'(x)$ are absolutely continuous on $\overline{G}$, $\mathcal{L}(g)(x)\in L_2(G)$, and $g(x)$ satisfies the boundary conditions (5). If $g(x)\in\mathcal{D}(L)$, then $L(g)(x)\overset{\text{def}}{=}\mathcal{L}(g)(x)$.) Denote by $\{v_n(x)\}_1^\infty$ the orthonormal (and complete in $L_2(G)$) system of eigenfunctions corresponding to this extension, and by $\{\lambda_n\}_1^\infty$ the corresponding system of non-negative eigenvalues enumerated in non-decreasing order. (By definition, $v_n(x)\in\mathcal{D}(L)$, and $v_n(x)$ satisfies the differential equation

$$-v_n''(x)+q(x)v_n(x)=\lambda_n v_n(x) \tag{7}$$

almost everywhere on G.) Then, the following propositions are true.

Proposition 1. [**4**]. *If $q(x)\in L_1(G)$, then there exists a constant $C_0>0$, independent of $n\in\mathbb{N}$, such that*

$$\max_{x\in\overline{G}}|v_n(x)|\le C_0, \qquad n\in\mathbb{N}. \tag{8}$$

Proposition 2. [**4**]. *If $q(x)\in L_p(G)$ $(p>1)$, then there exists a constant $A>0$ such that*

$$\sum_{t\le\sqrt{\lambda_n}\le t+1} 1\le A \tag{9}$$

for each $t\ge 0$, where A does not depend on t.

PROPOSITION 3. [**6**]. *Suppose* $q(x) \in C(\overline{G})$. *Then the eigenfunctions* $v_n(x)$ *have continuous second derivative on* $\overline{G}$, *satisfy the equation (7) everywhere on* G *and there exist constants* $\mu_0(G) > 0$, $C_j > 0$ *(*$j = 1, 2$*), independent of* $n \in \mathbb{N}$, *such that*

$$\max_{x \in \overline{G}} |v_n^{(j)}(x)| \le \begin{cases} C_j \lambda_n^{j/2} & \text{if} \quad \lambda_n > \mu_0(G), \\ C_j & \text{if} \quad 0 \le \lambda_n \le \mu_0(G). \end{cases} \tag{10}$$

PROPOSITION 4. [**9**]–[**10**]. (a) *Suppose* $q(x) \in L_p(G)$ *(*$1 < p \le 2$*), and* $h(x) \in \overset{\circ}{W}{}_1^{(1)}(G)$. *If* $h'(x)$ *is a bounded, piecewise monotone function on its domain, or* $h'(x) \in BV(\overline{G})$, *then for* $x \in \overline{G}$ *the equality*

$$h(x) = \sum_{n=1}^{\infty} h_n v_n(x), \quad \text{where} \quad h_n = \int_a^b h(x) v_n(x)\, dx,$$

holds, and the series are absolutely and uniformly convergent on $\overline{G}$.

(b) *Let* $q(x) \in L_p(G)(1 < p \le 2)$, *and* $h(x) \in \mathcal{D}(L)$. *Then for* $x \in \overline{G}$ *the equalities*

$$h(x) = \sum_{n=1}^{\infty} h_n v_n(x), \qquad h'(x) = \sum_{n=1}^{\infty} h_n v_n'(x)$$

hold, and the series are absolutely and uniformly convergent on $\overline{G}$.

(c) *Let* $q(x) \in AC(\overline{G})$, $h(x) \in W_1^{(3)}(G)$ *and* $h(x)$ *satisfies the boundary conditions (5). If* $\mathcal{L}(h)(a) = 0 = \mathcal{L}(h)(b)$, *and* $\mathcal{L}(h)'(x)$ *is a bounded, piecewise monotone function on its domain, or* $\mathcal{L}(h)'(x)$ *belongs to* $BV(\overline{G})$, *then for* $x \in \overline{G}$ *the equalities*

$$h^{(j)}(x) = \sum_{n=1}^{\infty} h_n v_n^{(j)}(x), \qquad j = 0, 1, 2,$$

hold, and the series converge absolutely and uniformly on $\overline{G}$.

We should give here some comments on the above propositions. First, Propositions 1–3 and Proposition $4(a)$ are also valid when the functions $v_n(x)$ ($n \in \mathbb{N}$) do not (necessarily) satisfy any boundary conditions. (In that case, $\{v_n(x)\}_1^{\infty}$ is called a *fundamental system of functions* of the operator (4), in the sense of Il'in [**4**]). Second, Propositions 1–4 are also valid in the general case of an *arbitrary* self-adjoint extension L of the operator (4). (Then, only a finite number of negative eigenvalues of L can exist; some obvious minor changes in formulation of Propositions 1–3 are needed.)

For the sake of simplicity we will work with a non-negative operator L, and estimates (10) will be used supposing that $\mu_0(G) = 1$.

1. Proof of Theorem 1

1. The Fourier scheme.. Let $\{v_n(x)\}_1^\infty$ be the orthonormal and complete in $L_2(G)$ system of eigenfunctions corresponding to a non-negative self-adjoint extension L of the operator (4), and let $\{\lambda_n \geq 0\}_1^\infty$ be the corresponding system of eigenvalues. (It is known that $\lambda_n \to +\infty$ as $n \to \infty$.) If we denote

$$\varphi_n = \int_a^b \varphi(x) v_n(x)\, dx, \quad f_n(t) = \int_a^b f(x,t) v_n(x)\, dx, \quad t \in [0,T],$$

then, by Proposition 4(a)–(b), the equalities

$$\varphi(x) = \sum_{n=1}^{\infty} \varphi_n v_n(x), \quad f(x,t) = \sum_{n=1}^{\infty} f_n(t) v_n(x), \quad t \in [0,T], \tag{11}$$

hold uniformly on the closed interval $\overline{G}$, and the series are absolutely convergent.

Let us look for the solution of (1)–(3) in the form

$$u(x,t) = \sum_{n=1}^{\infty} v_n(x) w_n(t),$$

where $w_n = w_n(t)$ are unknown functions to be defined. Applying to the problem the formal scheme of the Fourier method, and using decompositions (11) we obtain, as it is known, the following representation of the solution:

$$u(x,t) = \sum_{n=1}^{\infty} v_n(x) \left[\varphi_n e^{-\lambda_n t} + \int_0^t f_n(\tau) e^{-\lambda_n (t-\tau)} d\tau \right]. \tag{12}$$

The sum of this series formally satisfies the differential equation (1), the initial condition (2) and the boundary conditions (3).

The proof that formally defined function (12) is the s-classical solution will be based on uniform convergence on $\overline{\Omega}$ of the series appearing in the equalities

$$\frac{\partial u}{\partial x}(x,t) = \sum_{n=1}^{\infty} v_n'(x) \left[\varphi_n e^{-\lambda_n t} + \int_0^t f_n(\tau) e^{-\lambda_n (t-\tau)} d\tau \right], \tag{13}$$

$$\frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty} v_n(x) \left[-\lambda_n \varphi_n e^{-\lambda_n t} - \lambda_n \int_0^t f_n(\tau) e^{-\lambda_n (t-\tau)} d\tau + f_n(t) \right], \tag{14}$$

which are just formal at the moment. The precise meaning of the previous sentence is as follows.

For all $\lambda_n \geq 1$ and $(x,t) \in \overline{\Omega}$ we have the inequality

$$\left| \varphi_n e^{-\lambda_n t} + \int_0^t f_n(\tau) e^{-\lambda_n (t-\tau)} d\tau \right| \leq \lambda_n |\varphi_n| e^{-\lambda_n t} + \lambda_n \left| \int_0^t f_n(\tau) e^{-\lambda_n (t-\tau)} d\tau \right| + |f_n(t)|.$$

Hence, from the (supposed) absolute and uniform convergence (on $\overline{\Omega}$) of the series

$$\tag{15} \begin{gathered} \sum_{n=1}^{\infty} \lambda_n v_n(x) \varphi_n e^{-\lambda_n t}, \qquad \sum_{n=1}^{\infty} f_n(t) v_n(x), \\ \sum_{n=1}^{\infty} \lambda_n v_n(x) \int_0^t f_n(\tau) e^{-\lambda_n (t-\tau)} d\tau \end{gathered}$$

it will follow that the series (12) and (14) converge absolutely and uniformly on $\overline{\Omega}$. So, $u(x,t) \in C(\overline{\Omega})$. Now, by equality (12) and the uniform convergence of series (14), we get equality (14), and the continuity of $u'_t(x,t)$ on $\overline{\Omega}$. Also, the first decomposition (11) and equality (12) give us equality $u(x,0) = \varphi(x)$, $x \in \overline{G}$.

Further, by virtue of equality (12) and the (supposed) uniform convergence of series (13), we can conclude that equality (13) holds on $\overline{\Omega}$, and $u'_x(x,t) \in C(\overline{\Omega})$. Then from (12)–(13) it follows that $u(x,t)$ satisfies the boundary conditions (3).

Finally, the convergence of series (15) and the inequality

$$\begin{aligned} &|v''_n(x)| \left| \varphi_n e^{-\lambda_n t} + \int_0^t f_n(\tau) e^{-\lambda_n (t-\tau)} d\tau \right| \\ &\qquad \le (1 + |q(x)|) |v_n(x)| \cdot \left(\lambda_n |\varphi_n| e^{-\lambda_n t} + \lambda_n \left| \int_0^t f_n(\tau) e^{-\lambda_n (t-\tau)} d\tau \right| + |f_n(t)| \right) \end{aligned}$$

(holding, by equation (7), for all $\lambda_n \ge 1$ and $(x,t) \in \overline{\Omega}$) imply that the series appearing in the (formal) equality

$$\tag{16} \frac{\partial^2 u}{\partial x^2}(x,t) = \sum_{n=1}^{\infty} v''_n(x) \left[\varphi_n e^{-\lambda_n t} + \int_0^t f_n(\tau) e^{-\lambda_n (t-\tau)} d\tau \right]$$

converges absolutely and uniformly on $\overline{\Omega}$. As a consequence, by equality (13), we have that equality (16) is not just formal, and $u''_{x^2}(x,t) \in C(\overline{\Omega})$. Now, using (12)–(14), (16) and the second decomposition (11), we can prove that $u(x,t)$ satisfies the equation (1) at every point $(x,t) \in \Omega$.

In the next two subsections the convergence of series (13) and (14) will be established.

2. The convergence of series (13).. It suffices to show that the series

$$\tag{17} \sum_{n=1}^{\infty} v'_n(x) \varphi_n e^{-\lambda_n t}, \qquad \sum_{n=1}^{\infty} v'_n(x) \int_0^t f_n(\tau) e^{-\lambda_n (t-\tau)} d\tau$$

converge absolutely and uniformly on $\overline{\Omega}$.

Since for all $n \in \mathbb{N}$ and $(x,t) \in \overline{\Omega}$ we have

$$|v'_n(x) \varphi_n e^{-\lambda_n t}| \le |v'_n(x)| |\varphi_n|,$$

the convergence of the first series (17) follows, by Proposition 4(b), from the absolute and uniform convergence on $\overline{G}$ of the series

$$\sum_{n=1}^{\infty} \varphi_n v_n'(x).$$

In the case of the second series (17) we will use the following arguments. The conditions imposed on function $f_x'(x,\tau)$ imply the estimate

$$|f_n(\tau)| \le K(G, f_x', q) \cdot 1/\lambda_n, \qquad \lambda_n \ne 0, \tag{18}$$

where $K(G, f_x', q)$ depends on neither $\tau \in [0,T]$, nor $n \in \mathbb{N}$ (see the proof of estimate (13) in **[9]**; there we only need $q(x) \in L_1(G)$). Note that the proof of Proposition 4(a) is based on the estimate mentioned. Now, we have on $\overline{\Omega}$, by estimates (8) and (10):

$$\left| v_n'(x) \int_0^t f_n(\tau) e^{-\lambda_n (t-\tau)} d\tau \right| \le \begin{cases} (b-a) C_0 C_1 T \|f\|_{C(\overline{\Omega})} & \text{if} \quad \lambda_n \in [0,1], \\ C_1 K(G, f_x', q) \cdot \lambda_n^{-3/2} & \text{if} \quad \lambda_n > 1, \end{cases}$$

where $\|f\|_{C(\overline{\Omega})} \overset{\text{def}}{=} \max_{(x,t)\in\overline{\Omega}} |f(x,t)|$. Hence, according to estimate (9), for every $(x,t) \in \overline{\Omega}$ the following holds:

$$\begin{aligned} \sum_{n=1}^{\infty} \left| v_n'(x) \int_0^t f_n(\tau) e^{-\lambda_n (t-\tau)} d\tau \right| &= \sum_{0 \le \sqrt{\lambda_n} \le 1} (\cdot) + \sum_{\sqrt{\lambda_n} > 1} (\cdot) \\ &\le (b-a) A C_0 C_1 T \|f\|_{C(\overline{\Omega})} + C_1 K(G, f_x', q) \cdot \sum_{\sqrt{\lambda_n} > 1} \frac{1}{\lambda_n^{3/2}}. \end{aligned}$$

But using estimate (9) again, we obtain

$$\sum_{\sqrt{\lambda_n} > 1} \frac{1}{\lambda_n^{3/2}} = \sum_{k=1}^{\infty} \Bigg(\sum_{k < \sqrt{\lambda_n} \le k+1} \frac{1}{\lambda_n^{3/2}} \Bigg) \le A \cdot \sum_{k=1}^{\infty} \frac{1}{k^3}. \tag{19}$$

Therefore, the second series (17) and, consequently, the series (13) converge absolutely and uniformly on the closed rectangle $\overline{\Omega}$.

3. The convergence of series (14).. It suffices to prove that the three series (15) converge absolutely and uniformly on $\overline{\Omega}$.

In the case of the first series (15) the proof is based on an appropriate estimate for φ_n. Using the integration by parts and the fact that functions $\varphi(x)$, $v_n(x)$ belong to the domain of the self-adjoint operator L considered, we first obtain the equalities

$$\begin{aligned} \varphi_n = \int_a^b \varphi(x) v_n(x)\, dx &= \frac{1}{\lambda_n} \cdot \int_a^b \varphi(x) \big(-v_n''(x) + q(x) v_n(x) \big)\, dx \\ &= \frac{1}{\lambda_n} \cdot \int_a^b \mathcal{L}(\varphi)(x) v_n(x)\, dx, \quad \lambda_n \ne 0. \end{aligned} \tag{20}$$

Then we use the differential equation (7), the integration by parts again and equalities $\mathcal{L}(\varphi)(a) = 0 = \mathcal{L}(\varphi)(b)$. So we can get the equality

$$\varphi_n = \frac{1}{\lambda_n^2}\int_a^b \mathcal{L}(\varphi)'(x)v_n'(x)\,dx + \frac{1}{\lambda_n^2}\int_a^b \mathcal{L}(\varphi)(x)q(x)v_n(x)\,dx, \lambda_n \neq 0,$$

wherefrom, by conditions imposed on function $\mathcal{L}(\varphi)'(x)$, the necessary estimate results:

$$|\varphi_n| \leq K(G, \mathcal{L}(\varphi)', q) \cdot 1/\lambda_n^2,$$

where $K(G, \mathcal{L}(\varphi)', q)$ is independent of $n \in \mathbb{N}$ (see the proof of estimate (20) in [**9**]; there we need $q(x) \in AC(\overline{G})$). Note that the proof of Proposition 4(c) is based on the estimate mentioned. Now, for any $(x,t) \in \overline{\Omega}$ the following holds:

$$\begin{aligned}\sum_{n=1}^{\infty}\left|\lambda_n v_n(x)\varphi_n e^{-\lambda_n t}\right| &= \sum_{0\leq\sqrt{\lambda_n}\leq 1}(\cdot) + \sum_{\sqrt{\lambda_n}>1}(\cdot)\\ &\leq AC_0^2\|\varphi\|_{L_1(G)} + C_0K(G, \mathcal{L}(\varphi)', q)\cdot\sum_{\sqrt{\lambda_n}>1}\frac{1}{\lambda_n}\\ &\leq D_1 + D_2\cdot\sum_{k=1}^{\infty}\frac{1}{k^2},\end{aligned}$$

(see (19); the constants D_1, D_2 have the obvious meaning). Thus, the first series (15) converges absolutely and uniformly on $\overline{\Omega}$.

By Proposition 4(a), the second series (15) converges absolutely and uniformly on $\overline{G}$ for every $t \in [0,T]$. But using estimates (8)–(10) and (18), we can obtain the absolute and uniform convergence on $\overline{\Omega}$ of the series. This follows from

$$\begin{aligned}\sum_{n=1}^{\infty}|f_n(t)v_n(x)| &= \sum_{0\leq\sqrt{\lambda_n}\leq 1}(\cdot) + \sum_{\sqrt{\lambda_n}>1}(\cdot)\\ &\leq (b-a)AC_0^2\|f\|_{C(\overline{\Omega})} + C_0K(G, f_x', q)\cdot\sum_{\sqrt{\lambda_n}>1}\frac{1}{\lambda_n}.\end{aligned}$$

We prove the convergence of the third series (15) in the following way:

$$\begin{aligned}\sum_{n=1}^{\infty}\left|\lambda_n v_n(x)\int_0^t f_n(\tau)e^{-\lambda_n(t-\tau)}d\tau\right| &= \sum_{0\leq\sqrt{\lambda_n}\leq 1}(\cdot) + \sum_{\sqrt{\lambda_n}>1}(\cdot)\\ &\leq (b-a)AC_0^2T\|f\|_{C(\overline{\Omega})} + C_0K(G, f_x', q)\cdot\sum_{\sqrt{\lambda_n}>1}\frac{1}{\lambda_n}.\end{aligned}$$

(The estimates (8)–(9) and (18) are used here.) This ensures the absolute and uniform convergence on $\overline{\Omega}$ of the series considered.

4. Uniqueness of the solution.. As we already mentioned in Introduction, the uniqueness of the s-solution will be proved by a method used in [**5**].

Suppose that there exist two s-classical solutions $u_1(x,t), u_2(x,t)$ of the problem (1)–(3). Then the function

$$u(x,t) \overset{\text{def}}{=} u_1(x,t) - u_2(x,t)$$

is a s-classical solution of the problem

$$\begin{gathered} \frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) + q(x)u(x,t) = 0, \quad (x,t) \in \Omega, \\ u(x,0) = 0, \quad a \le x \le b, \\ u(x,t) \text{ satisfies the boundary conditions (3).} \end{gathered} \tag{21}$$

Also, for every $t \in [0,T]$ the function $u_t(x) \overset{\text{def}}{=} u(x,t)$ satisfies conditions of Proposition 4(b). Hence, for every $(x,t) \in \overline{\Omega}$ the equality

$$u(x,t) = \sum_{n=1}^{\infty} c_n(t) v_n(x),$$

holds, where the series is uniformly convergent on $\overline{G}$ and

$$c_n(t) = \int_a^b u(x,t) v_n(x)\, dx, \qquad n \in \mathbb{N}.$$

From $u(x,t), u'_t(x,t) \in C(\overline{\Omega})$ it follows that $c_n(t) \in C^{(1)}[0,T]$ and

$$c'_n(t) = \int_a^b \frac{\partial u}{\partial t}(x,t) v_n(x)\, dx.$$

The function $c_n(t)$ satisfies on $(0,T)$ the differential equation

$$c'_n(t) + \lambda_n c_n(t) = 0.$$

This follows from

$$\begin{aligned} \int_a^b \frac{\partial u}{\partial t}(x,t) v_n(x)\, dx &= \int_a^b \left[\frac{\partial^2 u}{\partial x^2}(x,t) - q(x)u(x,t)\right] v_n(x)\, dx \\ = \int_a^b u(x,t)\big[v''_n(x) - q(x)v_n(x)\big]\, dx &= -\lambda_n \cdot \int_a^b u(x,t) v_n(x)\, dx. \end{aligned}$$

(The first equality is a consequence of the differential equation (21); the second one holds because the functions $u_t(x), v_n(x)$ belong to the domain of the operator

L, and the third equality follows from the differential equation (7).) Therefore, we have

$$c_n(t) = B_n e^{-\lambda_n t}, \qquad t \in (0,T),$$

where B_n is an arbitrary real constant.

Now, using the initial condition (21) and continuity of $c_n(t)$ on $[0,T]$, for any $n \in \mathbb{N}$ we can obtain the equalities

$$B_n = \lim_{t\to 0} c_n(t) = \int_a^b u(x,0)v_n(x)\,dx = 0,$$

and conclude that $c_n(t) = 0$ for all $t \in [0,T], n \in \mathbb{N}$. But this means that $u(x,t) = 0$ for all $(x,t) \in \overline{\Omega}$.

5. The a priori estimate.. Let $u(x,t)$ be the s-classical solution of the problem (1)–(3). Then we can write $u(x,t) = u_1(x,t) + u_2(x,t)$, where

$$\begin{aligned} u_1(x,t) &= \sum_{n=1}^{\infty} v_n(x)\varphi_n e^{-\lambda_n t}, \\ u_2(x,t) &= \sum_{n=1}^{\infty} v_n(x) \cdot \int_a^b f_n(\tau) e^{-\lambda_n (t-\tau)} d\tau. \end{aligned} \tag{22}$$

For the initial function $\varphi(x)$ the equalities (20) hold. Hence, using those equalities and the estimates (8)–(9), for every $(x,t) \in \overline{\Omega}$ we have

$$\begin{aligned} |u(x,t)| &\le \sum_{n=1}^{\infty} \left| v_n(x)\varphi_n e^{-\lambda_n t} \right| = \sum_{0 \le \sqrt{\lambda_n} \le 1} (\cdot) + \sum_{\sqrt{\lambda_n} > 1} (\cdot) \\ &\le AC_0^2 \cdot \int_a^b |\varphi(x)|\,dx + C_0 \cdot \sum_{\sqrt{\lambda_n}>1} \frac{|\mathcal{L}(\varphi)_n|}{\lambda_n} \\ &\le AC_0^2 \|\varphi\|_{L_1(G)} + AC_0 \cdot \left(\sum_{k=1}^{\infty} \frac{1}{k^2} \right) \left(\|\varphi''\|_{L_1(G)} + \|q\|_{C(\overline{G})} \cdot \|\varphi\|_{L_1(G)} \right), \end{aligned}$$

wherefrom it follows that the estimate

$$\max_{(x,t)\in\overline{\Omega}} |u(x,t)| \le D_1 \left(\|\varphi\|_{L_1(G)} + \|\varphi''\|_{L_1(G)} \right) \tag{23}$$

holds, with the constant $D_1 > 0$ defined by

$$D_1 \overset{\text{def}}{=} \max \left\{ AC_0^2 + AC_0 \|q\|_{C(\overline{G})} \pi^2/6, AC_0 \pi^2/6 \right\}.$$

In the case of function $u_2(x,t)$ we can write

$$|u_2(x,t)| \le \sum_{n=1}^{\infty}\left|v_n(x)\cdot\int_a^b f_n(\tau)e^{-\lambda_n(t-\tau)}d\tau\right| = \sum_{0\le\sqrt{\lambda_n}\le 1}(\cdot)+$$
$$+\sum_{\sqrt{\lambda_n}>1}(\cdot) \le AC_0^2T(b-a)\|f\|_{C(\overline{\Omega})} + C_0^2(b-a)\|f\|_{C(\overline{\Omega})}\cdot\sum_{\sqrt{\lambda_n}>1}\frac{1}{\lambda_n},$$

for every $(x,t)\in\overline{\Omega}$. Having in mind estimate (9), we see that the estimate

$$\max_{(x,t)\in\overline{\Omega}}|u_2(x,t)| \le D_2\|f\|_{C(\overline{\Omega})} \tag{24}$$

holds if we put $D_2 \stackrel{\text{def}}{=} AC_0^2(b-a)\left(T+\pi^2/6\right)$.

Now, from (23)–(24) we obtain the final a priori estimate

$$\max_{(x,t)\in\overline{\Omega}}|u(x,t)| \le D\left(\|\varphi\|_{L_1(G)}+\|\varphi''\|_{L_1(G)}+\|f\|_{C(\overline{\Omega})}\right),$$

where $D \stackrel{\text{def}}{=} \max\{D_1,D_2\}$.

Proof of Theorem 1 is completed.

2. Proof of Theorem 2

1. Two lemmas.. Let us rewrite the formal solution (12) in the form

$$u(x,t)=u_1(x,t)+u_2(x,t),$$

where $u_1(x,t)$ and $u_2(x,t)$ are defined formally by equalities (22). Then the assertions of Theorem 2 concerning the existence and properties of the classical solution will result from the following two lemmas.

Lemma 1. *Let us assume: 1) $q(x)\in C(\overline{G})$;*

2) $\varphi(x)\in \overset{\circ}{W}{}_1^{(1)}(G)$, and $\varphi(x)$ satisfies the boundary conditions (5); $\varphi'(x)$ is a bounded, piecewise monotone function on its domain, or $\varphi'(x)$ belongs to $BV(\overline{G})$.

Then the equality

$$u_1(x,t)=\sum_{n=1}^{\infty}v_n(x)\varphi_n e^{-\lambda_n t} \tag{25}$$

holds uniformly on $\overline{\Omega}$, and the equalities

$$\begin{aligned}\frac{\partial u_1}{\partial t}(x,t)&=-\sum_{n=1}^{\infty}\lambda_n v_n(x)\varphi_n e^{-\lambda_n t},\\ \frac{\partial u_1}{\partial x}(x,t)&=\sum_{n=1}^{\infty}v_n'(x)\varphi_n e^{-\lambda_n t},\\ \frac{\partial^2 u_1}{\partial x^2}(x,t)&=\sum_{n=1}^{\infty}v_n''(x)\varphi_n e^{-\lambda_n t}\end{aligned} \tag{26}$$

hold uniformly on $\overline{\Omega}_\epsilon = [a,b]\times[\epsilon,T]$, *where* $\epsilon \in (0,T)$ *is an arbitrary number. Also, series (25)–(26) are absolutely convergent.*

LEMMA 2. *Let the following conditions be satisfied: 1)* $q(x) \in C(\overline{G})$;

2) $f(x,t) \in C(\overline{\Omega})$; $f(x,t) \in \overset{\circ}{W}{}_1^{(1)}(G)$ *for every* $t \in [0,T]$; $f'_x(x,t)$ *is bounded on* $\mathcal{D}(f'_x)$ *and possesses the property* $(\mathcal{A})$, *or* $f'_x(x,t)$ *possesses the property* $(\mathcal{B})$.
Then the equalities

$$
u_2(x,t) = \sum_{n=1}^{\infty} v_n(x) \int_0^t f_n(\tau) e^{-\lambda_n (t-\tau)} d\tau,
\tag{27}
$$
$$
\frac{\partial u_2}{\partial x}(x,t) = \sum_{n=1}^{\infty} v'_n(x) \int_0^t f_n(\tau) e^{-\lambda_n (t-\tau)} d\tau,
$$

$$
\frac{\partial u_2}{\partial t}(x,t) = -\sum_{n=1}^{\infty} v_n(x) \left[\lambda_n \int_0^t f_n(\tau) e^{-\lambda_n (t-\tau)} d\tau - f_n(t)\right],
\tag{27}
$$
$$
\frac{\partial^2 u_2}{\partial x^2}(x,t) = \sum_{n=1}^{\infty} v''_n(x) \int_0^t f_n(\tau) e^{-\lambda_n (t-\tau)} d\tau
$$

hold uniformly on $\overline{\Omega}$, *and the series are absolutely convergent.*

Indeed, having Lemma 1 proved, we see that equalities (26) hold on Ω, so one can immediately check that the function $u_1(x,t)$ belongs to the classes described in 1^*_u), satisfies the equation (1) (with $f = 0$) on Ω in the ordinary sense and satisfies the boundary conditions (3) for any $t \in (0,T]$. Also, equality (25) and the first decomposition (11) give us equality $u_1(x,0) = \varphi(x)$ on $\overline{G}$, wherefrom it follows that $u_1(x,t)$ satisfies the conditions (3) for $t = 0$. Hence, $u_1(x,t)$ is a classical solution of the problem (1)–(3) with $f = 0$.

On the other hand, Lemma 2 shows us that the function $u_2(x,t)$ is an s-classical solution of the problem (1)–(3) with $\varphi = 0$. Therefore, $u(x,t)$ will be the classical solution of the problem (1)–(3) possessing the properties described in the theorem.

In next two sections we will give the proof of the lemmas. The uniqueness of the solution and the a priori estimate will be considered in sections 4 and 5 respectively.

2. Proof of Lemma 1.. The series (25) converges absolutely and uniformly on $\overline{\Omega}$. This is implied by the inequality

$$
\sum_{n=1}^{\infty} \left|v_n(x)\varphi_n e^{-\lambda_n t}\right| \le \sum_{n=1}^{\infty} |v_n(x)||\varphi_n|
$$

because of the absolute and uniform convergence on $\overline{G}$ of the majorizing series, which follows from Proposition $4(a)$. Hence, equality (25) holds on $\overline{\Omega}$, and $u_1(x,t) \in C(\overline{\Omega})$.

The second part of Lemma 1 is based on the following proposition: For every $\epsilon > 0$ there exists a constant $K > 0$ such that the estimate

$$e^{-\lambda_n t} \le K/\lambda_n^{3/2} \tag{28}$$

holds for all $\lambda_n > 1$, $t \ge \epsilon$ (see [**1**, p. 151]). Now, using estimates (8)–(10),(28) and the Bessel inequality, we obtain on $\overline{\Omega}$ that

$$\begin{aligned}\sum_{n=1}^{\infty} \lambda_n |v_n(x)\varphi_n| e^{-\lambda_n t} &= \sum_{0\le\sqrt{\lambda_n}\le 1} (\cdot) + \sum_{\sqrt{\lambda_n}>1} (\cdot)\\ &\le AC_0^2\|\varphi\|_{L_1(G)} + C_0K\cdot \sum_{\sqrt{\lambda_n}>1} \frac{|\varphi_n|}{\lambda_n^{1/2}}\\ &\le D_3 + D_4\Big(\sum_{\sqrt{\lambda_n}>1}\frac{1}{\lambda_n}\Big)^{1/2}\Big(\sum_{\sqrt{\lambda_n}>1}|\varphi_n|^2\Big)^{1/2}\\ &\le D_3 + A^{1/2}D_4\|\varphi\|_{L_2(G)}\cdot\Big(\sum_{k=1}^{\infty}\frac{1}{k^2}\Big)^{1/2}.\end{aligned} \tag{29}$$

This means that the first series (26) converges absolutely and uniformly on $\overline{\Omega}_\epsilon$. So, by virtue of equality (25), the first equality (26) holds on $\overline{\Omega}_\epsilon$; consequently, it holds on Ω. Also, $(u_1)'_t(x,t) \in C(\Omega)\cap C(\overline{\Omega}_\epsilon)$.

Differentiating formally equality (25) with respect to x, we obtain the series $\sum_{n=1}^{\infty} v'_n(x)\varphi_n e^{-\lambda_n t}$ which can be majorized on $\overline{\Omega}_\epsilon$ in the following way:

$$\begin{aligned}\sum_{n=1}^{\infty} |v'_n(x)\varphi_n| e^{-\lambda_n t} &= \sum_{0\le\sqrt{\lambda_n}\le 1} (\cdot) + \sum_{\sqrt{\lambda_n}>1} (\cdot)\\ &\le AC_0C_1\|\varphi\|_{L_1(G)} + C_0C_1K\|\varphi\|_{L_1(G)}\cdot\sum_{\sqrt{\lambda_n}>1}\frac{1}{\lambda_n}.\end{aligned}$$

(The estimates (8)–(10) and (28) are used here.) Hence, the series converges absolutely and uniformly on $\overline{\Omega}_\epsilon$, wherefrom it results, by equality (25), that the second equality (26) holds on $\overline{\Omega}_\epsilon$. Consequently, this equality is true on Ω, and $(u_1)'_x(x,t) \in C(\Omega)\cap C(\overline{\Omega}_\epsilon)$.

Finally, for the series $\sum_{n=1}^{\infty} v''_n(x)\varphi_n e^{-\lambda_n t}$ we obtain, according to estimates (8), (10) and (28), that

$$\begin{aligned}\sum_{n=1}^{\infty} |v''_n(x)\varphi_n| e^{-\lambda_n t} &= \sum_{0\le\sqrt{\lambda_n}\le 1} (\cdot) + \sum_{\sqrt{\lambda_n}>1} (\cdot)\\ &\le AC_0C_2\|\varphi\|_{L_1(G)} + C_2\cdot\sum_{\sqrt{\lambda_n}>1}\lambda_n|\varphi_n|e^{-\lambda_n t}\\ &\le D_5 + C_2K\cdot\sum_{\sqrt{\lambda_n}>1}\frac{|\varphi_n|}{\lambda_n^{1/2}},\end{aligned}$$

where $(x,t) \in \overline{\Omega}_\epsilon$. Since the convergence of the majorizing series can be proved as in (29), by the second equality (26) we conclude that the third equality (26) holds on $\overline{\Omega}_\epsilon$, the series being absolutely and uniformly convergent. Hence, this equality holds on Ω, and $(u_1)''_{x^2}(x,t) \in C(\Omega) \cap C(\overline{\Omega}_\epsilon)$.

Lemma 1 is proved.

3. Proof of Lemma 2.. The proof is completely contained in the proof of Theorem 1. Indeed, by the estimates (8) and (10), we see that the first and the fourth series (27) have the same majorizing series as the third series (15). Further, the convergence of the third series (27) follows from the convergence of the second and the third series (15). Finally, the second series (27) and the second series (17) are the same.

REMARK 3. Conditions 2) imposed in Theorem 2 can be replaced by the following ones: $\varphi(x) \in W_2^{(2)}(G)$, and $\varphi(x)$ satisfies the boundary conditions (5). In this case it will follow from Proposition 4(b) that the second equality (26) holds *on the whole* $\overline{\Omega}$. Hence, then we will have

$$\frac{\partial u}{\partial x}(x,t) = \sum_{n=1}^{\infty} v_n'(x)\left[\varphi_n e^{-\lambda_n t} + \int_0^t f_n(\tau)e^{-\lambda_n(t-\tau)}d\tau\right]$$

uniformly on $\overline{\Omega}$, and the series will converge absolutely on this set.

4. Uniqueness of the solution.. Proof is the same as in the case of s-classical solutions. Namely, the only difference is that now $c_n(t)$ belongs to the class $C[0,T] \cap C^{(1)}(0,T)$, but this does not interfere the proof.

5. The a priori estimate.. Let $u(x,t)$ be the classical solution of the problem (1)–(3), and let $u_1(x,t)$, $u_2(x,t)$ be defined by equalities (22). Denote by $\mathcal{D}(\varphi')$ the domain of $\varphi'(x)$. Then the estimate

$$|\varphi_n| \le D_3 \frac{M(\varphi') + \|\varphi\|_{L_1(G)}}{\lambda_n}, \qquad \lambda_n \ne 0, \tag{30}$$

holds, where $D_3 > 0$ does not depend on $\varphi(x)$, and either

$$M(\varphi') \stackrel{\text{def}}{=} n(\varphi', \mathcal{D}(\varphi')) \cdot \sup_{x \in \mathcal{D}(\varphi')} |\varphi'(x)|$$

if $\varphi'(x)$ is bounded and piecewise monotone on $\mathcal{D}(\varphi')$, or $M(\varphi') \stackrel{\text{def}}{=} V_a^b(\varphi')$ if $\varphi'(x) \in BV(\overline{\Omega})$ (see the proof of Proposition 4 in [9]).

Using estimate (30), one can obtain the estimate

$$\max_{(x,t)\in\overline{\Omega}} |u_1(x,t)| \le D_4 \left(\|\varphi\|_{L_1(G)} + M(\varphi')\right), \tag{31}$$

where $D_4 \stackrel{\text{def}}{=} \max\{AC_0^2 + D_3 AC_0\pi^2/6, D_3 AC_0 \pi^2/6\}$. Proof of this estimate has the same "structure" as the proof of estimate (23).

In the case of function $u_2(x,t)$ the estimate (24) holds, with the same constant D_2. Hence, from (24) and (31) the following a priori estimate

$$\max_{(x,t)\in\overline{\Omega}} |u(x,t)| \le D\left(\|\varphi\|_{L_1(G)} + M(\varphi') + \|f\|_{C(\overline{\Omega})}\right)$$

results, with $D \stackrel{\text{def}}{=} \max\{D_2, D_4\}$.

Proof of Theorem 2 is completed.

3. Proof of Theorem 3

1. Auxiliary propositions.. Let us start again from the decomposition $u(x,t) = u_1(x,t) + u_2(x,t)$ of the formal solution (12), the functions $u_1(x,t)$ and $u_2(x,t)$ being defined by (22).

Assumptions 1) and 2) of Theorem 3 imply that for $u_1(x,t)$ Lemma 1 is valid. That is why this function is a classical solution of (1)–(3) with $f = 0$. Hence, the other assertions of Theorem 3, except the uniqueness and the a priori estimate, will follow from

LEMMA 3. *Let us assume: 1) $q(x) \in C(\overline{G})$;*
2) $f(x,t) \in C(\overline{\Omega})$, and $f(x,t)$ satisfies the Hölder condition on $[0,T]$, with an exponent $\alpha \in (1/2, 1]$, uniformly with respect to $x \in \overline{G}$.
Then the equality

$$u_2(x,t) = \sum_{n=1}^{\infty} v_n(x) \int_0^t f_n(\tau) e^{-\lambda_n (t-\tau)} d\tau \tag{32}$$

holds uniformly on $\overline{\Omega}$, and the equalities

$$\begin{aligned} \frac{\partial u_2}{\partial t}(x,t) &= \sum_{n=1}^{\infty} v_n(x) \frac{d}{dt} \int_0^t f_n(\tau) e^{-\lambda_n (t-\tau)} d\tau, \\ \frac{\partial u_2}{\partial x}(x,t) &= \sum_{n=1}^{\infty} v_n'(x) \int_0^t f_n(\tau) e^{-\lambda_n (t-\tau)} d\tau \end{aligned} \tag{33}$$

hold uniformly on $\overline{\Omega}_\epsilon$ for every $\epsilon \in (0,T)$. The series are absolutely convergent.

Also, $(u_2)''_{x^2}(x,t) \in C(\Omega)$, and $u_2(x,t)$ satisfies the equation (1) on Ω in the ordinary sense.

This lemma will be proved by a generalization of the Chernyatin method. The major characteristic of the method is the following: Differing from the classical procedure for justification of the Fourier method, proof of existence and continuity of $(u_2)''_{x^2}(x,t)$ is not based on the direct differentiation of the second series (33) (because f is not sufficiently smooth), but on the following

PROPOSITION 5. [**7**, Lemma], [**2**, p. 347]. *If for every $t \in [\epsilon, T]$ the series*

$$\sum_{n=1}^{\infty} v_n''(x) \int_0^t f_n(\tau) e^{-\lambda_n (t-\tau)} d\tau$$

converges in $L_2(G)$ to the function $v(x,t) \in C(\overline{\Omega}_\epsilon)$, then $u_2(x,t)$ has the partial derivative $(u_2)''_{x^2}(x,t)$ on $\overline{\Omega}_\epsilon$, and $(u_2)''_{x^2}(x,t) = v(x,t)$.

2. An asymptotic formula.. We prove Lemma 3 by applying an appropriate modification of the Chernyatin method: it turns out that in the general case considered it is possible to use successfully estimates (8)–(10) instead of specific asymptotics for eigenfunctions and eigenvalues of the operator generated by (4) and the zero boundary conditions, which were originally applied in [**7**].

To keep the paper self-contained, in this section we will completely expose the first step of the proof, as it was done in [**7**]: it is necessary to establish an asymptotic formula for the functions

$$F_n(t) \stackrel{\text{def}}{=} \int_0^t f_n(\tau) e^{-\lambda_n (t-\tau)} d\tau, \qquad t \in [\epsilon, T], \tag{34}$$

where $\epsilon \in (0,T)$ is an arbitrary number. In order to do that, we will approximate $f_n(\tau)$ by a piecewise linear function $\psi_n(\tau) \in C[0,t]$ on every closed interval $[0,t] \subseteq [0,T]$.

Let us fix $n \in \mathbb{N}$ and $t \in (0,T]$. Suppose $\lambda_n \neq 0$, and define points $\tau_i^n \in [0,t]$ $(i = 0, 1, \dots, m_n)$ by

$$\tau_0^n = 0, \quad \tau_{m_n}^n = t, \quad \tau_i^n = \tau_{i-1}^n + \delta_n, \qquad \delta_n = t/m_n,$$

where $m_n \in \mathbb{N}$ will be defined below. Then define

$$\begin{gathered} \psi_n(\tau_0^n) = f_n(\tau_0^n), \quad \psi_n(\tau_i^n) = f_n(\tau_i^n), \\ a_i^n = \frac{\psi_n(\tau_i^n) - \psi_n(\tau_{i-1}^n)}{\delta_n}, \\ (\forall \tau \in [\tau_{i-1}^n, \tau_i^n]) \quad \psi_n(\tau) = \psi_n(\tau_{i-1}^n) + a_i^n (\tau - \tau_{i-1}^n), \end{gathered} \tag{35}$$

$$(\forall \tau \in [0,t]) \quad \psi_n'(\tau) = \begin{cases} a_i^n & \text{if} \quad \tau \in [\tau_{i-1}^n, \tau_i^n), \\ a_{m_n}^n & \text{if} \quad \tau = \tau_{m_n}^n. \end{cases} \tag{36}$$

The number m_n will be defined by demanding that

$$(\forall \tau \in [0,t]) \; |\psi_n(\tau) - f_n(\tau)| < 1/\lambda_n^\alpha. \tag{37}$$

Using condition (6) and estimates (8), we obtain

$$(\forall t, t' \in [0,T]) \quad |f_n(t) - f_n(t')| \leq D|t - t'|^\alpha, \tag{38}$$

where $D = BC_0(b-a)$. Now, by virtue of (38) and the corresponding equalities (35), for every $\tau \in [0,t]$ we have the inequlities

$$|\psi_n(\tau) - f_n(\tau)| \le |\psi_n(\tau_{i-1}^n) - f_n(\tau)| + |\psi_n(\tau) - \psi_n(\tau_{i-1}^n)| \le 2D\delta_n^\alpha,$$

supposing that $\tau \in [\tau_{i-1}^n, \tau_i^n]$. So we see that condition (37) will be satisfied if we put

$$m_n \stackrel{\text{def}}{=} [t(2D)^{1/\alpha}\lambda_n] + 1, \tag{39}$$

where $[\rho]$ denotes the entire part of a number ρ. It is convenient to write (37) in the following form

$$f_n(\tau) = \psi_n(\tau) + r_n(\tau) \cdot 1/\lambda_n^\alpha, \tag{40}$$

where $r_n(\tau)$ is continuous, and $|r_n(\tau)| < 1$ on $[0,t]$ for all $t \in (0,T]$ and $n \in \mathbb{N}$.

Let us estimate the coefficients a_i^n. By (35), (38)–(39) we have

$$|a_i^n| \le D\delta_n^{\alpha-1} = D\left(\frac{[t(2D)^{1/\alpha}\lambda_n] + 1}{t}\right)^{1-\alpha} \le D\lambda_n^{1-\alpha}\left((2D)^{1/\alpha} + \frac{1}{t\lambda_n}\right)^{1-\alpha}.$$

Since the number $\epsilon \in (0,T)$ is arbitrarily fixed, the above inequalities imply that there exists a constant $K(\alpha,\epsilon) > 0$, not depending on t and n, such that the estimate

$$|a_i^n| \le K(\alpha,\epsilon) \cdot \lambda_n^{1-\alpha} \tag{41}$$

holds for every $t \in [\epsilon, T]$.

Return now to the integral (34). Using (40), the integration by parts and the first mean-value formula for integrals, we obtain

$$\begin{aligned} F_n(t) &= \int_0^t \psi_n(\tau)e^{-\lambda_n(t-\tau)}d\tau + \frac{r_n(\theta_n)}{\lambda_n^\alpha}\int_0^t e^{-\lambda_n(t-\tau)}d\tau \\ &= \frac{f_n(\tau)}{\lambda_n}e^{-\lambda_n(t-\tau)}\Big|_0^t - \frac{1}{\lambda_n}\int_0^t \psi_n'(\tau)e^{-\lambda_n(t-\tau)}d\tau + O\big(\lambda_n^{-(1+\alpha)}\big), \end{aligned} \tag{42}$$

where $\theta_n \in [0,t]$, and $\big|O\big(\lambda_n^{-(1+\alpha)}\big)\big| \le 2\lambda_n^{-(1+\alpha)}$. By virtue of (36) and (41), we can estimate the last integral in (42):

$$\left|\int_0^t \psi_n'(\tau)e^{-\lambda_n(t-\tau)}d\tau\right| = \left|\sum_{i=1}^{m_n} a_i^n \int_{\tau_{i-1}^n}^{\tau_i^n} e^{-\lambda_n(t-\tau)}d\tau\right| \le K(\alpha,\epsilon) \cdot \frac{1}{\lambda_n^\alpha}.$$

From this estimate and equalities (42) the final asymptotic formula for the integrals (34) follows:

$$(\forall t \in [\epsilon, T]) \quad F_n(t) = \frac{f_n(t)}{\lambda_n} - \frac{f_n(0)}{\lambda_n}e^{-\lambda_n t} + O\big(\lambda_n^{-(1+\alpha)}\big), \tag{43}$$

where $\big|O\big(\lambda_n^{-(1+\alpha)}\big)\big| \le \big(2 + K(\alpha,\epsilon)\big)\lambda_n^{-(1+\alpha)}$.

3. Proof of Lemma 3.. The series (32) converges absolutely and uniformly on $\overline{\Omega}$. By estimates (8)–(9), this follows from

$$\sum_{n=1}^{\infty}|F_n(t)v_n(x)| = \sum_{0\le\sqrt{\lambda_n}\le 1}(\cdot) + \sum_{\sqrt{\lambda_n}>1}(\cdot)$$
$$\le (b-a)AC_0^2T\|f\|_{C(\overline{\Omega})} + 2(b-a)C_0^2\|f\|_{C(\overline{\Omega})}\cdot\sum_{\sqrt{\lambda_n}>1}\frac{1}{\lambda_n}.$$

Hence the sum $u_2(x,t)$ of the series belongs to the class $C(\overline{\Omega})$.

Consider the existence and continuity of the derivative $(u_2)'_x(x,t)$. Let ϵ be an arbitrary number from the interval $(0,T)$. Differentiating formally equality (32) with respect to x, by virtue of (10), (34) and (43), we obtain the following equalities on $\overline{\Omega}_\epsilon$:

$$\begin{aligned}(u_2)'_x(x,t) &= \sum_{0\le\sqrt{\lambda_n}\le 1}F_n(t)v'(x) + \sum_{\sqrt{\lambda_n}>1}F_n(t)v'_n(x)\\ &= \sum_{0\le\sqrt{\lambda_n}\le 1}F_n(t)v'_n(x) + \sum_{\sqrt{\lambda_n}>1}\frac{f_n(t)}{\lambda_n}v'_n(x)\\ &\quad -\sum_{\sqrt{\lambda_n}>1}\frac{f_n(0)}{\lambda_n}v'_n(x)e^{-\lambda_n t} + \sum_{\sqrt{\lambda_n}>1}O\big(\lambda_n^{-(1+\alpha)}\big)v'_n(x).\end{aligned}\tag{44}$$

The finite series $\sum_{0\le\sqrt{\lambda_n}\le 1}|F_n(t)v'_n(x)|$ can be bounded by $(b-a)AC_0C_1T\|f\|_{C(\overline{\Omega})}$. The other series converge absolutely and uniformly on $\overline{\Omega}_\epsilon$. Indeed, according to estimates (10), for every $(x,t)\in\overline{\Omega}_\epsilon$ the estimates

$$\left|\frac{f_n(t)}{\lambda_n}v'_n(x)\right| \le C_1\frac{|f_n(t)|}{\lambda_n^{1/2}},$$
$$\big|O\big(\lambda_n^{-(1+\alpha)}\big)v'_n(x)\big| \le \big(2+K(\alpha,\epsilon)\big)\frac{C_1}{\lambda_n^{1/2+\alpha}}$$

are true. Therefore,

$$\sum_{\sqrt{\lambda_n}>1}\left|\frac{f_n(t)}{\lambda_n}v'_n(x)\right| \le C_1\Big(\sum_{n=1}^{\infty}|f_n(t)|^2\Big)^{1/2}\cdot\Big(\sum_{\sqrt{\lambda_n}>1}\frac{1}{\lambda_n}\Big)^{1/2},$$

$$\sum_{\sqrt{\lambda_n}>1}\big|O\big(\lambda_n^{-(1+\alpha)}\big)v'_n(x)\big| \le \big(2+K(\alpha,\epsilon)\big)C_1\cdot\sum_{\sqrt{\lambda_n}>1}\frac{1}{\lambda_n^{1/2+\alpha}}$$
$$\le D_6A\sum_{k=1}^{\infty}\frac{1}{k^{1+2\alpha}},$$

where D_6 has the obvious meaning. Having in mind the uniform convergence of the series $\sum_{n=1}^{\infty} |f_n(t)|^2$ on $[0,T]$, we can conclude that the first and the third series converge absolutely and uniformly on $\overline{\Omega}_\epsilon$. Then the convergence of the second series follows immediately from the convergence of the first one. Therefore, it is proved that $\sum_{n=1}^{\infty} F_n(t)v_n'(x)$ converges absolutely and uniformly on $\overline{\Omega}_\epsilon$. This and equality (32) imply that $(u_2)_x'(x,t)$ exists on $\overline{\Omega}_\epsilon$, and the second equality (33) holds on this closed rectangle. As a consequence, we have $(u_2)_x'(x,t) \in C(\Omega) \cap C(\overline{\Omega}_\epsilon)$.

Let us establish now the existence and continuity of the partial derivative $(u_2)_t'(x,t)$ on $\overline{\Omega}_\epsilon$. From (34) it follows $F_n'(t) = f_n(t) - \lambda_n F_n(t)$. Using this and (43), we write the formal equalities on $\overline{\Omega}_\epsilon$

$$
\begin{aligned}
(45)\quad (u_2)_t'(x,t) &= \sum_{n=1}^{\infty} F_n'(t)v_n(x) \\
&= \sum_{0\le\sqrt{\lambda_n}\le 1} \left(f_n(t) - \lambda_n F_n(t)\right) v_n(x) + \sum_{\sqrt{\lambda_n}>1} f_n(0)e^{-\lambda_n t}v_n(x) - \sum_{\sqrt{\lambda_n}>1} O(\lambda_n^{-\alpha})v_n(x).
\end{aligned}
$$

The finite sum $\sum_{0\le\sqrt{\lambda_n}\le 1} |f_n(t) - \lambda_n F_n(t)| v_n(x)|$ can be bounded by the number $(b-a)(1+T)AC_0^2\|f\|_{C(\overline{\Omega})}$. The first series following this summ converges absolutely and uniformly on $\overline{\Omega}_\epsilon$. This can be easily proved by using estimates (8)–(9) and (28). For the second series we have

$$
\begin{aligned}
\sum_{\sqrt{\lambda_n}>1} \left|O\big(\lambda_n^{-\alpha}\big)v_n(x)\right| &\le \big(2 + K(\alpha,\epsilon)\big)C_0 \cdot \sum_{\sqrt{\lambda_n}>1} \frac{1}{\lambda_n^{\alpha}} \\
&\le A\big(2 + K(\alpha,\epsilon)\big)C_0 \cdot \sum_{k=1}^{\infty} \frac{1}{k^{2\alpha}},
\end{aligned}
$$

where the majorizing numerical series converges because $\alpha \in (1/2, 1]$. Hence, we proved that the series $\sum_{n=1}^{\infty} F_n'(t)v_n(x)$ converges (absolutely and) uniformly on $\overline{\Omega}_\epsilon$, wherefrom the existence of $(u_2)_t'(x,t)$ and the first equality (33) on $\overline{\Omega}_\epsilon$ follow. Also, $(u_2)_t'(x,t) \in C(\Omega) \cap C(\overline{\Omega}_\epsilon)$.

It remains to consider the existence and continuity of the partial derivative $(u_2)_{x^2}''(x,t)$. We will start from the series $\sum_{n=1}^{\infty} F_n(t)v_n''(x)$. By equation (7), we can first write the equalities

$$
\begin{aligned}
F_n(t)v_n''(x) &= q(x)F_n(t)v_n(x) - \lambda_n F_n(t)v_n(x) \\
&= q(x)F_n(t)v_n(x) + F_n'(t)v_n(x) - f_n(t)v_n(x),
\end{aligned}
$$

where $(x,t) \in \overline{\Omega}_\epsilon$, and then the formal equality

$$
(46)\quad \sum_{n=1}^{\infty} F_n(t)v_n''(x) = q(x) \cdot \sum_{n=1}^{\infty} F_n(t)v_n(x) + \sum_{n=1}^{\infty} F_n'(t)v_n(x) - \sum_{n=1}^{\infty} f_n(t)v_n(x).
$$

Now, for every $t \in [\epsilon, T]$ the three series on the right-hand in (46) converge in $L_2(G)$ to the functions $q(x)u_2(x,t), (u_2)'_t(x,t), f(x,t)$ respectively. Since the function

$$v(x,t) = q(x)u_2(x,t) + (u_2)'_t(x,t) - f(x,t)$$

is continuous on $\overline{\Omega}_\epsilon$, we see that the first series (46) satisfies the condition imposed in Proposition 5. That is why there exists the partial derivative $(u_2)''_{x^2}(x,t)$ on $\overline{\Omega}_\epsilon$, and the equality

$$\frac{\partial^2 u_2}{\partial x^2}(x,t) = q(x)u_2(x,t) + \frac{\partial u_2}{\partial t}(x,t) - f(x,t) \tag{47}$$

holds on this closed rectangle. Since the number $\epsilon \in (0,T)$ is arbitrary, we have $(u_2)''_{x^2}(x,t) \in C(\Omega)$, and equality (47) shows that $u_2(x,t)$ satisfies the equation (1) on Ω (in the ordinary sense).

Proof of Lemma 3 is completed.

Note that Remark 3 remains to be valid in the case considered too.

4. On uniqueness and a priori estimate.. In previous sections we have established the existence (and some properties) of a classical solution $u(x,t)$. But this solution is unique, and for it the a priori estimate $(*)$ holds. Proof of these assertions is given in sections 4–5 §2.

Theorem 3 is proved.

5. On Remarks 1–2.. Remark 1 is obvious: the boundary conditions imposed on $\varphi(x)$ are, in fact, the compatibility conditions with the boundary conditions imposed on $u(x,0)$.

Remark 2 is based on the following: If we suppose that $\alpha_{11}\beta_{21} - \alpha_{21}\beta_{11} \neq 0$, then Proposition 4 and all the estimates of the Fourier coefficients, used in our proofs, remain to be valid (see Remark 4 in [**10**]).

REMARK 4. The part of results presented in this paper, contained in Theorems 1 and 2, was reported on the Voronezh Spring Mathematical Seminar devoted to "*Contemporary Methods in Theory of Boundary Problems*", held in Voronezh (Russia), May 3rd–9th, 1998.

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Matematički fakultet
Studentski trg 16/IV
11001 Beograd, p.p. 550
Yugoslavia
lazetic@matf.bg.ac.yu

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