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CONVERGENCE STRUCTURES AND S-ASYMPTOTIC BEHAVIOUR OF FOURIER HYPERFUNCTIONS

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Abstract. Some structural theorems for the convergence in the space of Fourier hyperfunctions are proved and applied to the S-asymptotic behaviour of elements in this space.

Preliminaries

We denote by $\mathbf{D}^n$ the compactification of $\mathbf{R}^n, \mathbf{D}^n = \mathbf{R}^n \cup S_\infty^{n-1}$ and supply it with the usual topology. The sheaves $\tilde{\mathcal{O}}$ and $\mathcal{Q}$ on $\mathbf{D}^n + i\mathbf{R}^n$ are defined as follows (cf. [**4**], [**5**]). For any open set $U \subset \mathbf{D}^n + i\mathbf{R}^n$, and $\delta \geq 0$ $\tilde{\mathcal{O}}^{-\delta}(U)(\tilde{\mathcal{O}}^0(U) = \tilde{\mathcal{O}}(U))$ consists of those elements of $\mathcal{O}(U \cap C^n)$ which satisfy $|F(z)| \leq C_{V,\varepsilon} \exp(-(\delta - \varepsilon) \times |\operatorname{Re} z|)$ uniformly for any open set $V \subset C^n$, $\bar{V} \subset U$, and for every $\varepsilon > 0$. Hence, $\tilde{\mathcal{O}}|_{C^n} = \mathcal{O}$. The derived sheaf $\mathcal{H}^n_{\mathbf{D}^n}(\tilde{\mathcal{O}})$, denoted by $\mathcal{Q}$, is called the sheaf of Fourier hyperfunctions. It is a flabby sheaf on $\mathbf{D}^n$ [**5**]. We need only the space of global sections $\mathcal{Q}(\mathbf{D}^n)$.

Let I_k, $k = 1, \dots$ be open intervals, neighbourhoods of $0 \in \mathbf{R}$, let $I = I_1 \times \cdots \times I_n$ be a convex neighbourhood of $0 \in \mathbf{R}^n$ and $U_j = \{(\mathbf{D}^n + iI) \cap \{\operatorname{Im} z_j \neq 0\}\}$, $j = 1, \dots, n$. The family $\{\mathbf{D}^n + iI,\ U_j;\ j = 1, \dots, n\}$ gives a relative Leray covering for the pair $\{\mathbf{D}^n + iI,\ (\mathbf{D}^n + iI) \setminus \mathbf{D}^n\}$ relative to the sheaf $\tilde{\mathcal{O}}$. Thus

$$\mathcal{Q}(\mathbf{D}^n) = \tilde{\mathcal{O}}((\mathbf{D}^n + iI)\#\mathbf{D}^n) \Big/ \sum_{j=1}^{n} \tilde{\mathcal{O}}((\mathbf{D}^n + iI)\#_j\mathbf{D}^n),$$

where $(\mathbf{D}^n + iI)\#\mathbf{D}^n = U_1 \cap \cdots \cap U_n$ and

$$(\mathbf{D}^n + iI)\#_j\mathbf{D}^n = U_1 \cap \cdots \cap U_{j-1} \cap U_{j+1} \cap \cdots \cap U_n.$$

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Similarly $\mathcal{Q}^{-\delta}$, $\delta > 0$, is defined using $\tilde{\mathcal{O}}^{-\delta}$ instead of $\tilde{\mathcal{O}}$ (cf. Definition 8.2.5. in [**4**]).

We shall use the notation Λ for the set of n-vectors with entry $\{-1, 1\}$; the corresponding open orthants in $\mathbf{R}^n$ will be denoted by Γ_σ, $\sigma \in \Lambda$. By Γ we denote a convex cone in $\mathbf{R}^n$.

A global section $f = [F] \in \mathcal{Q}(\mathbf{D}^n)$ is defined by $F \in \tilde{\mathcal{O}}((\mathbf{D}^n + iI)\#\mathbf{D}^n)$; $F = (F_\sigma)$, where $F_\sigma \in \tilde{\mathcal{O}}(\mathbf{D}^n + iI_\sigma)$, $\mathbf{D}^n + iI_\sigma$ is an infinitesimal wedge of type $\mathbf{R}^n + i\Gamma_\sigma 0$, $\sigma \in \Lambda$, $I_\sigma = I \cap \Gamma_\sigma$. F is the *defining function* for f.

Recall the topological structure of $\mathcal{Q}(\mathbf{D}^n)$. Let $f = [F] \in \mathcal{Q}(\mathbf{D}^n)$ and $F \in \tilde{\mathcal{O}}((\mathbf{D}^n + iI)\#\mathbf{D}^n)$. Then, a family of semi-norms is defined by $P_{K,\varepsilon}(F) = \sup\limits_{z \in \mathbf{R}^n + iK} |F(z)\exp(-\varepsilon|\operatorname{Re} z|)|$, $\varepsilon > 0$, $K \subset\subset I \setminus \{0\}$; $\tilde{\mathcal{O}}((\mathbf{D}^n + iI)\#\mathbf{D}^n)$ is a Fréchet and Montel space, as well as $\mathcal{Q}(\mathbf{D}^n)$.

Let $f = [F] \in \mathcal{Q}(\mathbf{D}^n)$. Then we associate to f, $f(x) \cong \sum\limits_{\sigma \in \Lambda} \operatorname{sgn}\sigma F_\sigma(x + i\Gamma_\sigma 0)$, $F_\sigma \in \tilde{\mathcal{O}}(\mathbf{D}^n + iI_\sigma)$ (cf. [**4**, Theorem 8.5.3 and Definition 8.3.1]).

The Fourier transform of $\mathcal{Q}(\mathbf{D}^n)$ is defined by the use of functions $\chi_\sigma = \chi_{\sigma_1} \cdots \chi_{\sigma_n}$, where $\sigma_k = \pm 1$, $k = 1, \dots, n$, $\sigma = (\sigma_1, \dots, \sigma_n)$ and $\chi_1(t) = e^t/(1+e^t)$, $\chi_{-1}(t) = 1/(1+e^t)$, $t \in \mathbf{R}$. Let $u(x) \cong \sum\limits_{\sigma \in \Lambda} U_\sigma(x + i\Gamma_\sigma 0) = \sum\limits_{\sigma \in \Lambda} \sum\limits_{\tilde{\sigma} \in \Lambda} (\chi_{\tilde{\sigma}} U_\sigma)(x + i\Gamma_\sigma 0)$, where $\chi_{\tilde{\sigma}} U_\sigma \in \mathcal{Q}(\mathbf{D}^n + iI_\sigma)$, $\sigma, \tilde{\sigma} \in \Lambda$ and decreases exponentially along the real axis outside the closed $\tilde{\sigma}$-th orthant.

The Fourier transform of u is defined by

$$\mathcal{F}(u) \cong \sum_{\sigma \in \Lambda} \sum_{\tilde{\sigma} \in \Lambda} \mathcal{F}(\chi_{\tilde{\sigma}} U_\sigma)(\xi - i\Gamma_{\tilde{\sigma}} 0)$$

$$\sum_{\sigma \in \Lambda} \sum_{\tilde{\sigma} \in \Lambda} \int\limits_{\operatorname{Im} z = y^\sigma} e^{iz\zeta} (\chi_{\tilde{\sigma}} U_\sigma)(z)\,dx,\ y^\sigma \in I_\sigma,\ \ \zeta = \xi + i\eta,$$

where $\mathcal{F}(\chi_{\tilde{\sigma}} U_\sigma) \in \tilde{\mathcal{O}}(\mathbf{D}^n - iI_{\tilde{\sigma}})$ and $|\mathcal{F}(\chi_{\tilde{\sigma}} U_\sigma)(z)| = O(e^{-w|x|})$ for a suitable $w > 0$ along the real axis outside the closed σ-orthant. $\mathcal{F}$ is an automorphism of $\mathcal{Q}(\mathbf{D}^n)$.

A continuous function v defined on $\mathbf{R}^n$ is of *infra-exponential growth* if for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that $|v(x)| \le C_\varepsilon e^{\varepsilon|x|}$, $x \in \mathbf{R}^n$. By $\|v\|_\varepsilon = \sup\limits_{x \in \mathbf{R}^n} e^{-\varepsilon|x|}|v(x)|$, $\varepsilon > 0$ we define a family of seminorms in the space of functions of infra-exponential growth. By ℓv we denote the Fourier hyperfunction defined by v.

An infinite-order differential operator $J(D) = \sum\limits_{|\alpha| \ge 0} b_\alpha D^\alpha$ with $\lim\limits_{|\alpha| \to \infty} \sqrt{[|\alpha|]|b_\alpha|\alpha!}$ $= 0$ is called a local operator. $J(D)$ acts on $\mathcal{Q}$ as a sheaf homomorphism and continuously on $\mathcal{Q}(\mathbf{D}^n)$.

We shall use the following proposition proved in [**7**] and [**8**].

Proposition 1. *Let I be a convex neighbourhood of $0 \in R^n$ and $I_\sigma = I \cap \Gamma_\sigma$, $\sigma \in \Lambda$. Let $\{f_h;\ h \in \Gamma\}$ be a family in $\mathcal{Q}(\mathbf{D}^n)$ such that $f_h \cong \sum_{\sigma\in\Lambda} G_{h,\sigma}(x + i\Gamma_\sigma 0)$, where $G_{h,\sigma} \in \tilde{\mathcal{O}}(\mathbf{D}^n + iI_\sigma)$, $h \in \Gamma$, $\sigma \in \Lambda$.*

A necessary and sufficient condition that f_h converges in $\mathcal{Q}(\mathbf{D}^n)$ to $f \cong \sum_{\sigma\in\Lambda} G_\sigma(x + i\Gamma_\sigma 0)$ as $\|h\| \to \infty$, $h \in \Gamma$ is the existence of families $\{F_{h,\sigma}; h \in \Gamma\} \subset \tilde{\mathcal{O}}(\mathbf{D}^n + iI_\sigma)$, $\sigma \in \Lambda$, such that

1) $F_h = (F_{h,\sigma})$ belongs to the same class as $G_h = (G_{h,\sigma})$, $h \in \Gamma$;

2) For every $\sigma \in \Lambda$, $F_{h,\sigma}$ converges to F_σ in $\tilde{\mathcal{O}}(\mathbf{D}^n + iI_\sigma)$ as $\|h\| \to \infty$, $h \in \Gamma$, where $F = (F_\sigma)$ belongs to the same class as $G = (G_\sigma)$.

Convergence structure for Fourier hyperfunctions

We shall prove some structural theorems for the convergence in the space of Fourier hyperfunctions $\mathcal{Q}(\mathbf{D}^n)$. First we shall give a modified form to Theorem 1.3 in [**3**]. We give the complete proof of it although we shall use the same method of the proof as in the theorem mentioned above.

Let $\mathcal{B}[K]$ be the set of hyperfunctions with supports in the compact set $K \subset \mathbf{R}^n$.

Proposition 2. *Let K' be a compact set in $\mathbf{R}^n$ and K the convex hull of K'. Then for every $u = [U] \in \mathcal{B}[K]$ there exist an elliptic local operator $J_0(D)$ and a function $v \in \mathbf{C}^\infty \cap \mathbf{L}^2$ with the properties:*

a) $|v(x)| \le C_\varepsilon e^{-(1-\varepsilon)|x|}$, $x \in \mathbf{R}^n$, for every $\varepsilon > 0$ and $\ell v \in \mathcal{Q}^{-1}(\mathbf{D}^n)$

b) $v(x) = (u * \ell g)(x)$, where the function g has the same property a) as the function v.

c) $u = J_0(D)v$.

Proof. The Fourier transform $\hat{u}$ of u is an entire function and satisfies the following growth condition

$$|u(\zeta)| \le C \exp\Big(\frac{|\zeta|}{\varphi(|\zeta|)} + H_K(\operatorname{Im}\zeta)\Big),$$

where $\varphi(r)$ is a monotone increasing function of $r > 0$ and satisfies: $\varphi(0) = 1$, $\varphi(r) \to \infty$ when $r \to \infty$; $H_K(\eta) = \sup_{x\in K} x\eta$. (cf. Lemma 1.1 in [**3**]). We take an elliptic local operator $J(D)$ which corresponds to the chosen φ as it is done in Lemma 1.2 in [**3**]. Now we can consider $\hat{u}$ as a Fourier hyperfunction with defining function $U = (\hat{u}(\zeta), 0, \dots, 0)$. The corresponding function to $J(D)$, $J(\zeta)$ is an entire function, $|J(\zeta)| \le C_\varepsilon e^{\varepsilon|\zeta|}$ for every $\varepsilon > 0$. Also, for any prescribed positive constants A, C, we have

$$J(\zeta) \ge C \exp(A|\zeta|/\varphi(|\zeta|)) \text{ for } |\eta| \le 1$$

or

$$|J^{-2}(\zeta)| \le C^{-2}\exp(-2A|\zeta|/\varphi(|\zeta|)),\ |\eta| \le 1.$$

We can take without loss of generality that $|\zeta|/\varphi(|\zeta|) \ge c|\zeta|^{\gamma}$ with some $c > 0$ and $0 < \gamma < 1$. We have now the following estimate:

$$|\hat{u}(\zeta)/J^2(\zeta)| \le C\exp(-c|\zeta|^{\gamma}),\ \ |\eta| \le 1. \tag{1}$$

Let $V = \hat{u}/J^2$, then $\hat{u} = J^2V$ and $u = J^2(D)\mathcal{F}^{-1}(V)$. Since $\hat{u}$ is also an entire function, $\hat{u}/J^2$ belongs to $\mathcal{O}(\mathbf{D}^n + i\{|\eta| < 1\})$. By Theorem 8.2.6 in [**4**], $\mathcal{F}^{-1}(V) \in \mathcal{Q}^{-1}(\mathbf{D}^n)$. Consequently, $v = \mathcal{F}^{-1}(V) \in \mathcal{Q}^{-1}(\mathbf{D}^n)$.

Now we prove that v is rapidly decreasing. For V we know that $V = \hat{u}/J^2 = (\hat{u}(\zeta)/J^2(\zeta), 0, \dots,)$. Then

$$\mathcal{F}^{-1}(V) \simeq \sum_{\sigma\in\Lambda} \mathcal{F}^{-1}(\chi_\sigma(\zeta)\hat{u}(\zeta)/J^2(\zeta)) = \sum_{\sigma\in\Lambda}\mathcal{F}^{-1}(V)_\sigma(x - i\Gamma_\sigma 0)$$

Consider $\mathcal{F}^{-1}(V)_\sigma(z)$ for a $\sigma \in \Lambda$ and $z \in \mathbf{R}^n - iI_\sigma$:

$$\Big|\mathcal{F}^{-1}(V)_\sigma(z)\Big| = \frac{1}{(2\pi)^n}\Big|\int_{\mathbf{R}^n} e^{-\zeta z i}\ \frac{\hat{u}(\zeta)}{J^2(\zeta)}\ \chi_\sigma(\zeta)\ d\xi\Big|$$

$$\le \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} e^{\xi y + \eta x}\Big|\ \frac{\hat{u}(\zeta)}{J^2(\zeta)}\ \chi_\sigma(\zeta)\Big)\Big|\ d\xi,\ \ \zeta = \xi + i\eta$$

uniformly for $|\eta| \le (1-\varepsilon)$ for every $\varepsilon > 0$. Thus we can write $\eta = -(1-\varepsilon)x/|x|$ and

$$\Big|\mathcal{F}^{-1}(V)_\sigma(z)\Big| \le \frac{1}{(2\pi)^n}e^{-(1-\varepsilon)|x|}\int_{\mathbf{R}^n} e^{\xi y}\Big|\ \frac{\hat{u}(\zeta)}{J^2(\zeta)}\ \chi_\sigma(\zeta)\Big|\ d\xi,$$

where the integral is convergent since $y \in -I_\sigma$. It follows that $\mathcal{F}^{-1}(V)_\sigma(x+iy)$, $y \in -I_\sigma$, $\sigma \in \Lambda$, can be extended to real axis and that $v(x)$ is rapidly decreasing.

Furthermore we can give another analytic form to ℓv. For the holomorphic function $J^{-2}(\zeta),\ |\eta| < 1$, we can take

$$|J^{-2}(\zeta)| \le C\exp(-|\zeta|^{\gamma}),\ 0 < \gamma < 1,\ |\eta| < 1. \tag{2}$$

By Theorem 8.2.6 in [**4**], $J^{-2}(\zeta)$ is the Fourier transform of a $g \in \mathcal{Q}^{-1}(\mathbf{D}^n)$. By the same reasons as for v we have $|g(x)| \le Ce^{-(1-\varepsilon)|x|}$ for every $\varepsilon > 0,\ x \in \mathbf{R}^n$. By the Paley–Wiener–Ehrenpreis theorem, $\hat{u}(\zeta)$ is an entire function of ζ and for any $\varepsilon > 0$ and any $|\eta| \le \delta,\ \delta > 0$, there exists $C_{\varepsilon,\delta}$ such that:

$$|\hat{u}(\zeta)| \le C_{\varepsilon,\delta}e^{\varepsilon|\zeta|},\ \ \ |\eta| \le \delta. \tag{3}$$

Consequently, $\hat{u}(\zeta)$ is a slowly increasing analytic function and (cf. Theorem 8.4.3 in [**4**]), $\mathcal{F}(u * g) = \hat{u} \cdot J^{-2}$.

Since $(\hat{u} \cdot J^{-2})(\xi) \in \mathbf{L}^2$, then $(u * g)(x) \in \mathbf{L}^2$, as well. We know that $\mathcal{B}[K] \subset \mathcal{Q}^{-1}(\mathbf{D}^n)$, then by Theorem 8.2.8 in [**4**], $u * g \in \mathcal{Q}^{-1}(\mathbf{D}^n)$.

Now we have in correspondence to [**3**]:

1) $\ell v \in \mathcal{Q}^{-1}(\mathbf{D}^n)$ is defined by the function v belonging $\mathbf{L}^2 \cap \mathbf{C}^\infty$ (cf. [**3**]).

2) $v(x) = (u * \ell g)(x)$, where $\ell g \in \mathcal{Q}^{-1}(\mathbf{D}^n)$, $|g(x)| \leq Ce^{-(1-\varepsilon)|x|}$, for every $\varepsilon > 0$, $x \in \mathbf{R}^n$.

3) $|v(x)| \leq Ce^{-(1-\varepsilon)|x|}$, for every $\varepsilon > 0$, where $C > 0$. □

COROLLARIES OF PROPOSITION 2. 1) *If we take in Proposition* 2, $u = \delta$, *and* $v = q$, *then*

$$\delta = J_0(D)\ell q, \tag{4}$$

where J_0 *and* q *have the following properties:*

$J_0(D) = J(D)J(D)$, $J(D)$ *is an elliptic local operator constructed by any monotone increasing function* $\varphi(r)$, *of* $r > 0$ *with the properties:* $\varphi(0) = 1$, $\varphi(r) \to \infty$ *(see Lemma* 1.2 *in* [**3**]*).*

$\ell q \in \mathcal{Q}^{-1}(\mathbf{D}^n)$, $q \in \mathbf{L}^2 \cap \mathbf{C}^\infty$ *and* $|q(x)| \leq Ce^{-(1-\varepsilon)|x|}$, $x \in \mathbf{R}^n$, *for every* $\varepsilon > 0$. *Also* $\mathcal{F}(q)(\zeta) = J^{-2}(\zeta)$,

Proof. In the proof of Proposition 2 we have only to take $\mathcal{F}(u) = \mathcal{F}(\delta) = 1$. Then (1) is satisfied for every J, we constructed using the function φ mentioned above, because of (2).

2) *If* $f \in \mathcal{Q}(\mathbf{D}^n)$, *then there exist an elliptic local operator* $J_0(D)$ *and a function* $v \in \mathbf{C}^\infty$ *which is of "infra exponential" growth such that* $f = J_0(D)v$, $\ell v \in \mathcal{Q}(\mathbf{D}^n)$.

Assertion 2) was first proved in [**3**] but without the property of v to be of "infra exponential" growth. The complete assertion 2) was proved in [**2**]. We shall show how it can be proved using 1).

Proof. By (4), by properties of q and Proposition 8.4.8 in [**4**]

$$f = (J_0(D)q) * f = J_0(D)(\ell q * f).$$

It remains only to show that $(\ell q * f)$ is a function v with the required properties. Because of 1) we can choose for J the same function as it is done in [**2**].

Let $(\ell q * f) = v$. By Proposition 8.4.3 in [**4**]

$$\mathcal{F}(\ell q * f)(\zeta) = \hat{v}(\zeta) = J^{-2}(\zeta)\mathcal{F}(f)(\zeta) = \sum_{\sigma, \tilde{\sigma} \in \Lambda} J^{-2}\mathcal{F}(\chi_{\tilde{\sigma}} F_\sigma)(x - i\Gamma_{\tilde{\sigma}} 0).$$

or $\mathcal{F}^{-1}\, \mathcal{F}(\ell q * f) = v$.

Let $\eta \in I_{\tilde{\sigma}}$. We consider

$$v_{\sigma,\tilde{\sigma}}(z) = \frac{1}{(2\pi)^n} \int\limits_{Im\zeta=\eta} e^{-iz\zeta} \frac{\mathcal{F}(\chi_{\tilde{\sigma}} F_\sigma)(\zeta)}{J^2(\zeta)} d\zeta, \quad \sigma, \tilde{\sigma} \in \Lambda.$$

Then for every $\sigma, \tilde{\sigma}$

$$|v^{(p)}_{\sigma,\tilde{\sigma}}(z)| \leq C e^{x\eta} \int\limits_{\mathbf{R}^n} e^{y\zeta} e^{-|\zeta|^\gamma} |\zeta|^p d\zeta, \; z \in \mathbf{R}^n + iI_\sigma, \; p = 0, 1, \ldots$$

This implies that $v^{(p)}_{\sigma,\tilde{\sigma}}(z)$, $z \in \mathbf{R}^n + iI_\sigma$, $p = 0, 1, \ldots$ and $\sigma, \tilde{\sigma} \in \Lambda$, is continuable to a continuous function $v^{(p)}_{\sigma,\tilde{\sigma}}(x)$ up to real axis and $|v_{\sigma,\tilde{\sigma}}(x)| \leq C_\varepsilon e^{\varepsilon|x|}$ for every $\varepsilon > 0$. The function $v(x) = \sum\limits_{\sigma,\tilde{\sigma} \in \Lambda} v_{\sigma,\tilde{\sigma}}(x)$ has the required properties. □

THEOREM 1. *Let I be a convex neighbourhood of $0 \in \mathbf{R}^n$ and $I_\sigma = I \cap \Gamma_\sigma$, $\sigma \in \Lambda$. Let $\{f_h;\ h \in \Gamma\}$ be a family in $\mathcal{Q}(\mathbf{D}^n)$ such that*

$$f_h \cong \sum_{\sigma \in \Lambda} F_{\sigma,h}(x + i\Gamma_\sigma 0), \; F_{\sigma,h} \in \tilde{\mathcal{O}}(\mathbf{D}^n + iI), \; h \in \Gamma, \; \sigma \in \Lambda. \tag{5}$$

A necessary and sufficient condition that f_h converges in $\mathcal{Q}(\mathbf{D}^n)$ when $h \in \Gamma$, $\|h\| \to \infty$, is that $f_h * g$ converges in $\mathcal{Q}(\mathbf{D}^n)$, $h \in \Gamma$, $\|h\| \to \infty$, for every $g = [G] \in \mathcal{Q}^{-\delta}(\mathbf{D}^n)$ and for every $\delta > 0$. More precisely:

a) If $f_h \to u = [U]$ in $\mathcal{Q}(\mathbf{D}^n)$, $h \in \Gamma$, $\|h\| \to \infty$, then $f_h * g \to u * g$ in $\mathcal{Q}(\mathbf{D}^n)$, $h \in \Gamma$, $\|h\| \to \infty$; $g \in \mathcal{Q}^{-\delta}(\mathbf{D}^n)$, $\delta > 0$.

b) If $f_h * g \to v_g$ in $\mathcal{Q}(\mathbf{D}^n)$, $h \in \Gamma$, $\|h\| \to \infty$, for every $g \in \mathcal{Q}^{-1}(\mathbf{D}^n)$, then there exists an elliptic local operator $J_0(D)$ such that $f_h \to J_0(D)v_g$ in $\mathcal{Q}(\mathbf{D}^n)$, $h \in \Gamma$, $\|h\| \to \infty$.

Proof. a) By Proposition 1 we can suppose that $\{F_h; h \in \Gamma\}$ is such that $F_h \to F \in [U]$ in $\tilde{\mathcal{O}}((\mathbf{D}^n + iI) \neq \mathbf{D}^n)$, $h \in \Gamma$, $\|h\| \to \infty$. By Proposition 8.4.3 in [**4**] there exists $f_h * g$, $f_h * g \in \mathcal{Q}(\mathbf{D}^n)$, $h \in \Gamma$, and

$$(f_h * g)(x) = \int\limits_{\mathbf{R}^n} f_h(x - \zeta) g(\zeta) d\zeta = \sum_{\sigma,\tilde{\sigma} \in \Lambda} F_{\sigma,h}(x + i\Gamma_\sigma 0) * G_{\tilde{\sigma}}(x + i\Delta_{\tilde{\sigma}} 0)$$

where

$$F_{\sigma,h}(x + i\Gamma_\sigma 0) * G_{\tilde{\sigma}}(x + \Delta_{\tilde{\sigma}} 0) = \int\limits_{\mathbf{R}^n} F_{\sigma,h}(z - \xi - i\eta_{\tilde{\sigma}}) G_{\tilde{\sigma}}(\xi + i\eta_{\tilde{\sigma}}) d\xi. \tag{6}$$

$\eta_{\tilde{\sigma}} \in I_{\tilde{\sigma}}$, $\Delta_{\tilde{\sigma}}$ is also an open $\tilde{\sigma}$ orthant in $\mathbf{R}^n$ and z be in $\mathbf{R}^n + i(I_\sigma + I_{\tilde{\sigma}})$.

Let us consider only (6) for a fixed $\sigma,\ \tilde\sigma \in \Lambda$. We will prove that

$$\lim_{h\in\Gamma,|h|\to\infty} F_{\sigma,h}(x+i\Gamma_\sigma 0) * G_{\tilde\sigma}(x+i\Delta_{\tilde\sigma}0) = $$
$$= F_\sigma(x+i\Gamma_\sigma 0) * G_{\tilde\sigma}(x+i\Delta_{\tilde\sigma}0). \text{ in } \mathcal{O}(\mathbf{D}^n + iI_\sigma).$$

Let $\varepsilon < \delta,\ K_\sigma$ be a compact set in I_σ and let $\eta_{\tilde\sigma}$ be chosen such that $y - \eta_{\tilde\sigma} \in I_\sigma$ when $y \in K_\sigma$. Suppose that for $\omega > 0$ we have chosen h_0 so that $P_{K_\sigma,\varepsilon}(F_{\sigma,h} - F_\sigma) < \omega$, $|h| \ge |h_0|$, $h \in \Gamma$. Then

$$\sup_{z\in\mathbf{R}^n+iK_\sigma} e^{-\varepsilon|x|}\left|\int_{\mathbf{R}^n} F_{\sigma,h}(z-\xi-i\eta_{\tilde\sigma})G_{\tilde\sigma}(\xi+i\eta_{\tilde\sigma})d\xi - \int_{\mathbf{R}^n} F_\sigma(z-\xi-i\eta_{\tilde\sigma})G_{\tilde\sigma}(\xi+i\eta_{\tilde\sigma})d\xi\right|$$
$$\le \int_{\mathbf{R}^n} \sup_{z\in\mathbf{D}^n+iK_\sigma} e^{-\varepsilon|x|-\varepsilon|\xi|}\Big|F_{\sigma,h}(z-\xi-i\eta) - F_\sigma(x-\xi-i\eta_{\tilde\sigma})\Big| e^{\varepsilon|\xi|}\big|G(\xi+i\eta_{\tilde\sigma})\big| d\xi$$
$$\le \omega \int_{\mathbf{R}^n} e^{-(\delta-\varepsilon)|\xi|} d\xi, \qquad |h| \ge |h_0|, \quad h \in \Gamma.$$

b) By Corollary 1) of Proposition 2 there exists a $g_0 \in \mathcal{Q}^{-1}(\mathbf{D}^n)$ such that

$$\delta = J_0(D)g_0,\ \ g_0 \in \mathcal{Q}^{-1}(\mathbf{D}^n).$$

Consequently

$$f_h = (J_0(D)g_0) * f_h = J_0(D)(g_0 * f_h).$$

Since $J_0(D)$ acts continuously on $\mathcal{Q}(\mathbf{D}^n)$, $f_h \to J_0(D)v_g$ in $\mathcal{Q}(\mathbf{D}^n)$, $h\in\Gamma$, $|h| \to \infty$.

Applications. S-Asymptotics of Fourier hyperfunctions

We shall apply the convergence structures that we proved to the asymptotic behaviour of Fourier hyperfunctions. There are many definitions of asymptotic behaviour of generalized functions. We mention two of them, which are most used: the quasi-asymptotics and S-asymptotics.

S-asymptotics was defined for distributions [**14**] for ultradistributions [**9, 10**] and for some other generalized functions [**17**]. It was applied in the quantum field theory [**1**], for Abelian and Tauberian type theorems, for solutions of partial differential equations,... [**11**]–[**14**]. It is easy to extend it to Fourier hyperfunctions [**18**].

In fact, the S-asymptotics of Fourier hyperfunctions extends the S-asymptotics of tempered distributions, because we have the continuous inclusion **S'** $\hookrightarrow \mathcal{Q}(\mathbf{D}^n)$ (see [**3**]).

Definition 1. Suppose that c is a positive continuous function defined on $\mathbf{R}^n$ and $f \in \mathcal{Q}(\mathbf{D}^n)$. f is said to have the S-asymptotics related to c in the cone $\Gamma \in \mathbf{R}^n$ if there exists

$$\lim_{h\in\Gamma,|h|\to\infty} \frac{f(\cdot+h)}{c(h)} = u \ \text{ in } \mathcal{Q}(\mathbf{D}^n), \quad u \ne 0.$$

For short, $f(x+h) \overset{s}{\sim} c(h) \cdot u(x)$, $h \in \Gamma$, $|h| \to \infty$, in $\mathcal{Q}(\mathbf{D}^n)$. We shall prove two simple properties of the S-asymptotics.

PROPOSITION 3. a) *Let $P(D)$ be a local operator and $f \in \mathcal{Q}(\mathbf{D}^n)$. If $f(x+h) \overset{s}{\sim} c(h) \cdot u(x)$, $h \in \Gamma$, $|h| \to \infty$ in $\mathcal{Q}(\mathbf{D}^n)$, then $P(D)f(x+h) \overset{s}{\sim} c(h) \cdot P(D)u(x)$, $h \in \Gamma$, $|h| \to \infty$ in $\mathcal{Q}(\mathbf{D}^n)$, as well.*

b) *Let* $\operatorname{supp} f \subset K$, *where K is a compact set in $\mathbf{R}^n$. Then $f \in \mathcal{Q}(\mathbf{D}^n)$ and $f(x+h) \overset{s}{\sim} c(h) \cdot 0$, $h \in \Gamma$, $|h| \to \infty$ in $\mathcal{Q}(\mathbf{D}^n)$ for every positive function c and every cone Γ.*

Proof. a) This property follows from the property of a local operator; $P(D)$ maps $\mathcal{Q}(\mathbf{D}^n)$ into $\mathcal{Q}(\mathbf{D}^n)$ and this mapping is continuous.

b) Let $\operatorname{supp} f \subset K$, where $K \subset \mathbf{R}^n$ is a compact set. We take γ_0 large enough and such that $h + K \subset \mathbf{R}^n \setminus K$, $|h| \geq \gamma_0$. By definition of the support of a hyperfunction, $f(x+h) = 0$, $|h| \geq \gamma_0$. $\square$

The next examples show that Definition 1 is not a trivial extension of the S-asymptotics of distributions. Let $P(D)$ be a local operator

$$\sum_{|\alpha| \geq 0} b_\alpha D^\alpha, \quad b_\alpha \neq 0.$$

The Fourier hyperfunction $f = 1 + P(D)\delta$ has the S-asymptotics related to $c = 1$ in any cone Γ and with the limit $u = 1$ but f is not a distribution. For the S-asymptotics of f it is enough to prove that

$$\lim_{h \in \Gamma, |h| \to \infty} P(D)\delta(x+h) = 0 \text{ in } \mathcal{Q}(\mathbf{D}^n).$$

Since $P(D)$ maps continuously $\mathcal{Q}(\mathbf{D}^n)$ into $\mathcal{Q}(\mathbf{D}^n)$, by Proposition 3. b) the above limit follows.

Since $P(D)\delta = \sum_{|\alpha| \geq 0} b_\alpha D^\alpha \delta$ is a distribution if and only if $b_\alpha \neq 0$ for a finite number of α, the Fourier hyperfunction $1 + P(D)\delta$ is not a distribution, but it has the S-asymptotics related to $c = 1$.

We can also find the coefficients b_α of the local operator $P(D)$ such that $f = 1 + P(D)\delta$ is not defined by an ultradistribution belonging to $\mathcal{D}^{(M_p)'}$ or $\mathcal{D}^{\{M_p\}'}$ (Beurling or Roumieu type) when $M_p = (p!)^s$, $s > 1$ (see [**18**]). For ultradistributions see [**6**].

A direct consequence of Theorem 1 is

THEOREM 2. *A necessary and sufficient condition that $f = [F] \in \mathcal{Q}(\mathbf{D}^n)$ has the S-asymptotics in $\mathcal{Q}(\mathbf{D}^n)$ related to c in the cone Γ is that $f * g$ has the same property, for every $g = [G] \in \mathcal{Q}^{-\delta}(\mathbf{D}^n)$, $\delta > 0$. More precisely:*

a) If $f(x+h) \overset{s}{\sim} c(h) \cdot u(x)$, $h \in \Gamma$, $|h| \to \infty$, in $\mathcal{Q}(\mathbf{D}^n)$, then $(f * g)(x+h) \overset{s}{\sim} c(h) \cdot (u * g)(x)$, $h \in \Gamma$, $|h| \to \infty$, in $\mathcal{Q}(\mathbf{D}^n)$ for every $g \in \mathcal{Q}^{-\delta}(\mathbf{D}^n)$, $\delta > 0$.

b) If $(f * g)(x+h) \overset{s}{\sim} c(h) \cdot v_g(x)$, $h \in \Gamma$, $|h| \to \infty$, *in* $\mathcal{Q}(\mathbf{D}^n)$ *for every* $g \in \mathcal{Q}^{-1}(\mathbf{D}^n)$, *then there exists an elliptic local operator* $J_0(D)$ *such that* $f(x+h) \overset{s}{\sim} c(h) \cdot J_0(D)v_g$, $h \in \Gamma$, $|h| \to \infty$, *in* $\mathcal{Q}(\mathbf{D}^n)$.

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