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# VASSILIEV INVARIANTS OF DOODLES, ORNAMENTS, ETC.

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ABSTRACT. In 1990 V. Vassiliev introduced the notion of finite order invariants of knots. These invariants may be thought about as polynomials in the functional space of all invariants. The order of invariants is defined by certain filtration of a resolution of the discriminant set, i.e. of the space of 'quasiknots' (smooth non-embeddings of the circle to the 3-space): the invariants of order $n$ are 0-cohomologies of the space of knots, dual in some sense to homology of the $n$-th term of the filtration. But after the works of Vassiliev [V90] and Kontsevich [K93] the study of the finite order invariants was reduced to the study of chord diagrams, which represent, in fact, transversal selfintersections of the discriminant, and the homological origins of the theory were nearly forgotten.

I'd like to remind the general construction of finite order invariants and the combinatorial objects appearing in the calculation of such invariants. Instead of classification of knots, several variants of classification of plane curves without triple points will be considered. These problems are, in a sense, more generic, because not only transversal selfintersections, but also more complicated singularities of the discriminant, should necessarily be considered.

On the other hand, diagrams other than the chord diagrams, relevant to classification of knots and plane curves will be constructed, and some recent results by M. Goussarov, M. Polyak, O. Viro, V. Vassiliev and myself will be formulated.

## 1. INTRODUCTION

This paper is a very extended version of my talk at the conference "Geometric Combinatorics" (Kotor, Yugoslavia, 1998). Nearly the first half of the paper introduces the necessary definitions and constructions, and the second half is devoted to the content of the talk itself. Since the talk was scheduled shorter than expected, only the last quarter of the paper was actually exposed.

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Let us start from a generic problem: given compact manifolds $M_1, \dots, M_k$ describe the topology of the space of smooth maps of their disjoint union $M_1 \sqcup \cdots \sqcup M_k$ to $\mathbf{R}^n$ (or another manifold) with no non-generic intersections of the images of the components (either including, or excluding self-intersections). Vassiliev theory [V90, V94] gives us a tool for calculation the cohomology of this space. We restrict ourselves with 0-cohomology group (i.e. the group of invariants) only.

Even if all $M_i$ are 1-dimensional (i.e. circles), this problem is highly non-trivial. For $n \geq 3$ the image of a generic map has no (self-)intersections at all, and the problem splits into three classical problems:

- classification of knots ($k = 1$);
- isotopy classification of links ($k > 1$, selfintersections are forbidden);
- homotopy classification of links ($k > 1$, selfintersections are allowed).

For $n = 2$ the image of a generic map has necessarily (self-)intersections, but it has no triple points. Such objects are less classical. They were introduced in [FT77, Kh94] as *doodles* (no triple points at all) and in [V93] as *ornaments* (no intersections of $\geq 3$ components). Again, the classification splits into the following three problems:

- classification of single-component doodles ($k = 1$);
- classification of multi-component doodles ($k > 1$);
- classification of ornaments ($k \geq 3$).

A general Vassiliev theory of finite order invariants can be applied in each case: see [V90] for knots, [V98] for single-component doodles and [V93] for ornaments. It gives an algorithm producing all finite order invariants, but two questions remain:

- how to describe finite order invariants more explicitly;
- whether finite order invariants classify everything.

The second question was answered positively for $n > 3$ [V90], for a special case of links — braids and string links [B-N95] —, and for doodles [M98]. A satisfactory answer to the first question for knots is given by Polyak-Viro construction [PV94] and Goussarov theorem [GPV98]. In the other cases the questions remain still open.

It appears surprisingly, that the calculation of finite order invariants of doodles and ornaments is more complicated than that for knots and links. The reason can be explained informally as follows. Since the codimension of a knot or link in $\mathbf{R}^3$ is equal to 2, one can suppose not only themselves, but also homotopy films between them, being generic enough. On the contrary, the homotopy films between doodles or ornaments sweep the whole areas in the plane. So there exist rich combinatorial structures, nearly hidden in Vassiliev theory for knots and links, and revealed for doodles and ornaments.

The goal of this paper is to compare briefly the theories of Vassiliev invariants for the six classification problems above, and to describe the known answers to the two questions above.

The structure of the paper is as follows. In sections 2 and 3 the elementary and homological theory of invariants of ornaments is introduced in details and the main results of papers [V93] and [M95] are summarized. These long definitions

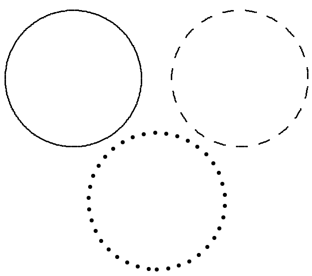


FIGURE 1.
Trivial 3-ornament(s). All eight oriented ornaments, obtained from this picture, are equivalent.

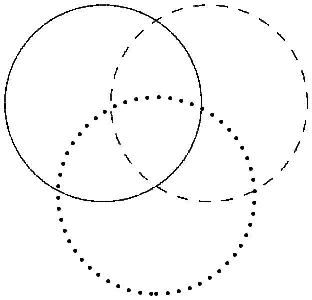


FIGURE 2.
The simplest non-trivial 3-ornament(s). All eight oriented ornaments, obtained from this picture, are pairwise non-equivalent.

and bulky constructions are supposed to be sufficient for not to repeat them for knots, links and doodles. In section 4 the differences between the case of ornaments and the cases of doodles, knots and links are outlined. In section 5 the Polyak-Viro construction [PV94, P94] of finite order invariants, its generalizations for ornaments and doodles [M96, M98], and some new results [GPV98, M98] are described. No proofs are given along the whole paper.

## 2. Elementary theory of finite order invariants of ornaments

**2.1. Ornaments and quasiornaments.** Denote by $C_k$ the disjoint union $c_1 \sqcup \cdots \sqcup c_k$ of $k$ circles.

**Definition 1** (See [V93]). *A* $k$-ornament *(or simply ornament) is a $C^\infty$-smooth map $C_k \to \mathbf{R}^2$ such that the images of no three different circles intersect at the same point in $\mathbf{R}^2$. Two ornaments are* equivalent, *if the corresponding maps $C_k \to \mathbf{R}^2$ can be connected by a homotopy $C_k \times [0,1] \to \mathbf{R}^2$ such that for any $t \in [0,1]$ the corresponding map of $C_k \times t$ is an ornament.*

See fig. 1 for example of a trivial 3-ornament, and figs. 2,3,4 for examples of non-trivial ones; all these ornaments are pairwise nonequivalent.

**Definition 2** (See [V93]). *A* $k$-quasiornament *(or simply* quasiornament*) is any $C^\infty$-smooth map $C_k \to \mathbf{R}^2$. The space of all $k$-quasiornaments is denoted by $\kappa_k$. The* discriminant $\Sigma \subset \kappa_k$ *is the space of all quasiornaments which are not ornaments, i.e., have forbidden triple points.*

**Definition 3** (See [V93]). *A $k$-ornament is* regular *if it is an immersion of $C_k$, and all the multiple points of the image of $C_k$ in $\mathbf{R}^2$ are simple transversal intersection points only. A $k$-quasiornament is* regular *if it is an immersion of $C_k$, and at any multiple point of the image all local components meet transversally.*

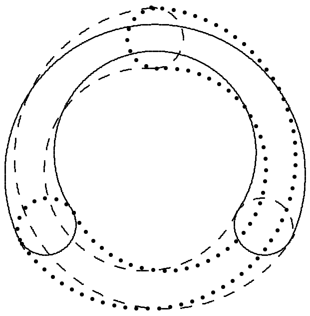


FIGURE 3.
Non-trivial 3-ornament(s)

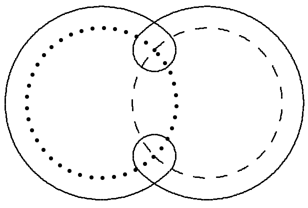


FIGURE 4.
Non-trivial 3-ornament(s),
trivial in $\mathbf{S}^2$

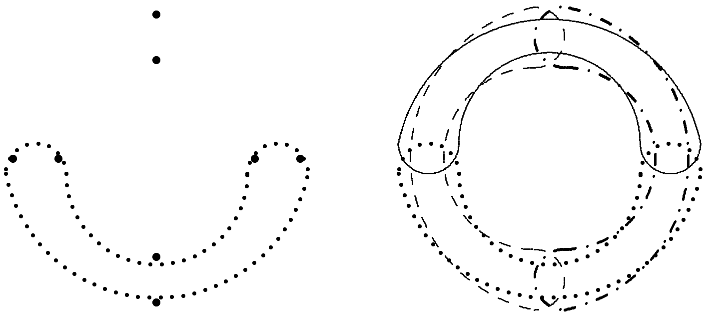


FIGURE 5. Brunnean 4-ornament(s) (the right picture)

Regular ornaments form a dense open set in the space of quasiornaments. Usually we consider regular ornaments only.

2.2. **Examples of invariants.** Let us fix an orientation of $\mathbf{R}^2$. Let $x$ be a point of $\mathbf{R}^2$, and $c_l$ be (the image of) the $l$-th component of a regular $k$-quasiornament. Recall that the index of a point $x \in \mathbf{R}^2$ with respect to a closed oriented curve $c : \mathbf{S}^1 \to \mathbf{R}^2$ not containing $x$ is the rotation number of the vector $c(t) - x$ when $t$ runs along $\mathbf{S}^1$. Denote by $\operatorname{ind}_l(x)$ the index of $x$ with respect to $c_l$ if $x \notin c_l$, and the smallest value of the index in a small neighborhood of $x$, if $x \in c_l$.

If $x$ is a simple intersection of the $i$-th component $c_i$ and $j$-th component $c_j$ of a regular $k$-quasiornament ($k$-ornament in particular), let $\sigma_{i,j}(x)$ be 1 or $-1$ depending on whether the orientation given by the tangent frame $(t_x(c_i), t_x(c_i))$ coincides with the orientation of $\mathbf{R}^2$ or not.

To any regular $k$-ornament $\phi$ and $1 \leq i < j \leq k$ there corresponds the following integer-valued function $I_{i,j}(b_1, \dots, b_k)$ of integer arguments:

$$I_{i,j}(b_1, \dots, b_k)(\phi) = \sum_{\substack{x \in c_i \cap c_j \\ \operatorname{ind}_l(x) = b_l, 1 \leq l \leq k}} \sigma_{i,j}(x).$$

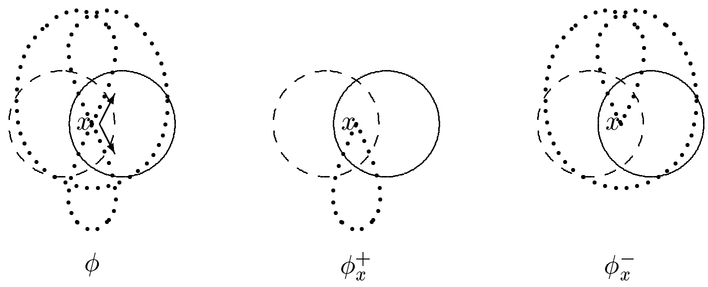



FIGURE 6. An ornament and its halves

**Theorem 1** (See [V93, Theorem 4]). *Each function $I_{i,j}(b_1,\dots,b_k)$ is an invariant of ornaments.*

For any $k$ integer non-negative exponents $\beta=(\beta_1,\dots,\beta_k)$ we can define functions $V_{i,j}(\beta)$ of $k$-ornaments:

$$V_{i,j}(\beta)=\sum_{b_1,\dots,b_k=-\infty}^{\infty}\binom{b_1}{\beta_1}\cdot\dots\cdot\binom{b_k}{\beta_k}\cdot I_{i,j}(b_1,\dots,b_k),$$

where $1\le i<j\le k$ and

$$\binom{b_l}{\beta_l}=\frac{b_l\cdot(b_l-1)\cdot\dots\cdot(b_l-\beta_l+1)}{\beta_l!},$$

no matter if $b_l$ is positive or negative. Obviously, $V_{i,j}(\beta)$ are integer-valued invariants. These *index momenta invariants* are the simplest finite order invariants (see theorem 3).

The integer-valued polynomials of invariants (e.g. the product of two invariants) are also invariants. A kind of antisymmetric product of invariants, which yields other invariants, is defined below.

Let $x$ be a selfintersection point of the $l$-th component $c_l$ of a regular ornament $\phi$. $x$ splits the image $\phi(c_l)$ into two closed curves having one non-smooth point each (namely, $x$ itself). Provided both $c_l$ and the plane $\mathbf{R}^2$ are oriented (as we always suppose), the pair of tangent vectors to $\phi(c_l)$ at $x$ allows us to call one of the two half-curves positive, and the other negative (see fig. 6). Let $\phi_x^+$ and $\phi_x^-$ be the ornaments obtained from $\phi$ by substitution of the image of $c_l$ by its positive and negative halves respectively. These ornaments are not regular, because they are not smooth at the point $x$, but they can be replaced by equivalent regular ornaments.

We define a binary operation of wedge product $\bigwedge_l$ on invariants of $k$-ornaments, where $l\le k$ indicates the component of the ornament. Let $I_1$ and $I_2$ be invariants, $\phi$ a regular ornament, and $x_1,\dots,x_n$ all selfintersections of $\phi(c_l)$. Then

$$\begin{aligned}(I_1\bigwedge_l I_2)(\phi)=\sum_{i=1}^{n}\ &(I_1(\phi)-I_1(\phi_{x_i}^+))(I_2(\phi)-I_2(\phi_{x_i}^-))\\ &-(I_1(\phi)-I_1(\phi_{x_i}^-))(I_2(\phi)-I_2(\phi_{x_i}^+)).\end{aligned}$$

**Theorem 2** (See [M95, Theorem 6]). *$I_1 \bigwedge_l I_2$ is an invariant of ornaments (and can be defined for non-regular ornaments too).*

**Example.** Let us consider the ornament $\phi$ from fig. 4: the first of the three components has two selfintersections with two little inner loops, the other two components are just circles with two mutual intersections each of which lies within one of the loops. It is easy to see that $(V_{2,3}(1,0,0) \bigwedge_1 V_{2,3}(2,0,0))(\phi) = \pm 2$, where the sign depends on the orientations of the curves. So the invariant $V_{2,3}(1,0,0) \bigwedge_1 V_{2,3}(2,0,0)$ does not belong to the subring generated by the invariants $V_{i,j}(\beta)$ above, since all $I_{i,j}(b_1, b_2, b_3)$ and hence all $V_{i,j}(\beta)$ vanish on this ornament.

### 2.3. Singular points and degeneration modes.

**Definition 4** (See [V93]). *A* standard singularity of complexity $j$ *is a pair of the form (a quasiornament $\phi : C_k \to \mathbf{R}^2$; a point $x \in \mathbf{R}^2$) such that $\phi^{-1}(x)$ consists of exactly $j+1$ points $z_1, \dots, z_{j+1}$, at least three of which belong to different components of $C_k$, the map $\phi$ close to all these points is an immersion, and the corresponding $j+1$ local branches of $\phi(C_k)$ are pairwise non-tangent at $x$. The integer vector $(\alpha_1(x), \dots, \alpha_k(x))$ where $\alpha_i(x)$ is the number of points of the $i$-th component of $C_k$ in $\phi^{-1}(x)$ is called* signature *of the singularity $x$. A regular quasiornament is called a* quasiornament of complexity $i$, *if all its forbidden points (i.e., the points, at which at least three different components meet) are standard singular points, and the sum of the complexities of these singularities is equal $i$.*

**Definition 5** (See [V93]). *Suppose that a regular quasiornament $\phi$ has $m$ singular points $x_1, \dots, x_m$ and $z_{1,1}, \dots, z_{m,j_m}$ are their pre-images. A* degeneration mode *of $\phi$ is some arbitrary order of marking all these points $z_{i,j}$ satisfying the following conditions: on any step we mark either some three points of $\phi^{-1}(x_l)$ for some $(x_l)$, belonging to some three different components of $C_k$, (if none point of $\phi^{-1}(x_l)$ is already marked) or one point (if some three or more other points with the same image are already marked).*

2.4. **Characteristic numbers.** The *characteristic numbers*, assigned by invariants to regular quasiornaments and their degeneration modes, are defined inductively by complexity of the quasiornaments.

The characteristic number assigned to an ornament is the value the invariant takes on it. The last step of the degeneration mode of the $k$-quasiornament is either marking of a triple of points on different components $c_i$, $c_j$ and $c_l$, or marking a single point on $c_l$. The marked point(s) are pre-images of some point $x \in \mathbf{R}^2$ of intersection of three or more components. Consider a local transversal perturbation of the $l$-th component of the $k$-quasiornament in a small neighborhood of one of the marked points, i.e. a shift of a small piece of component $c_l$ through the intersection point $x$. The perturbed quasiornaments are of smaller complexity than the initial one, and belong to two different equivalence classes. One of them can be called 'positive' and the other 'negative'. Namely, if $x$ is a simple intersection of $c_i$ and $c_j$, then the 'positive' $k$-quasiornaments are those with greater value of $(l-i)(l-j)\sigma_{i,j}(x)\operatorname{ind}_l(x)$, and if $x$ is a more complex intersection, the 'positive' $k$-quasiornaments are those with greater value of $\mathrm{ind}_l(x)$. The difference of the characteristic numbers of the 'positive' and 'negative' $k$-quasiornaments for the degeneration modes obtained from the given one by removing the last step becomes the characteristic numbers of the given $k$-quasiornament and the degeneration mode.

2.5. **Finite order invariants.**

**Definition 6** (See [V93]). *An invariant of ornaments is an* invariant of order $i$ *if all characteristic numbers of any regular quasiornament of complexity* $> i$*, assigned by this invariant, vanish.*

**Theorem 3** (See [V93, Theorem 5]). $V_{i,j}(\beta)$ *are invariants of order* $|\beta|+1$*, where* $|\beta| = \beta_1 + \cdots + \beta_k$.

**Theorem 4** (See [M95, Theorem 5]). *The product of invariants of orders* $i_1$ *and* $i_2$*, is an invariant of order* $i_1 + i_2$*. Integer-valued polynomials in finite order invariants are finite order invariants.*

**Theorem 5** (See [M95, Theorem 7]). *If* $I_1$ *and* $I_2$ *are invariants of orders* $i_1$ *and* $i_2$ *respectively, then* $I_1 \bigwedge_l I_2$*,* $1 \le l \le k$*, are invariants of order* $i_1 + i_2 + 1$.

To formulate more theorems about finite order invariants of ornaments we need more definitions.

2.6. **Configurations.** Let $A$ be a matrix of non-negative integer numbers $a_i^l$, where $1 \le i \le k$ and $1 \le l \le m$ ($k \ge 3$ is the number of components of the $k$-ornaments, and $m$ is arbitrary). Denote by $a^l$ the sum $a_1^l + \cdots + a_k^l$. We will consider only such matrices $A$, each row of which contains at least three nonzero elements. We will consider these matrices up to transposition of their rows, so we may assert that $a^1 \ge a^2 \ge \ldots \ge a^m \ge 3$. Denote by —A— the number $a^1 + \ldots + a^m$, and by $\#A$ the number of rows $a^l$ of the matrix, i.e. $\#A = m$.

**Definition 7** (See [V93]). *An* $A$-configuration *is a collection of* $|A|$ *pairwise different points in* $C_k$ *divided into groups of cardinalities* $a^1, \ldots, a^{\#A}$*, such that the number of points of the* $l$*-th group on the* $i$*-th component is equal to* $a_i^l$ *(and hence any group contains the points of at least three different components of* $C_k$*).* $A$ *is called the* signature *of the* $A$*-configuration. Two* $A$*-configurations are* equivalent *if they can be transformed one into the other by a diffeomorphism* $C_k \to C_k$ *which preserves ordering and orientations of all components of* $C_k$*. A quasiornament* $\phi : C_k \to \mathbf{R}^2$ respects *an* $A$*-configuration if it sends any of corresponding* $\#A$ *groups of points into one point in* $\mathbf{R}^2$*.* $\phi$ strictly respects *the* $A$*-configuration if, moreover, all these* $\#A$ *points in* $\mathbf{R}^2$ *are distinct, have no extra pre-images than these* $|A|$ *points, and* $\phi$ *has no extra points in* $\mathbf{R}^2$ *where three of more different components meet. A degeneration mode of an* $A$*-configuration is a degeneration mode of arbitrary quasiornament strictly respecting it.*

We call $\mathrm{compl}(A) = |A| - \#A$ the *complexity* of the $A$-configuration; it equals to the complexity $\mathrm{compl}(\phi)$ of any regular quasiornament $\phi$ strictly respecting it.

We call $\deg(\phi) = \deg(A) = |A| - 2\#A$ the *degree* of the regular quasiornament $\phi$ or of the $A$-configuration strictly respected by it; the degree is equal to the number of degeneration steps in any degeneration mode of $\phi$.

2.7. **Uniform invariants.** A subgroup (in fact, subring) of finite order invariants of ornaments is described below. These invariants have a simple explicit combinatorial representation. Only these invariants have analogues for maps of collections of multi-dimensional manifolds.

**Definition 8.** *An invariant $I$ of order $i$ is a*
- *single-group invariant, if the characteristic numbers of $I$ vanish for all regular $k$-quasiornaments with more than one singular point and all their degeneration modes;*
- *single-group invariant of signature $\alpha = (\alpha_1, \dots, \alpha_k)$, $|\alpha| = \alpha_1 + \dots + \alpha_k = i$ (all $\alpha_j$ are non-negative integers), if it is a single-group invariant and the characteristic numbers of $I$ vanish for all regular $k$-quasiornaments of complexity $i$ and their degeneration modes except quasiornaments with exactly one singular point of signature $\alpha$.*
- *$m$-group invariant, if the characteristic numbers of $I$ vanish for all regular $k$-quasiornaments with more than $m$ singular points and all their degeneration modes;*
- *$m$-group invariant of signature $A = \{\alpha^1, \dots, \alpha^m\}$, where each $\alpha^i = (\alpha^i_1, \dots, \alpha^i_k)$ is a signature of a singularity, and $|A| = |\alpha^1| + \dots + |\alpha^m| = i$, if it is a $m$-group invariant and the characteristic numbers of $I$ vanish for all regular $k$-quasiornaments of complexity $i$ and their degeneration modes except quasiornaments with exactly $m$ singular points of signatures $\alpha^1, \dots, \alpha^m$.*

The existence and non-uniqueness (in general case) of $m$-group invariants of any given signature and some bounds for the number of them are shown below (see theorem 7).

**Definition 9.** *An invariant $I$ of order $i$ is a* uniform invariant *if the characteristic number assigned by $I$ a $k$-quasiornaments $\phi$ of complexity $i$ and its degeneration mode $\Gamma$ depends on the signature $A$ of the configuration $J$ strictly respected by $\phi$ and on $\Gamma$, but not on $J$ and $\phi$ themselves.*

Independence of the characteristic numbers of the highest possible complexity of the quasiornament strictly respecting the same configuration (or equivalent configurations) is a general fact (see [V93, Theorem 6]). This definition requires the characteristic numbers of (the quasiornaments strictly respecting) nonequivalent configurations of the same signature $A$ to be equal to each other. Since all single-group configurations of the same signature are equivalent, all single-group invariants are uniform.

The following proposition gives the way to construct other uniform invariants.

**Proposition 1** (See [M95, Proposition 1]). *If $I_1$ is a uniform $m_1$-group invariant of order $i_1$, and $I_2$ is a uniform $m_2$-group invariant of order $i_2$, then the product $I_1 \cdot I_2$ is a uniform $m_1 + m_2$-group invariant of order $i_1 + i_2$.*

Both the parameters $\beta$ of the invariants $V_{i,j}(\beta)$ and the signatures $\alpha$ of singular points are $k$-dimensional integer vectors. Let $\epsilon_i$ denote the $i$-th vector of the standard base of this space, $\alpha \leq \beta$ means $\alpha_i \leq \beta_i$ for each $i$, and so on. Denote by $n(\beta)$ and $l(\beta)$ the number of positive components $\beta_i$ and the index of the last positive component respectively.

**Theorem 6** (See [M95, Theorem 3]).
(i) *Each $V_{i,j}(\beta)$ is a uniform single-group invariant of signature $\beta + \epsilon_i + \epsilon_j$.*
(ii) *For each $i$, $1 \leq i \leq k$ and each $\beta$,*

$$\sum_{\substack{1\leq j<i\leq k \\ \beta_j>0}} V_{j,i}(\beta-\epsilon_j) = \sum_{\substack{1\leq i<j\leq k \\ \beta_j>0}} V_{i,j}(\beta-\epsilon_j).$$

*There are no more non-trivial linear relations between the invariants $V_{i,j}(\beta)$.*

Denote by $v^q$ the invariant $V_{i^q,j^q}(\beta^q - \epsilon_{i^q} - \epsilon_{j^q})$ where $q = 1, \dots, p$, $i^q < j^q$, and all the triples $\beta^q, i^q, j^q$ are different for different $q$; let $m^q$ be positive integers. Denote $(i^1, \dots, i^r)$ by $I$, $(j^1, \dots, j^r)$ by $J$, $(m^1, \dots, m^r)$ by $M$, $|M|$ by $m$, $\{\underbrace{\beta^1, \dots, \beta^1}_{m^1}, \dots, \underbrace{\beta^r \dots, \beta^r}_{m^r}\}$ by $B$, and $\prod_{q=1}^r \binom{v^q}{m^q}$ by $V_{I,J}^M(B)$.

**Theorem 7** (See [M95, Theorem 4]).
(i) *$V_{I,J}^M(B)$ is a uniform $m$-group invariant of signature $B$.*
(ii) *Each uniform $m$-group invariant of signature $B$ is an integer linear combination of $V_{I,J}^M(B)$ for different $(I,J)$ and uniform $m$-group invariants of orders $< |B|$. Each uniform $m$-group invariant of order $n$ is an integer linear combination of $V_{I,J}^M(B)$ for different $(B, I, J, M)$ with $|M| \leq m$ and $|B| = n$, and uniform $m$-group invariants of orders $< n$.*
(iii) *Polynomials of $V_{i,j}(\beta)$ are uniform finite order invariants of ornaments and there are no other uniform finite order invariants.*

On the other hand, invariants of shape $I_1 \bigwedge_l I_2$ (see theorem 2) are non-uniform.

## 3. Homological theory of finite order invariants of ornaments

The skeleton of the homological theory of invariants of ornaments is as follows. Classes of equivalent ornaments are connected components of the set of ornaments. Numerical invariants of ornaments, i.e. functions, constant on the equivalency classes, form the zero-dimensional cohomology group of this set. This group can be expressed in terms of homology of one-point compactification of the discriminant by a sort of Alexander duality, so the study of invariants is dual to the study of the highest closed (Borel-Moore) homology group of the discriminant.

We work here with the infinite-dimensional space of all quasiornaments as if it were a finite-dimensional space of a very large dimension. A technique of finite-dimensional approximations which justifies this approach is developed in [V93].

In [V93, section 7] a resolution $\sigma$ of the discriminant $\Sigma$ was constructed. One-point compactification of the resolution is homotopy equivalent to one-point compactification of the discriminant itself. It has the natural filtration $\emptyset = \sigma_1 \subset$

$\sigma_2 \subset \sigma_3 \subset \dots$ (*main filtration*), corresponding to complexity of quasiornaments (which is the same as the complexity $|A| - \#A$ of $A$-configurations strictly respected by the quasiornaments). Each quotient space $\sigma_i/\sigma_{i-1}$ of the main filtration has the *auxiliary filtration* corresponding to the total number of points $|A|$ of the $A$-configuration. The auxiliary filtrations are finite and the corresponding spectral sequence calculates the homology of $\sigma_i/\sigma_{i-1}$. The main filtration is infinite, so the problem of convergence arises. Still, the corresponding spectral sequence calculates a subgroup of homology group of $\sigma$. The invariants dual to this subgroup are finite order invariants in the sense of section 2 (see theorem 8).

The exact definitions and theorems of the homological theory of invariants of ornaments are given below. The elementary definitions and results stated above are just translations of the homological ones (see propositions 2 and 3). Some auxiliary constructions are introduced first.

3.1. **Order complexes of configurations.** Given an $A$-configuration $J$, two triples consisting each of a simplicial complex, its subcomplex and quotient complex are associated with $J$. They can be defined as well for an abstract $|A|$-element set $\Theta$ subdivided into two families of subsets: *circles* of different colors $\theta_1, \dots, \theta_k$ and *groups* $\theta^1, \dots, \theta^{\#A}$ such that $\#\theta^j = a^j$ and each group intersects at least 3 circles. We will call such sets *abstract $A$-configurations* and denote $|A|$ and $\#A$ by $|\Theta|$ and $\#\Theta$ respectively.

Recall general definition of order complex: given a partially ordered set $P$, its order complex is the simplicial complex generated by all simplices $(p_1, p_2, \dots, p_n)$ where $p_i \in P$ and $p_1 < p_2 < \dots < p_n$.

For each $A$-configuration $J$ the space $\chi(J)$ of all quasiornaments respecting $J$ is obviously an affine subspace of codimension $2(|A| - \#A)$ in the space of all quasiornaments. Let $J_1, \dots, J_N$ be all (3)-subconfigurations (i.e. triples of points of different colors) of $J$. Then the spaces $\chi(J_i)$ and their intersections form a partially ordered set (with traditionally 'reverse' order: $V < W$ iff $W \subset V$).

**Definition 10** (Cf. [V93, section 7.1-7.3]). *The order complex of this partially ordered set is called the* order complex of $A$-configuration $J$ *and is denoted by $\Delta(J)$. Denote by $\Lambda(J)$ the subcomplex in $\Delta(J)$ generated by all simplices, not containing the maximal element $\chi(J_1) \cap \chi(J_2) \cap \dots \cap \chi(J_N)$, and by $\Xi(J)$ the corresponding quotient complex $\Delta(J)/\Lambda(J)$. This complex is called the* quotient complex *of $A$-configuration $J$.*

3.2. **Resolution space.**

**Definition 11** (Cf. [V93, section 7.2, 7.4 and proposition 13]). *The* resolution $\sigma$ *of the discriminant $\Sigma$ is the naturally topologized space of the pairs $(\phi, \delta_{J(\phi)})$, where $\phi \in \Sigma$ is a quasiornament, $J(\phi)$ is the $A$-configuration strictly respected by it, and $\delta_{J(\phi)} \in \Delta(J(\phi))$.*

Obviously, $\sigma$ allows also a representation as the space of the triples $(J, \xi_J, \phi)$ where $J$ is an $A$-configuration, $\xi_J \in \Delta(J) \setminus \Lambda(J)$, and a quasiornament $\phi$ respects $J$ [V93, section 7.4]. The complexity and the total number of points of $J$ determine the main and the auxiliary filtrations of the resolution of the discriminant. Denote the $i$-th term of the main filtration of $\sigma$ by $\sigma_i$.

3.3. **Spectral sequences.** A homological spectral sequence corresponds to each of the filtrations above. For some reasons they were turned to cohomological spectral sequences in [V93] and we preserve this tradition here. Namely, if $E^r_{p,q}$ is a homological spectral sequence of the main filtration, then rename the term $E^r_{p,q}$ to $E_r^{-p,\infty-1-q}$. $\infty-1$ here is the dimension of $\Sigma$; more accurately, $\infty-q$ means codimension $q$ in any sufficiently large finite-dimensional approximating space. This cohomological spectral sequence is called the *main spectral sequence.*

Similarly, if $G^r_{s;p,q}$ is the homological spectral sequence of an auxiliary filtration converging to $E^1_{s,p+q-s}$, its terms are renamed to $G_r^{-s;-p,\infty-1-q}$ and converge to $E_1^{-s,\infty-1-q-p+s}$.

**Theorem 8** (See [V93, Theorem 8]). *The main diagonal $\{E_r^{-i,i}\}$ of the main spectral sequence $E_r^{p,q}$ converges to the subgroup of finite order invariants in $H^*(\kappa_k\setminus\Sigma)$; the group $E_\infty^{-i,i}$ is naturally isomorphic to the quotient group of the group of invariants of order $\le i$ modulo the invariants of order $\le i-1$.*

3.4. **Blown configuration space.** To simplify the calculation of the spectral sequences we go from the infinite-dimensional resolution space to a homotopy equivalent locally finite-dimensional cellular complex.

Let us define the *blown configuration space* $\sigma^{(c)}$ as the space of the pairs $(J,\xi_J)$ where $J$ is a configuration and $\xi_J\in\Xi(J)$. This space splits into finite-dimensional subspaces, each of which corresponds to a class of configurations equivalent to some $A$-configuration $J$ ( $J$-block). Each $J$-block has a structure of fiber bundle whose base is the class of $A$-configurations equivalent to $J$ and fiber is $\Xi(J)$; its dimension is $|A|+(|A|-2\#A-1)=2(|A|-\#A)-1$. On the other hand, the blown configuration space has naturally main and auxiliary filtrations mentioned on page 110. Namely, the $s$-th term $\sigma_s^{(c)}$ of the main filtration is the union of $J$-blocks of $A$-configurations of complexity $|A|-\#A\le s$, and $p$-th term $\sigma_{s,p}^{(c)}$ of the auxiliary filtration of $\sigma_s^{(c)}/\sigma_{s-1}^{(c)}$ is the union of $J$-blocks of $A$-configurations of complexity $|A|-\#A=s$ consisting of $|A|\le p$ points. So main and *auxiliary spectral sequences* may be constructed.

Omitting the component $\phi$ of the triples $(J,\xi_J,\phi)$ we define the natural projection of the resolution $\sigma$ to the blown configuration space $\sigma^{(c)}$. To each $J$-block in $\sigma^{(c)}$ there corresponds the cellular subcomplex in $\sigma$ which is the direct product of the $J$-block by an affine subspace of codimension $2\,\mathrm{compl}(J)$ in the space of quasiornaments. So, $\sigma$ and $\sigma^{(c)}$ are homotopy equivalent (moreover, the projection providing the homotopy equivalence is a proper map and preserves the filtrations), and the results formulated in terms of the resolution $\sigma$ in [V93] are rewritten in terms of the blown configuration space $\sigma^{(c)}$ below.

3.5. **Homology of the blown configuration space and finite order invariants.** Denote by $\bar H_*(\ldots)$ the closed homology groups (also known as Borel-Moore homology, or homology of one-point compactification) of the given space. Denote by $\infty$ the dimension of the space of $k$-ornaments, or more accurately, a large enough generic subspace of it (see [V93]). Then $\sigma$ contains no cells of bigger dimensions.

**Proposition 2** (See [V93, section 7.4]). *The group of invariants of order $i$ is isomorphic by Alexander duality to the group $\bar{H}_{\infty-1}(\sigma_i)$. This isomorphism maps an invariant of order $i$ to a linear combination of cycles of the maximal dimension of one-point compactification of $\sigma_i$.*

In other words, the group of invariants of order $i$ is isomorphic to the highest closed homology group of $\sigma_i$, or equally the highest closed homology group $\bar{H}_{2i-1}(\sigma_i^{(c)})$ of $\sigma_i^{(c)}$. Since $\sigma_i^{(c)}$ is a finite-dimensional cellular complex, the calculation of invariants of any given order becomes just a (rather complicated) technical problem.

Given a quasiornament $\phi$, each cell of maximal dimension in $\Xi(J(\phi))$ corresponds to the simplex of maximal dimension in $\Delta(J(\phi))$; the latter corresponds obviously to the degeneration mode of $\phi$. This correspondence is a bijection.

**Proposition 3** (See [V93, proposition 17]). *Let the pair $(\phi, \xi_{J(\phi)})$ belong to a cell of maximal dimension in $\sigma_i \setminus \sigma_{i-1}$ (or, equivalently, $\xi_{J(\phi)}$ belong to a cell of maximal dimension in $\sigma_i^{(c)} \setminus \sigma_{i-1}^{(c)}$), and the quasiornament $\phi$ be regular. Let $I$ be an invariant of order $i$. Then the characteristic number, corresponding to $I$, $\phi$ and a degeneration mode of $\phi$ is equal to the multiplicity with which the cell corresponding to the degeneration mode participates in the cycle corresponding to $I$.*

### 3.6. Auxiliary spectral sequences and homology of $J$-blocks.

**Proposition 4** (See [V93, section 6]). *The auxiliary spectral sequences of the blown configuration space are exactly the same as the auxiliary spectral sequences defined on page 111 for the resolution of the discriminant. $G_1^{-s;-p,q} = H_{2s-1+p-q}(\sigma_{s,p}^{(c)}, \sigma_{s,p-1}^{(c)})$. $G_1^{-s;-p,q}$ is identically zero if $p < 3$, $s < 2$, $p < s+1$, $p < 3(p-s)$ (all these are simple relations between $|A|$ and $\#A$), or $-p+q < 0$ or $-p+q > 2s-1$ (the minimal and maximal possible codimensions in $\sigma_{s,p}^{(c)}$). $E_1^{-p,q}$ is identically zero if $p < 2$ or $-p+q < 0$. See fig. 7.*

The boundary of a $J$-block in $\sigma^{(c)}$ consists of the blocks obtained by merging together several (collections of) adjacent in $C_k$ points of the $A$-configurations. For instance, if $J$-block belongs to some term of the main and auxiliary filtrations but not to previous terms, then the part of its boundary, belonging to the same term of the main filtration and the previous term of the auxiliary one, consists of blocks, obtained by merging together pairs of points, lying adjacently on the same component of $C_k$ but belonging to different groups of the $A$-configuration. Such a *pair-merging map* induces the map of homology groups of the blocks (also called pair-merging map).

**Proposition 5** (See [V93, section 6.4]). *The differential*

$$d_1^{-s;-p,q} : G_1^{-s;-p,q} \to G_1^{-s;-p+1,q}$$

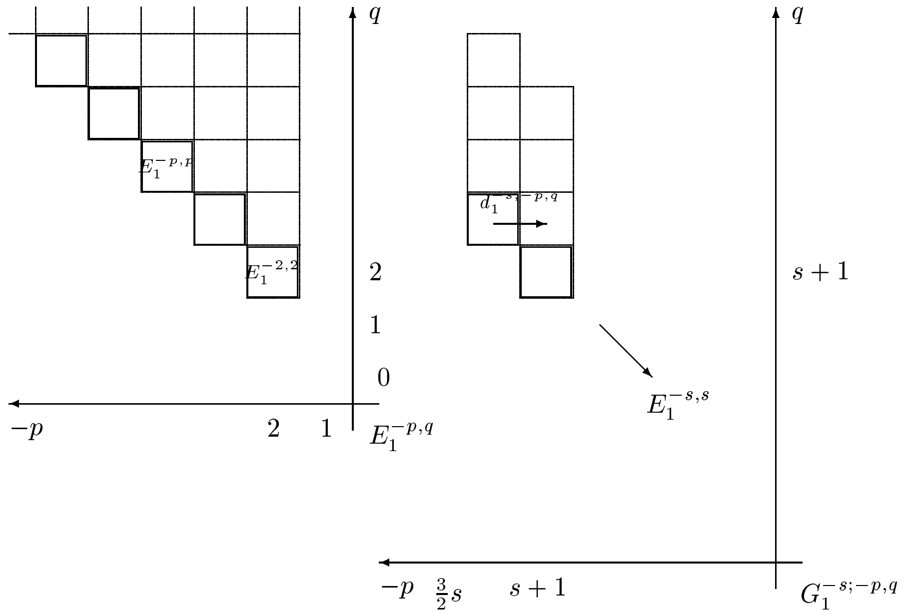



FIGURE 7. Main and auxiliary spectral sequences.
Only framed cells can be non-trivial.

*is the direct sum of all possible pair-merging maps*

$$H_{2s-1+p-q}(\sigma^{(c)}_{s,p}, \sigma^{(c)}_{s,p-1}) \to H_{2s-2+p-q}(\sigma^{(c)}_{s,p-1}, \sigma^{(c)}_{s,p-2}),$$

*or, equally,*

$$\bar{H}_{2s-1+p-q}(\sigma^{(c)}_{s,p}/\sigma^{(c)}_{s,p-1}) \to \bar{H}_{2s-2+p-q}(\sigma^{(c)}_{s,p-1}/\sigma^{(c)}_{s,p-2}).$$

The most interesting case for calculation of invariants is when $p = q$: in this case the highest closed homology group $\bar{H}_{2s-1}(\sigma^{(c)}_{s,p}/\sigma^{(c)}_{s,p-2})$ maps to the next-to-highest group $\bar{H}_{2s-2}(\sigma^{(c)}_{s,p-1}/\sigma^{(c)}_{s,p-2})$.

3.7. **Calculation of finite order invariants.** The results of the previous subsections suggest the following way of calculation of finite order invariants.

1. To calculate homology of the quotient complexes of configurations.
2. To calculate closed homology of $J$-blocks. Each $J$-block is a fiber bundle, its base is a rather simple manifold, and the fiber is a quotient complex. Hence the first terms $G_1^{-s;-p,q}$ of the auxiliary spectral sequences will be calculated.
3. To calculate the auxiliary spectral sequences, i.e. to calculate closed homology of $\sigma^{(c)}_s/\sigma^{(c)}_{s-1}$. Hence the first terms $E_1^{-s;-p,q}$ of the main spectral sequence will be calculated.
4. To calculate the main spectral sequence, i.e. to calculate closed homology of $\sigma^{(c)}_s$.

In fact, at each step one need to calculate more or less explicitly the generators of the groups for subsequent use. On the other hand, only the highest and next-to-highest homology groups participate in calculation of the diagonal cells of the spectral sequences, and only the latter give invariants.

In [M95] the first of these steps was performed thoroughly, the second one — for the highest dimension, the third one — for certain cases and for 3-ornaments only, and the forth step was not performed at all. A careful study of the first term of the main spectral sequence sometimes allowed one to guess a subgroup of invariants of the same order and the same dimension and avoid the honest calculation of the main spectral sequence. For instance, this was the way, how the uniform invariants and their correspondence to the configurations (theorems 6 and 7) were discovered.

3.7.1. *Calculation of homology of quotient complexes.* Homology groups of order complexes of partition lattices generalizing complexes $\Lambda(\Theta)$ (remind that $\Xi(\Theta)$ is the suspension of $\Lambda(\Theta)$) were investigated in [BW92]. It was shown that they are free Abelian groups and a recurrent formula for their dimensions was obtained. The combinatorial meaning of these dimensions and the generators of homology groups are shown in propositions 6 and 7.

The simplest is the case, corresponding to 3-ornaments, where the abstract $A$-configuration $\Theta$ is split into exactly three subsets $\theta_1$, $\theta_2$, and $\theta_3$ of points of different colors. We will call such $A$-configurations *$A^3$-configurations.*

The most interesting is the case of the homology groups of the highest possible dimension. In fact, only highest and next-to-highest homology groups are used in further calculations.

**Proposition 6** (See [M95, Propositions 11, 21 and 22])**.**
(i) *If $\Theta$ is an $A^3$-configuration then the highest homology group of its quotient complex $H_{|\Theta|-2\#\Theta-1}(\Xi(\Theta))$ is one-dimensional.*
(ii) *If $\Theta$ is an $A$-configuration of one group of $m = |\Theta|$ points of $k \geq 3$ colors, then* $\dim H_{m-3}(\Xi(\Theta)) = \binom{k-1}{2}$.
(iii) *If $\Theta$ is any $A$-configuration, $\theta^1, \dots, \theta^{\#\Theta}$ are its groups, and $k^i \geq 3$ is the number of colors of the points of $\theta^i$, then*

$$\dim H_{m-3}(\Xi(\Theta)) = \prod_{i=1}^{\#\Theta} \binom{k^i - 1}{2}.$$

The explicit formulas for the generators of these groups are omitted here. Given an $A^3$-configuration $\Theta$, denote the generator of $H_{|\Theta|-2\#\Theta-1}(\Xi(\Theta))$ by $\xi(\Theta)$.

**Proposition 7** (See [M95, Propositions 12 and 24])**.**
(i) *The dimension of any homology group of the quotient complex of an $A^3$-configuration $\Theta$ $H_{|\Theta|-2\#\Theta-p}(\Xi(\Theta))$, $p > 1$ is equal to the number of partitions of $\Theta$ into $p$ subconfigurations $\Theta_1, \dots, \Theta_p$. To each such partition the generator $\Sigma^{p-1} \otimes_{j=1}^{p} \xi(\Theta_j)$ corresponds.*
(ii) *The dimension of any homology group $H_{|\Theta|-2\#\Theta-p}(\Xi(\Theta))$, $p > 1$, of an $A$-configuration $\Theta$ is equal to the sum over all partitions of $\Theta$ into $p$ subconfigurations*

$\Theta_1, \dots, \Theta_p$ *of dimensions of their highest homology groups:*

$$\dim H_{|\Theta|-2\#\Theta-p}(\Xi(\Theta)) = \sum_{\Theta_1,\dots,\Theta_p} \sum_{j=1}^{p} \dim H_{|\Theta_j|-2\#\Theta_j-1}(\Xi(\Theta_j))$$

*To each such partition the generators* $\Sigma^{p-1} \otimes_{j=1}^{p} \xi_{i_j}(\Theta_j)$ *correspond, where* $\xi_{i_j}(\Theta_j)$ *are the generators of* $H_{|\Theta_j|-2\#\Theta_j-1}(\Xi(\Theta_j))$.

$\Sigma^n$ stands for $n$-fold suspension here.

3.7.2. *Calculation of the highest homology groups of $J$-blocks.*

**Proposition 8** (See [M95, Proposition 14]). *The dimension of the highest closed homology group of a $J$-block of an $A$-configuration $J$ is equal to the dimension of the highest homology group of $\Xi(J)$. The dimension of the highest closed homology group of the $J$-block of a one-group $A$-configuration $J$ of $n$ colors is equal to $\binom{n-1}{2}$. The dimension of the highest closed homology group of the $J$-block of an $m$-group $A$-configuration $J$with groups of $n^1, \dots, n^m$ colors is equal to $\prod_{j=1}^{m} \binom{n^j-1}{2}$. For instance, the highest closed homology groups of $J$-blocks of $A^3$-configurations are one-dimensional.*

**Remark.** The numbers $k^j$ of colors in the proposition above might be less than the total number $k$ of colors of the ornament: $3 \le k^j \le k$.

3.7.3. *Calculation of the auxiliary spectral sequences.*

**Proposition 9** (See [M95, Proposition 15]). *For each two-group $A^3$-configuration $J$ of complexity $s$ the algebraic boundary of $J$-block $E^J$ in $\sigma_s^{(c)}/\sigma_{s-1}^{(c)}$ is zero; or equally the differential $d_1^{-s;-p,p}$ of the auxiliary spectral sequence (see proposition 5) vanishes on them.*

The same is obvious for one-group $J$-blocks. The examples in [M95] show that this is not true for $J$-blocks of 'generic' configurations with more than two groups.

On the other hand, the simplest three-group configurations (9 points split into three groups three points each, the complexity equals to 6) form four different $J$-blocks, and $d_r^{-6;-9,9} = 0$ for any $r$ [M95, Appendix B]. Hence these $J$-blocks give four-dimensional subgroup of $E_1^{-6,6}$. This is the only subgroup of $E_1^{-p,p}$, $p \le 6$, for which not enough base invariants of order $p$ (namely, 1 instead of 4) are found yet.

## 4. Finite order invariants of doodles, knots and links versus invariants of ornaments

This section is much less formal than the previous sections: otherwise it would have been too long.

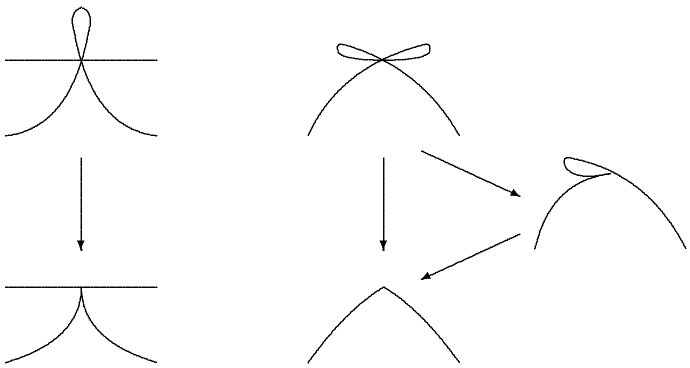


FIGURE 8. When the small loops are contracted, the quasidoodles tend to doodles.

### 4.1. **Invariants of doodles.**

4.1.1. *Elementary theory.* The only refinement, needed to apply the elementary theory of finite order invariants of ornaments to doodles, is the extension of the definition of the characteristic numbers to selfintersection points. Namely, it should be defined, which of the perturbations of a triple point, belonging to only one or two components of the doodle, are positive, and which are negative. If $x$ is a selfintersection of the $j$-th component and the image of the $l$-th component passes through $x$, then the positive quasidoodle is the one with greater value of $\mathrm{ind}_l(x)$. If $x$ is a triple selfintersection of one component, the routing along the component in its positive direction induces cyclic order of the edges of the little 'vanishing' triangle. Routing the triangle in this order induces the direction of each edge, which may coincide or not with its native direction (as a part of the oriented curve). Positive quasidoodles are those with even number of edges, for which the two directions coincide (see [A93]).

Of course, each invariant of ornaments is an invariant of doodles of the same order. Many new finite order invariants of doodles are introduced in [M98] and shown in section 5.2. They are enough to classify doodles.

4.1.2. *Homological theory.* The homological theory of doodles is more complicated than the corresponding theory for ornaments. The discriminant is not closed in the space of doodles (see fig. 8), so the Alexander duality cannot be applied. The closure of the discriminant should be considered instead of the discriminant itself. This means, that not only triple points, but also their degenerations — intersection points with vanishing first derivative of at least one component and points with vanishing both first and second derivatives — are forbidden. The complexity of each of these singularities is equal to 2, and the complexity of more complicated singularities can be defined inductively. In terms of configurations this means, that each point is marked as either simple, double or triple point, when there is no restriction on the derivative at this point, the first derivative vanishes, or the first

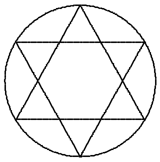


FIGURE 9. The configuration, generating the invariants of single-component doodles of order 4: two interlaced groups of three points each.

two derivatives vanish, respectively. As for ornaments, the complexity of a configuration is one-half the codimension of the subspace of quasidoodles, respecting this configuration. The resolution of the closure of the discriminant is defined exactly like for ornaments. Again, the complexity defines the main filtration of the resolution, and the number of geometrically distinct points of the configuration defines the auxiliary filtration of each stratum. Then the main and auxiliary spectral sequences can be written, and so on.

Only very first steps has been undertaken in this theory. In [V98] the invariants of single-component doodles of orders $\leq 4$ are calculated. There are no invariants of orders $< 4$, and the group of invariants of order 4 is one-dimensional and generated by the configuration, shown in fig. 9. More interesting results are obtained in [V98] for triple point free immersions.

4.2. **Invariants of knots and links.** In this section 'link' means 'link or knot' unless knots are mentioned explicitly. The sense of prefix 'quasi', verb 'respect', the regularity assumption and so on are the same as that for ornaments and doodles.

4.2.1. *Elementary theory.* Homotopy classification of links is similar to the classification of ornaments, and isotopy classification of knots and links is similar to the classification of single- and multi-component doodles, respectively. The definitions of configurations, their complexity, degeneration modes, characteristic numbers and the order of invariants can be transferred to knots and links as follows. The definition of configurations, their signatures and degeneration modes is repeated word by word with the number 3 (triple points are forbidden for ornaments and doodles) changed to 2 (double points are forbidden). The complexity $\mathrm{compl}(A)$ of an $A$-configuration is equal to one-third of the codimension of the space of quasilinks respecting the configuration and remains equal to $|A| - \#A$. The degree $\deg(A)$, equal to the number of degeneration steps, becomes also equal to $|A| - \#A$. The characteristic numbers are defined by the same inductive procedure, but the rules, which perturbation of a quasilink is positive, and which is negative, differ.

$A$-configurations with $a^1 = a^2 = \cdots = a^m = 2$ are called *chord diagrams* and depicted traditionally by the set of chords, connecting the points within each of $m$ groups of the configuration. For chord diagrams and quasilinks respecting them the positive and negative perturbations are defined in fig. 10. Then the characteristic numbers are obviously independent of the order of marking the intersection points,

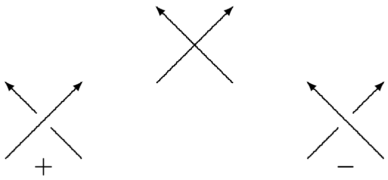


FIGURE 10. Positive and negative local perturbations of a quasiknot (quasilink).

and may be thought about as the single characteristic number. The choice of positive and negative perturbations for quasiornaments with triple and other multiple points is complicated. The tradition is to ignore the corresponding complicated configurations at all, and define the order of invariants via their characteristic numbers on chord diagrams only. This tradition is justified by the following theorem.

**Theorem 9.** *The characteristic numbers, assigned by a finite order invariant of links to regular quasilinks with no triple points, determine the invariant.*

It follows from the algorithm of calculation of the value of a finite order invariant, given in [V90], and a simple fact, that for any generic homotopy in the space of quasilinks, strictly respecting a chord diagram, no other singularities, than one additional intersection point, appear at any instant.

**Example.** Very few finite order invariants of links can be presented as explicitly, as invariants $V_{i,j}(\beta)$ of ornaments. Single-group and uniform invariants can be defined for links by definitions 8 and 9. All single-group invariants are spanned by linking numbers of pairs of components of the link. All uniform invariants are polynomials in these linking numbers.

4.2.2. *Homological theory.* The resolution of the discriminant and its main and auxiliary filtrations can be constructed for links exactly like for ornaments[1]. Like in the case of doodles, in the problem of isotopic classification of links the discriminant is not closed, and its closure should be considered instead. The boundary of the discriminant is formed by non-immersions. The configurations, describing the filtrations of the closure of the discriminant, besides groups of two or more points may contain 'double' points (within or out of the groups), where the first derivative vanishes.

Almost all necessary calculations of finite order invariants — much more than for ornaments — were performed for (long) knots in [V90] and could be easily modified for links. Avoiding technical details, exact formulas, numbers and theorems, and

[1]In the original paper [V93] and in the consequent editions (e.g. [V94]) another resolution of the discriminant is used, but both the resolutions are homotopy equivalent and the homotopy equivalence preserves the filtrations. Moreover, for the sake of simplicity, finite order invariants of long knots (embeddings $\mathbf{R}^1 \to \mathbf{R}^3$, coinciding with the identical embedding of a fixed line near infinity), are considered in [V93] instead of knot invariants. It is easy to see, that the spaces of invariants of knots, long knots, and knots in $\mathbf{S}^3$ are naturally isomorphic, but neither for higher cohomology groups, nor for invariants of links this trick works.

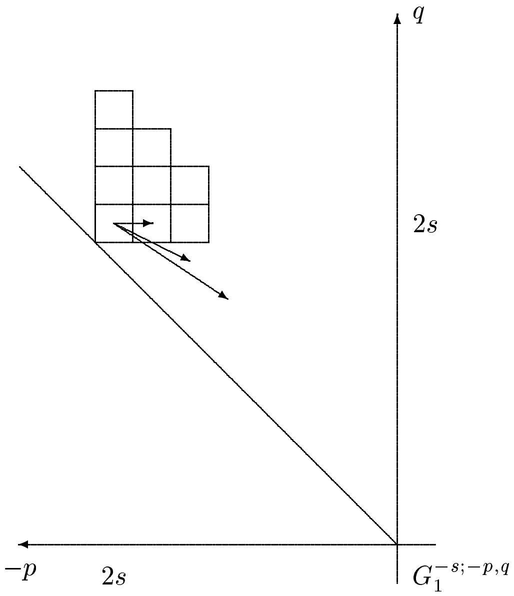



FIGURE 11. The support of an auxiliary spectral sequence for knots or links.

'rewriting the history' — imagining the classical work [V90] written in more modern technique — we describe the skeleton of these calculations.

**Calculation of homology groups $H_*(\Xi(\Theta))$ of the quotient order complexes of configurations.** The exact answer was obtained in [V90] in equivalent terms of homology of complexes of connected graphs. The follows features of the answer matter:

- all reduced homology groups are trivial except the highest $H_{\deg(\Theta)-1}(\Xi(\Theta))$;
- it $\Theta$ is a chord diagram, then $H_{\deg(\Theta)-1}(\Xi(\Theta)) = \mathbf{Z}$.

**Calculation of closed homology groups of $J$-blocks and of the first term $G_1^{-s;-p,q}$ of the auxiliary spectral sequences.** The support of the first term of each auxiliary spectral sequence is 'cut from below' as shown in fig. 11[2]. Only its left-down cell $G_1^{-s;-2s,2s}$ of $G_1^{-s;-p,q}$ lies on the main diagonal, and only it can give candidates to be invariants. This cell is the direct sum of the highest closed homology groups of all $J$-blocks, generated by chord diagrams of complexity $s$ (i.e. diagrams of $s$ chords). Each of the summands is isomorphic to $H_{s-1}(\Xi(J)) = \mathbf{Z}$, and the generator of this subgroup can be identified with the chord diagram.

**Calculation of the main diagonal $G_\infty^{-s;-p,p}$ of the auxiliary spectral sequences, and of the main diagonal $E_1^{-s,s}$ of the main spectral sequence.** Only the first differential $d_1^{-s;-2s,2s}$ can be non-trivial. It was calculated in [V90]. A chord diagram can degenerate in two ways: either two ends of two chords merge

[2]For long knots it consists of its bottom row only.

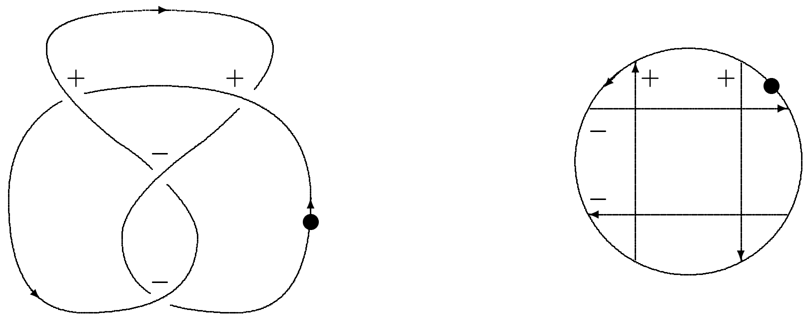



FIGURE 12. A diagram of the 'figure eight' knot and its Gauss diagram.

together and a group of three point is formed, or (in the isotopy theory only) two ends of the same chord merge together and a double point appears. The boundary of the $J$-block, generated by a chord diagram with $s$ chords, consists of $J$-blocks, generated by diagrams with either $s-2$ chords and one three-point group, or $s-1$ chords and one double point, respectively. Each of these boundary blocks produce a well-known 4-term or 1-term relation on the generators of $G_1^{-s;-2s,2s}$. The quotient group is $E_1^{-s,s}$.

**Calculation of the main diagonal $E_\infty^{-s,s}$ of the main spectral sequence.** All differentials are trivial and $E_\infty^{-s,s} = E_1^{-s,s}$ [K93].

This is the way, how the study of finite order invariants of links and knots was reduced to the study of chord diagrams modulo 4-term or 1-term relations. An alternative way, based on reverse auxiliary filtrations and leading to another formalism, is described in [V97].

## 5. DIAGRAM INVARIANTS

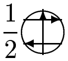
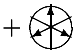
5.1. **Gauss diagram formulas for knots and links (after** [PV94]**).** Knots are usually depicted by regular projections onto planes — diagrams — with over- and underpasses marked. Each of the over- and underpasses can be marked with '+' or '−' sign according to fig. 10. A diagram is a generic immersion of a circle to a plane with finitely many transversal selfintersections. Such an immersion is characterized by its *Gauss diagram*: the circle with two pre-images of each selfintersection connected with a chord. Each chord may be oriented from the upper branch to the lower one and supplied with the sign defined above (see fig. 12 for an example).

An *arrow diagram* is an oriented circle with several arrows, connecting its points. It is not necessarily a Gauss diagram of any curve. A *representation* $\mu$ of an arrow diagram $X$ in another arrow diagram $Y$ is a homotopy class of embeddings $X \to Y$, mapping the circle onto the circle, arrows to arrows, and preserving all the orientations. For a representation $\mu : X \to G(\phi)$ in a Gauss diagram of a knot $\phi$ we define its sign $\sigma(\mu) = \prod_x \sigma(\mu(x))$ as the product over all arrows of $X$ of the signs of their images. Denote by $\langle X, G(\phi)\rangle$ the sum

$$\langle X, G(\phi)\rangle = \sum_{\mu: X \to G(\phi)} \sigma(\mu). \tag{1}$$

A knot have many non-isomorphic Gauss diagrams, but sometimes the sum (1) or a linear combination $\alpha_1\langle X_1, G(\phi)\rangle + \cdots + \alpha_m\langle X_m, G(\phi)\rangle$ of such sums is an invariant of knots. Then it is denoted by $\langle \alpha_1 X_1 + \cdots + \alpha_m X_m, \phi\rangle$.

**Example.** Let $v_3$ be the Vassiliev invariant of order 3, which takes values 0 on the unknot, $+1$ on the right trefoil and $-1$ on the left trefoil. then

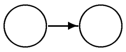
$v_3(\phi) = \Big\langle \frac{1}{2}$  $+$  $, \phi \Big\rangle.$

The definitions of representations, their signs and $\langle X, Y\rangle$ can be naturally transferred to *based arrow diagrams* (diagrams with a point marked on the circle, which must be respected by representations, but does not affect their signs) and to *multi-component arrow diagrams* (Gauss diagrams of links). Again, sometimes formula (1) gives an invariant of links or of knots with marked point, independent on this point.

**Example.** Let $v_2$ be the Vassiliev invariant of order 2, which takes values 0 on the unknot 1 on the trefoil. then

$v_2(\phi) = \Big\langle$  $, \phi \Big\rangle.$

**Example.** The linking number $v_1$ of a two-component oriented link $\phi$ (the unusual notation should remind, that the linking number is a Vassiliev invariant of order 1) satisfies the equality.

$v_1(\phi) = \Big\langle$  $, \phi \Big\rangle.$

**Remark.** Based knots, based Gauss diagrams and based arrow diagrams may be thought about as long knots and the relevant diagrams. Since classifications of long knots and of usual compact knots are equivalent, marking a point on a knot is harmless. On the contrary, based diagrams for links look suspicious.

The general relations between these diagram-generated invariants and finite order invariants are given by the following theorems.

**Theorem 10.** *If $\langle \alpha_1 X_1 + \cdots + \alpha_m X_m, \cdot\rangle$ is an invariant and each of the arrow diagrams $X_1, \ldots, X_m$ has no more than $n$ arrows, then $\langle \alpha_1 X_1 + \cdots + \alpha_m X_m, \cdot\rangle$ is an invariant of order $n$.*

**Theorem 11** (See [GPV98]). *Each finite order invariant of knots is generated by (based) arrow diagrams (i.e. can be represented as $\langle \alpha_1 X_1 + \cdots + \alpha_m X_m, \cdot\rangle$).*

**Remark.** There is no analogue of theorem 11 for links yet.

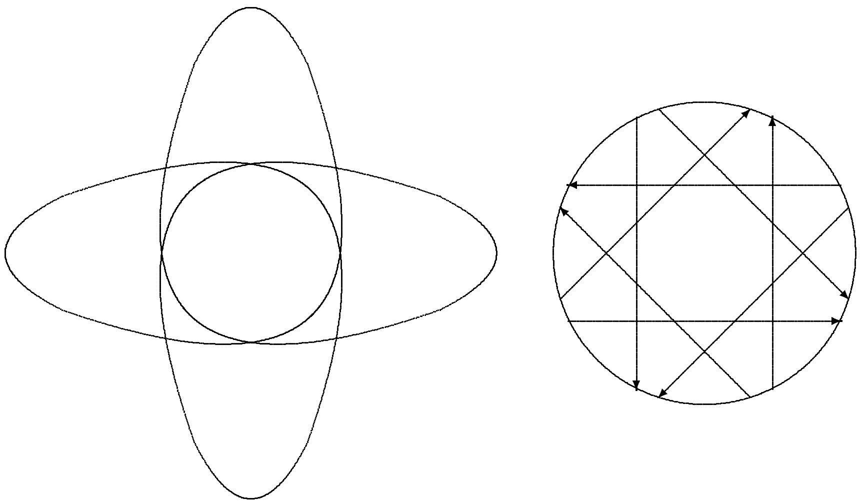


FIGURE 13. A non-trivial 1-doodle and its arrow diagram

5.2. **Gauss diagram formulas for doodles and ornaments.** Arrow diagrams and Gauss diagrams are applicable to curves, doodles and ornaments in the plane; formula (1) can be written and sometimes it gives invariants. But, preserving the general scheme, one needs to refine the definition of the orientations of the arrows, since there are no overpasses and underpasses in the plane.

In [P94] based plane curves, their Gauss diagrams and based arrow diagrams are considered. The complement of the oriented circle to the base point is an oriented line, all the arrows are directed forward and then can be thought about as chords. The sign of an arrow (chord) is defined like in fig. 10. It is $+1$ if the frame formed by the two tangent vectors in the intersection point, first of which corresponds to the arrowtail, is oriented positively, and $-1$ if negatively.

An alternative possibility is used in [M96, M98]. Each arrow of a Gauss diagram is directed that way, which provides its sign to be $+1$ (see fig. 13 for an example). So all the signs become $+1$ and can be ignored, and there is no more difference between Gauss diagrams and arrow diagrams. But the representations are now allowed to reverse the directions of arrows. The sign of a representation $\mu$ is defined as the product $\sigma(\mu) = \prod_x \sigma(\mu, x)$, where $\sigma(\mu, x) = +1$ if $\mu$ preserves the direction of $x$ and $-1$ if reverts it. Such a diagram of the ornament or doodle $\phi$ will be denoted by $\mathrm{ADiag}(\phi)$.

Series of new finite order invariants of ornaments, which are hybridization of Vassiliev invariants $V_{i,j}(\beta)$ and Polyak-Viro invariants (1), are found in [M96] for ornaments and in [M98] for doodles. For instance, they prove nontriviality of

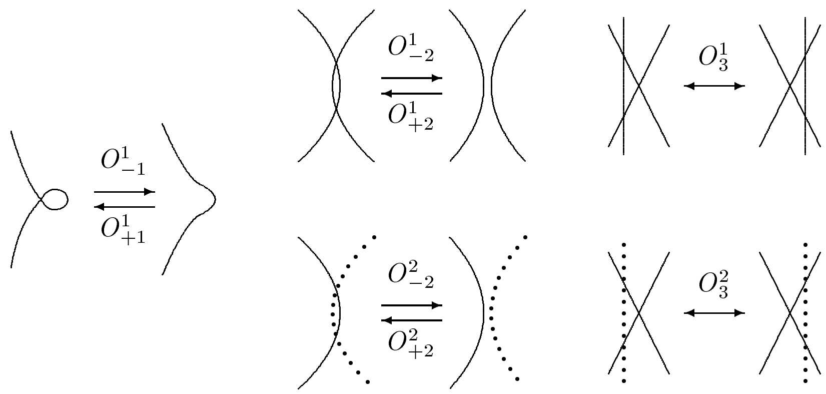



FIGURE 14. Allowed local moves for ornaments. The upper index indicates the number of the participating components, the lower index indicates the number of participating intersection points or its jump. Moves $O^*_{\pm *}$ are allowed for doodles, but moves $O^*_3$ are forbidden.

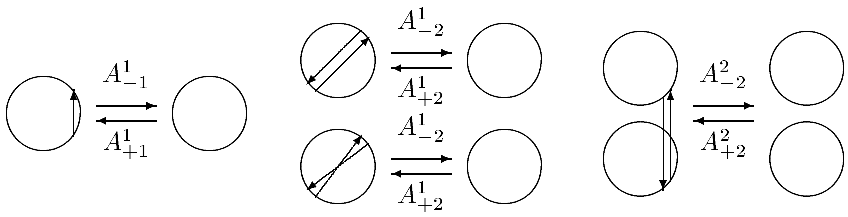



FIGURE 15. Allowed moves of arrow diagrams. Unchanged arrows and circles are not shown. The orientations of the circles are arbitrary.

ornaments shown in fig. 3 and fig. 5. For the sake of brevity we describe only the invariants of doodles, which are simpler and give more results.

5.2.1. *Reidemeister moves and minimal diagrams.* Given a generic path in the space of ornaments or doodles, almost all them along this path remain regular. At a finite number of singular instants one of the local moves shown in fig. 14 (left and middle pictures) occurs. Below we consider regular (quasi-) doodles only.

Let us define equivalence of arrow diagrams in a way, guaranteeing that the diagrams of equivalent doodles are equivalent. The equivalence is defined by the following *allowed moves*, corresponding to allowed local moves of doodles, shown in fig. 15 and 14 respectively.

$A^1_{-1}$: Deletion of an arrow of some of the circles, whose ends are neighboring (i.e. are not separated by the ends of other arrows).

$A^1_{+1}$: The move opposite to $A^1_{-1}$.

$A^1_{-2}$: Annihilation of two arrows of some of the circles, if the beginning of each arrow is a neighbor of the end of the other one.

$A^1_{+2}$**:** The move opposite to $A^1_{-2}$.

$A^2_{-2}$**:** Annihilation of two arrows connecting two circles, if the beginning of each arrow is a neighbor of the end of the other one.

$A^2_{+2}$**:** The move opposite to $A^2_{-2}$.

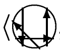
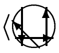
Like in fig. 14, the upper index of a move denotes the number of participating circles, and the lower one denotes the jump of the number of arrows. The equivalence class of its arrow diagram is obviously an invariant of the doodle.

**Definition 12.** *An arrow diagram is* minimal*, if no moves of types $A^*_{-*}$ are applicable to it.*

5.2.2. *Diagram-generated invariants.* A family of invariants of doodles, produced by formula (1), can be shown explicitly.

**Theorem 12.** *If $X$ is a minimal arrow diagram, then $\langle X, \cdot\rangle$ is an invariant of doodles.*

Theorem 10 remains true for doodles without any change.

**Example.** $\langle$$, \cdot\rangle$ is an invariant of order 4. If $\phi$ is the 1-doodle shown in fig. 13 (oriented arbitrarily), then $\langle$$, \phi\rangle = 8$. Cf. the very end of section 4.1.2.

5.2.3. *Generalized diagram-generated invariants.* The idea is as follows. The arrow diagram of a doodle may be also thought of as a 1-dimensional cell complex with each 1-cell oriented and $k$ oriented circles marked and numbered. There is a well-known coupling between 1-cycles and 0-chains in the plane, generated by the index $\mathrm{ind}_\phi(x)$ of a point with respect to a closed oriented curve. The doodle maps this complex to the plane and allows to pull this coupling back to the diagram and use it in formulas like (1).

To implement this idea we need to change the definition of $\mathrm{ind}_*(\cdot)$, given on page 104 to a more traditional one and extend it to piecewise-smooth curves and points on them.

**Definition 13.** *Let $\phi$ be a regular doodle, $\xi$ be a closed piecewise smooth curve, $\xi(\mathbf{S}^1) \subset \phi(C_k)$. Then the index $\mathrm{ind}_\xi(x)$ of a point $x \in \mathbf{R}^2$ is the same as defined on page 104 if $x \notin \xi(\mathbf{S}^1)$, and is the arithmetical mean of the indices of the points of two or four adjacent areas of $\mathbf{R}^2 \setminus \phi(C_k)$ if $x \in \xi(\mathbf{S}^1)$.*

The value of $\mathrm{ind}_\xi(x)$ is quarter-integer in general.

**Theorem 13.** *Let $X$ be a minimal arrow diagram with arrows $\{x^1, \dots, x^n\}$, and $\Xi = \{\xi_1, \dots, \xi_m\}$ a set of 1-cycles in $X$. If $F$ is a function of $m \times n$ variables $v^i_l$, $i = 1, \dots, n$, $l = 1, \dots, m$, actually depending only on those $v^i_l$, for which the cycle $\xi_l$ does not pass along the arrow $x^i$, then the sum*

(2)
$$\langle (X, \Xi, F), \phi\rangle = \sum_{\mu: X \to \mathrm{ADiag}(\phi)} \sigma(\mu) F(\mathrm{ind}_{\phi(\mu(\xi_1))}(\phi(\mu(x^1))), \dots, \mathrm{ind}_{\phi(\mu(\xi_m))}(\phi(\mu(x^n))))$$

*over all representations $X \to \mathrm{ADiag}(\phi)$ is an invariant of doodles.*

The assertion of minimalness of the diagram $X$ can be relaxed.

**Theorem 14.** *Let $X = \{x^1, \dots, x^n\}$ be an arrow diagram, $\Xi = \{\xi_1, \dots, \xi_m\}$ a set of 1-cycles in $X$, and $F$ a function of $m \times n$ variables $v_l^i$, $i = 1, \dots, n$, $l = 1, \dots, m$, such that*

- *$F$ depends only on those $v_l^i$, for which the cycle $\xi_l$ does not pass along the arrow $x^i$, and*
- *for any doodle $\phi$ and representation $\mu : X \to \mathrm{ADiag}(\phi)$ if $\phi \circ \mu$ maps some arrow(s) to the vertex (both vertices) of a loop or segment, vanishing at a move $O^*_{-*}$, then $F(\mathrm{ind}_{\phi(\mu(\xi_1))}(\phi(\mu(x^1))), \dots, \mathrm{ind}_{\phi(\mu(\xi_m))}(\phi(\mu(x^n)))) = 0$.*

*Then the sum (2) is an invariant of doodles.*

The proofs of theorems 12, 13 and 14 consist of direct check, that the proposed sums are invariant with respect to Reidemeister moves of doodles (fig. 14).

The analogue of theorem 10 looks like this.

**Theorem 15.** *If $X$ is an arrow diagram with $n$ arrows, $\Xi$ is a set of 1-cycles in $X$, and $F$ is a polynomial, then $\langle (X, \Xi, F), \cdot \rangle$ in theorems 13 and 14 is an invariant of order $\deg(F) + n$.*

5.2.4. *Finite order invariants classify doodles.* Theorems 14 and 15 give enough finite order invariants to prove the following theorem.

**Theorem 16.** *Non-equivalent $k$-doodles with $\leq n$ intersection points in each can be distinguished by invariants of order $6n(n+1) + 2k(n+2) - 2 + n$.*

**Remark.** There is no analogue of theorem 16 for ornaments yet.

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