PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 64 (78), 1998, 85–97

MICROLOCAL PROPERTIES OF ULTRADISTRIBUTIONS. COMPOSITION AND KERNEL TYPE OPERATORS

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Communicated by Mileva Prvanović

Abstract. The composition of an ultradistribution and a real analytic function as well as kernel type operators are analyzed. The suppleness of various spaces of microfunctions is given.

0. Introduction

The microlocal analysis of various classes of functions, generalized functions and hyperfunctions has a long history. Essentially it is related to the qualitative analysis of solutions of linear and non-linear equations and because of that it is a wast and reach theory.

Hörmander has studied in [**9**, Section 8] the wave front set of distributions and Roumieu type ultradistributions while Komatsu has investigated in [**12**] the microlocal properties of sheaves C^* and C_* which correspond to Beurling and Roumieu ultradistribution spaces and the spaces of corresponding ultradifferentiable functions. We refer to Sato, Kawai, Kashiwara [**21**] for the hyperfunction theory and the theory of microfunctions.

Note that Matsuzawa has developed in [**14**] the hyperfunction theory and the microlocal analysis by mean of the Gauss kernel. In this context we refer also to papers [**4**], [**5**]. The analysis of ultradistribution spaces is given in many papers (among others are [**2**], [**11–13**], [**15**], [**18**], [**20**] and [**22**]).

In this paper we continue our investigations of [**18**] and give the assertions which are known for distributions but need to be carefully examined for ultradistributions (see also [**11**] and [**19**]). In order to make clear our contribution to the theory, we note that uniform estimations related to all the derivatives of a function are the main problems in our proofs.

AMS Subject Classification (1991): 32A40, 46F15

In Sections 1 and 2 we recall the definitions of ultradistributional singular spectrums SS_*, SS^* ([**12**], [**6**]) and the definition and the properties of tempered ultradistribution spaces $\mathcal{S}'^*$ ([**3**], [**13**], [**16**]). The composition of a $u \in \mathcal{D}'^*$ with a real analytic function with the singular spectrum disjoint of the set of normals of f is given in Section 3. Note that this composition can not be made by using stationary phase arguments as in the distribution theory [**9**]. The microlocal analysis of a linear operator with a kernel $K \in \mathcal{D}'^*(X \times Y)$ is given in Section 4. For the decomposition of a microlocal support we refer to [**1**], [**6**], and [**7**]. In Section 5 we recall our definitions of various spaces of microfunctions and the results related to the suppleness of corresponding sheaves.

1. Notation and notions

As usual, M_p, $p \in \mathbb{N}_0$, denotes a sequence of positive numbers with $M_0 = 1$.

We refer to [**11**] for the meaning of conditions (M.1), (M.2)', (M.2), (M.3)' and (M.3). Also we use the following one [**15**]:

(M.1)* $\quad M^*_{p-1}M^*_{p+1} \le M^*_p$, $p \in \mathbb{N}$, where $M^*_0 = 1$, $M^*_p = M_p/p!$, $p \in \mathbb{N}$.

Let M_p satisfy (M.1) and (M.3)'. The associated function $M(\rho)$ and the growth function $\tilde{M}(\rho)$ related to M_p are defined by

$$M(\rho) = \sup_{p\in\mathbb{N}_0} \ln \frac{\rho^p}{M_p}, \quad \tilde{M}(\rho) = \sup_{p\in\mathbb{N}_0} \ln \frac{\rho^p}{M^*_p}, \ \rho > 0.$$

An open set in $\mathbb{R}^n$ will be always denoted by Ω; $K \subset\subset \Omega$ means that K is a compact subset of Ω. Recall,

$$\|\varphi\|_{K,h,M_p} = \sup_{x\in K,\alpha\in\mathbb{N}_0^n} \frac{|\ \varphi^{(\alpha)}(x)\ |}{h^{|\alpha|}M_{|\alpha|}}, \varphi \in C^\infty(\Omega).$$

We use the symbol $*$ for both (M_p) and $\{M_p\}$. For the definitions of $\mathcal{E}^*(\Omega)$, $\mathcal{D}^*_K(\Omega)$, $\mathcal{D}^*(\Omega)$ and their strong duals we refer to [**11**]. Throughout the paper we will assume that (M.1), (M.2)' and (M.3)' hold.

Presheaves $U \to \mathcal{E}^*(U)$, $U \to \mathcal{D}'^*(U)$, $U \subset \mathbb{R}^n$, are sheaves. These sheaves are known to be soft, that is, if f is a section on a closed set it can be extended on the whole space.

There exist injections

$$\mathcal{E}^* \hookrightarrow \mathcal{B}, \quad \mathcal{D}'^* \hookrightarrow \mathcal{B}.$$

Let N_p be a sequence of positive numbers which satisfies (M.1), (M.2)', (M.3)' and $N_0 = 1$. Then [**11**]

$$(N_p) \le (M_p) \ \ (\text{resp.}, \{N_p\} \le \{M_p\})$$

if there are constants $L > 0$ and $C > 0$ (resp., for every $\epsilon > 0$ there is $C_\epsilon > 0$) such that

$$N_p \leq CL^p M_p \ \text{ (resp., } N_p \leq C_\epsilon \epsilon^p M_p), \ \ p \in \mathbb{N}_0 .$$

Let $f \in \mathcal{D}'^{\dagger}$, where $\dagger = N_p \leq M_p$. Then $(x,\omega) \in S^*\Omega = \Omega \times S^{n-1}$ (the cosphere bundle) is not in $SS_* f$ (resp., not in $SS^* f$) iff there exist a neighborhood $U \subset \Omega$ of x and a conic neighborhood Γ of ω of the form

$$\Gamma = \{\xi \neq 0; \quad | \xi / | \xi | - \omega | < \eta\}$$

such that for every $\phi \in \mathcal{D}^*(U)$ the following holds:

In the (M_p)-case, for every $\epsilon > 0$ there is $C_\epsilon > 0$ such that

$$| \widehat{\phi f}(\xi) | \leq C_\epsilon e^{-M(\epsilon|\xi|)}, \quad \xi \in \Gamma$$

(resp., there are a $k > 0$ and a $C > 0$ such that

$$| \widehat{\phi f}(\xi) | \leq C e^{M(k|\xi|)}, \quad \xi \in \Gamma.)$$

In the $\{M_p\}$-case, there exist a $k > 0$ and a $C > 0$ such that

$$| \widehat{\phi f}(\xi) | \leq C e^{-M(k|\xi|)}, \quad \xi \in \Gamma$$

(resp., for every $\epsilon > 0$, there is a $C_\epsilon > 0$ such that

$$| \widehat{\phi f}(\xi) | \leq C_\epsilon e^{M(\epsilon|\xi|)}, \quad \xi \in \Gamma.$$

Note, the notion $SS_{\{M_p\}}$ is equivalent to Hörmander's notion WF_L.

By using the above definitions of singular spectrums we can formulate definitions (and we will do this) according to which $(x,\xi) \in \Omega \times (\mathbb{R}^n \setminus \{0\})$ is an element of a singular spectrum defined above if $(x, \xi/|\xi|)$ is an element of the same singular spectrum.

We denote by $\mathcal{B}(U)$ the space of Sato's hyperfunctions on an open set U in $\mathbb{R}^n$. It is known that $U \to \mathcal{B}(U)$ forms a sheaf and that this sheaf is flabby, that is, the restriction mappings $S : \mathcal{B}(U) \to \mathcal{B}(U_1)$ $(U_1 \subset U)$ are always surjective (cf. [**21**]).

For an open set U in a real analytic manifold M, let S^*U be the cosphere bundle of U and V be the complexification of U. Then $S^*_U V$ can be identified with $\sqrt{-1}S^*U$. In the last paragraph we will use the notation accomodated to Kaneko [**8**]. Recall, a hyperfunction f on U can be written as a formal sum of boundary values of holomorphic functions $F_j(z)$ defined on infinitesimal wedges $U + \sqrt{-1}\Gamma_j 0$, where Γ_j are open cones in $\mathbb{R}^n$, $f(x) = \sum_{j=1}^N F_j(x + \sqrt{-1}\Gamma_j 0)$. It is said that f is microanalytic at the point $(x, \sqrt{-1}\xi\infty) \in \sqrt{-1}S^*U$ if, for a suitable representation of f (given above) on a neighbourhood of $x, \Gamma_j \cap \{y \in \mathbb{R}^n; \langle \xi, y \rangle > 0\} = \emptyset$ holds for all $j \in \{1, \dots, N\}$. So, in the last paragraph we will use the identification of $\sqrt{-1}S^*U$ and S^*U and say that $(x, \sqrt{-1}\xi\infty) \in SSu$ if and only if $(x, \xi) \in SSu$. The same will be used for other types of singular supports. Thus, for an $f \in \mathcal{D}'^*(\Omega)$, $(x, \omega) \in S^*\Omega$ is not in SSf if $(x, \sqrt{-1}\omega\infty) \in S^*\Omega$ is not in $SS\{f\}$, where $\{f\}$ denotes the corresponding hyperfunction. This notion is equal to Hörmander's $WF_A f-$ the analytic wave front set of f [**9**, Definition 9.3.2., Theorem 9.6.3].

2. Tempered ultradistributions

Tempered ultradistribution spaces are introduced and studied in [**13**] and [**16**].

Let $m > 0$. The space of smooth functions φ on $\mathbb{R}^n$ which satisfy

$$\sigma_{m,2}(\varphi) = \Big(\sum_{\alpha,\beta \in \mathbb{N}_0^n} \int_{\mathbb{R}^n} \Big| \frac{m^{|\alpha+\beta|}}{M_{|\alpha|} M_{|\beta|}} (1 + |x|^2)^{|\beta|/2} \varphi^{(\alpha)}(x) \Big|^2 dx \Big)^{1/2} < \infty,$$

equipped with the topology induced by the norm $\sigma_{m,2}$, is denoted by $\mathcal{S}_2^{M_p,m}$. The strong duals of $\mathcal{S}^{(M_p)} = \operatorname{proj\,lim}_{m\to\infty} \mathcal{S}_2^{M_p,m}$ and $\mathcal{S}^{\{M_p\}} = \operatorname{ind\,lim}_{m\to 0} \mathcal{S}_2^{M_p,m}$ are called spaces of tempered ultradistributions of Beurling and Roumieu type. For every fixed $p \in [1, \infty]$, the family of norms $\{\sigma_{m,2};\ m > 0\}$ is equivalent to the family of norms $\{\sigma_{m,p};\ m > 0\}$ where instead of L^2 norm we put L^p norm.

$\mathcal{S}^{(M_p)}$ and $\mathcal{S}^{\{M_p\}}$ are (FS)- and (LS)-spaces, respectively. If (M.2) holds, they are (FN)- and (LN)- spaces, respectively (for these types of spaces, see [**8**]) and

$$\mathcal{D}^* \hookrightarrow \mathcal{S}^* \hookrightarrow \mathcal{E}^*, \ \ \mathcal{S}^* \hookrightarrow \mathcal{S},$$

where we have also that injections are continuous.

An $f \in \mathcal{D}'^*$ is in $\mathcal{S}'^*$ if and only if there exists a family $F_{\alpha,\beta} \in L^2(\mathbb{R}^n),\ \alpha, \beta \in \mathbb{N}_0^n$, such that

$$f = \sum_{\alpha,\beta \in \mathbb{N}_0^n} ((1 + |x|^2)^{\beta/2} F_{\alpha,\beta})^{(\alpha)} \ \text{ in } \ \mathcal{S}'^*,$$

and in case $\mathcal{S}'^{(M_p)}$, (resp., in the case $\mathcal{S}'^{\{M_p\}}$, for every $k > 0$) such that

$$\Big(\sum_{\alpha,\beta \in \mathbb{N}_0^n} \int_{\mathbb{R}^n} | \frac{M_{|\alpha|} M_{|\beta|}}{k^{|\alpha+\beta|}} F_{\alpha,\beta}(x)|^2 \Big)^{1/2} < \infty.$$

The corresponding structural theorems may be obtained by using the families of norms $\{ma_{m,p};\ m > 0\}$, $p \in [1, \infty]$.

If (M.1), (M.2) and (M.3) are assumed, then we have more precise structural characterisations [**16**].

The Fourier transformation is an isomorphism of $\mathcal{S}^*$ onto itself.

In the next theorem we use the notation N_p for the sequence of positive numbers and $N(\rho)$, $\tilde{N}(\rho)$ for the corresponding associated and growth function.

THEOREM 1. [**17**] *Assume that* $(M.1)^*, (M.2)$ *and* $(M.3)'$ *hold for* N_p $(N_0 = 1)$. *Let* Γ *be an open convex cone in* $\mathbb{R}^n$ *and* F *be an analytic function in*

$$Z = \{z \in \mathbb{C}^n;\ \ \operatorname{Im} z \in \Gamma,\ |\operatorname{Im} z| < a\}.$$

for some $a > 0$. *Moreover, assume*

$$|F(x+\sqrt{-1}y)| \le C_{a,b} e^{N(a|x|)+\tilde{N}(\frac{b}{|y|})},\ \ x+\sqrt{-1}y \in Z,$$

for some $a > 0,\ b > 0$ *and* $C_{a,b} > 0$, *in the* (N_p)*-case and for every* $a > 0,\ b > 0$ *there exists a* $C_{a,b} > 0$, *in the* $\{N_p\}$*-case. Then*

$$F(x+\sqrt{-1}y) \overset{\mathcal{S}'^{\dagger}}{\to} F(x+\sqrt{-1}0),\ \ y \to 0,\ \ y \in \Gamma.$$

In paricular

$$SS_{\dagger} \subset \mathbb{R}^n \times \Gamma^0 \setminus \{0\}$$

3. Composition with a real analytic function

We still assume that $(M.1)^*, (M.2)$ and $(M.3)'$ hold for M_p.

Let f be a real analytic mapping $X \to Y$, where X and Y are open sets in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. We are going to define the pullback$f^\star u$ of a $u \in \mathcal{D}'^{\dagger}$ with suitable properties of its singular spectrums. Denote

$$N_f = \{(f(x), \eta);\ \ {}^t f'(x)\eta = 0\}.$$

If γ is a closed conic subset of $Y \times (\mathbb{R}^n \setminus \{0\})$ (conic in the second variable), denote

$$\mathcal{D}'^*_{\gamma}(Y) = \{u \in \mathcal{D}'^*(Y);\ \ SSu \subset \gamma\},$$

$$f^\star\gamma = \{(x, {}^t f'(x)\eta);\ \ \ (f(x), \eta) \in \gamma\}.$$

For the composition of an ultradistribution with an analytic function we can not use the asymptotic expansion based on the stationary phase arguments as for distributions. Because of that the first part of the proof of the next theorem is completely different from the corresponding one for distributions [**9**].

THEOREM 2. *The pullback* $f^\star u$ *can be defined in one and only one way for all* $u \in \mathcal{D}'^*(Y)$ *with* $N_f \cap SSu = \emptyset$ *such that* $f^\star u = u \circ f$ *if* $u \in C^\infty(Y)$ *and for any closed conic subset* γ *of* $Y \times (\mathbb{R}^n \setminus \{0\})$ *with* $\gamma \cap N_f = \emptyset$, $u \in \mathcal{D}'^*_{\gamma}(Y)$ *implies*

$$SS(f^\star u) \subset f^\star(SSu)\ \ \text{and}\ \ SS_*(f^\star u) \subset f^\star(SS_* u). \tag{1}$$

Proof. Let Γ be an open convex cone in $\mathbb{R}^n$ such that $(Y \times \Gamma^0) \cap N_f = \emptyset$. Let $u \in \mathcal{D}'^*_\Gamma(Y)$. We define $f^\star u$ as an element from $\mathcal{D}'^*(X)$ as follows.

For every $x_0 \in X$ choose $h_0 \in \mathbb{R}^m$ such that $f'(x_0)h_0 \in \Gamma$. There exist an open set O_{x_0}, $x \in O_{x_0} \subset X$, an open convex cone Γ_{h_0} and $\epsilon_0 > 0$ such that

$$f(x + \sqrt{-1}h) \in Y + \sqrt{-1}\Gamma,\ h \in \Gamma_{h_0},\ |h| < \epsilon_0.$$

Theorem 3.9. in [**12**] and Theorem 1, imply that the following limit exists in $\mathcal{D}'^*(O_{x_0})$

$$f^\star u|_{O_{x_0}} = \lim_{\substack{h \to 0 \\ h \in \Gamma_{h_0}}} \Phi(f(\cdot + \sqrt{-1}h)), \tag{2}$$

where $\Phi(z) \in \mathcal{O}^*(Y + \sqrt{-1}\Gamma)$ and $u = \Phi(\cdot + \sqrt{-1}\Gamma 0)$.

The family of ultradistributions $\{f^\star u|_{O_{x_0}}; x_0 \in X\}$ has the property:

If $O_{x_0} \cap O_{x_1} \neq \emptyset$, $x_0, x_1 \in X$ then $f^\star u|_{O_{x_0}}$ and $f^\star u|_{O_{x_1}}$ coincide on $O_{x_0} \cap O_{x_1}$. Let us prove that for every $x \in O_{x_0} \cap O_{x_1}$ there is an $O_x \subset O_{x_0} \cap O_{x_1}$ and an h such that $f^\star u|_{O_{x_0}}$ and $f^\star u|_{O_{x_1}}$ coincide with $f^\star u|_{O_x}$ on O_x. Since $f'(x)h_0$ and $f'(x)h_1$ belong to Γ, this holds for $f'(x)((1-t)h_0 + th_1)$, for every $t \in [0,1]$. By the compactness of [0,1], there exist O_x and h such that the quoted assertion holds.

Thus, there exists an element in $\mathcal{D}'^*(X)$, denoted by $f^\star u$, which coincides with $f^\star u|_{O_{x_0}}$ for every x_0.

More generally, let $u \in \mathcal{D}'^*(Y)$ be such that $SSu \cap N_f = \emptyset$. There exist open convex cones Γ_j, $j = 1, \ldots, s$, such that

$$SSu \subset \bigcup_{i=1}^{s} Y \times \Gamma_j^0 \text{ and } \bigcup_{j=1}^{s} (Y \times \Gamma_j^0) \cap N_f = \emptyset.$$

Theorem 3.9. in [**12**] implies that there are holomorphic functions $\Phi_j \in \mathcal{O}^*(Y + \sqrt{-1}\Gamma_j)$, $j = 1, \ldots, s$, such that

$$u_j = \Phi_j(\cdot + \sqrt{-1}\Gamma_j 0),\ \ SSu_j \subset Y \times \Gamma_j,\ \ j = 1, \ldots, s,$$

(in the sense of convergence in $\mathcal{D}'^*(Y)$) and $u = \sum_{j=1}^{s} u_j$.

We define

$$f^\star u = \sum_{j=1}^{s} f^\star u_j. \tag{3}$$

In order to prove that this definition does not depend on the decomposition of u, assume that G_i, $i = 1, \ldots, p$, are convex open cones such that

$$\bigcup_{i=1}^{p} (Y \times G_i^0) \cap N_f = \emptyset,\ \bigcup_{i=1}^{p} Y \times G_i^0 \supset SSu,\ v_i = F_i(\cdot + \sqrt{-1}G_i 0)$$

and $u = \sum_{i=1}^{p} v_i$. We have to prove

$$\sum_{j=1}^{s} f^\star u_j = \sum_{i=1}^{p} f^\star v_i. \tag{4}$$

In the identity

$$\sum_{j=1}^{s} \Phi_j(\cdot + \sqrt{-1}\Gamma_j 0) = \sum_{i=1}^{p} F_i(\cdot + \sqrt{-1}G_i 0) \tag{5}$$

we may assume that Γ_j, $j = 1, \ldots, s$, are pairwise disjoint sets because if for some j_1 and j_2, $\Gamma_{j_1} \cap \Gamma_{j_2} \neq \emptyset$, then the left hand side sum is reduced by

$$\Phi_{j_1}(\cdot + \sqrt{-1}\Gamma_{j_1} 0) + \Phi_{j_2}(\cdot + \sqrt{-1}\Gamma_{j_2} 0) = (\Phi_{j_1} + \Phi_{j_2})(\cdot + \sqrt{-1}(\Gamma_{j_1} \cap \Gamma_{j_2})0).$$

In the same way we may assume that G_i, $i = 1, \ldots, p$, are pairwise disjoint sets.

Denote by Λ_j a subset of $\{1, \ldots, p\}$ which consists of i for which $\Gamma_j \cap G_i \neq \emptyset$, $j = 1, \ldots, s$. By using "Edge of the Wedge theorem" (see [**12**]) we have:

$$\Lambda_j = \emptyset \Rightarrow \Phi_j(\cdot + \sqrt{-1}\Gamma_j 0) \text{ is real analytic on } Y.$$

Put $d_j = \overline{\overline{\Lambda}}_j$. For every $i \in \{1, \ldots, p\}$ decompose $F_i(\cdot + \sqrt{-1}G_i 0)$

$$F_i(\cdot + \sqrt{-1}G_i 0) = \sum_{j \in V_i} F_{ij}(\cdot + \sqrt{-1}(G_i \cap \Gamma_j)0),$$

where V_i is the subset of $\{1, \ldots, s\}$ which consists of j for which $\Gamma_j \cap G_i \neq \emptyset$, $i = 1, \ldots, p$. Again,

$$V_i = \emptyset \Rightarrow F_i(\cdot + \sqrt{-1}G_i 0) \text{ is real analytic on } Y.$$

So, we rewrite (5) as

$$\sum_{j=1}^{s} d_j(\frac{1}{d_j}\Phi_j(\cdot + \sqrt{-1}\Gamma_j 0)) + \varphi = \sum_{j \in V_i \neq \emptyset} F_{ij}(\cdot + \sqrt{-1}(G_i \cap \Gamma_j)0) + \psi$$

where φ and ψ are real analytic and both sums have the same number of terms. Note, if $j \in V_{i_1} \cap V_{i_2}$, then $(G_{i_1} \cap \Gamma_{j_2}) \cap (G_{i_2} \cap \Gamma_{j_2}) = \emptyset$ and

$$\frac{1}{d_j}\Phi_j(\cdot + \sqrt{-1}\Gamma_j 0) - F_{i_1 j}(\cdot + \sqrt{-1}(\Gamma_j \cap G_{i_1})0)$$

is real analytic.

Thus, the change of variable defined by (3) on both side of (4) gives the same element in $\mathcal{D}'^*(X)$.

For the proof of (1) we use the notation from the beginning of this proof. From (2) it follows that

$$SS(f^\star u)|_{x_0} \subset \{(x_0,\xi);\ \langle h,\xi\rangle \geq 0,\ h \in \Gamma_{h_0}\},$$

where $SS(f^\star u)|_{x_0}$ is a subset of $SS(f^\star u)$ with the first projection x_0. From

$${}^t f'(x_0)\Gamma^0 = \{\xi;\ < h,\xi > \geq 0 \text{ if } f'(x_0)h \in \Gamma\} \ \ ([\mathbf{9}, \text{p. } 296]),$$

and $f'(x_0)\Gamma_{h_0} \subset \Gamma$ it follows

$$SS(f^\star u)|_{x_0} \subset \{(x_0,\ {}^t f'(x_0)\eta); \eta \in \Gamma^0 \setminus \{0\}\}.$$

By using the decomposition $u = \Sigma u_j$ in the neighborhood of $f(x_0)$ given in [**18**], with cones Γ_j such that Γ_j^0 are small enough, and Theorem 1, the proof of (1) follows.

4. The analysis of some operators

The six theorems which are to follow are well-known for distributions [**9**]. Their proofs are the same as for distributions.

Theorem 3. *If* $u \in \mathcal{D}'^*(X)$ *and* $v \in \mathcal{D}'^*(Y)$, *then*
i) $SS_*(u\otimes v) \subset SS_*u \times SS_*v \cup (\operatorname{supp} u \times \{0\}) \times SS_*v \cup SS_*u \times (\operatorname{supp} v \times \{0\})$.
ii) $SS(u\otimes v) \subset SSu \times SSv \cup (\operatorname{supp} u \times \{0\}) \times SSv \cup SSu \times (\operatorname{supp} v \times \{0\})$.

Theorem 4. *If* $u,v \in \mathcal{D}'^*(X)$ *and there are no points* (x,ξ) *such that*

$$(x,\xi) \in SS_*u \quad \textit{and} \quad (x,-\xi) \in SS_*v,$$

then by the pullback of the mapping

$$\delta : X \to X \times X,\ \delta(x) = (x,x)$$

is defined the product

$$uv = \delta^\star(u(x) \otimes v(y))$$

$\mathcal{D}'^*$ *and*

$$SS_*uv \subset \{(x,\xi+\eta);\ (x,\xi) \in SS_*u \ \ \textit{or}\ \xi = 0,$$
$$(x,\eta) \in SS_*v \quad \textit{or}\ \ \eta = 0\}.$$

The same holds for $SSuv$.

THEOREM 5. *Let* $u \in \mathcal{E}'^*(\mathbb{R}^n)$, $x' = (x_1, \dots, x_p)$, $x'' = (x_{p+1}, \dots, x_n)$ *and let*

$$u_1(x') = \int u(x', x'')dx''$$

be defined by

$$\langle u_1, \phi \rangle = \langle u(x', x''), \phi(x') \otimes 1_{x''} \rangle, \quad \phi \in \mathcal{D}^*(\mathbb{R}^n).$$

Then

$$SS_* u_1 \subset \{(x', \xi');\ ((x', x''), (\xi', 0)) \in SS_* u, \ \textit{for some } x''\}.$$

The same holds for SSu_1.

Proof. Since the proof of the theorem is slightly different from the proof of Theorem 8.4.5 in [**9**], we give details which are important in our assertion. Let $|\omega| = 1$, $\operatorname{supp} u = Q, \phi \in \mathcal{D}^*(\mathbb{R}^p)$. Put

$$\psi_\delta(x'') = \kappa(\delta x''),\ \delta > 0, x'' \in \mathbb{R}^{n-p},$$

where $\kappa \in \mathcal{D}^*(\mathbb{R}^{n-p})$ is equal to 1 in the unit ball $B(0,1)$. Take R such that $Q \subset B(0,R)$, and $\eta \in \mathcal{D}^*(\mathbb{R}^n)$ such that $\eta = 1$ in $B(0, 2R)$.

Note that

$$\langle u(t), K(\cdot + \sqrt{-1}\omega - t) \rangle$$

is analytic and exponentially decreasing if $|x''| > 2R$, and ω belongs to a suitable neighbourhood of $\{\omega;\ |\omega| \leq 1\}$. By letting $\delta \to 0$ in

$$\begin{aligned}\langle U(x + \sqrt{-1}\omega), \phi(x')\psi_\delta(x'') \rangle &= \langle \langle u(t), K(x + \sqrt{-1}\omega - t) \rangle, \eta(x)\phi(x')\psi_\delta(x'') \rangle \\ &\quad + \langle \langle u(t), K(x + \sqrt{-1}\omega - t) \rangle, (1 - \eta(x))\phi'(x')\psi_\delta(x'') \rangle,\end{aligned}$$

it follows

$$\begin{aligned}\langle u_1, \phi \rangle &= \int_{|\omega|=1} \langle U(\cdot + \sqrt{-1}\omega), \phi(x') \otimes 1_{x''} \rangle d\omega = \\ &= \int_{|\omega|=1} \langle U_1(\cdot + \sqrt{-1}\omega'), \phi \rangle d\omega,\end{aligned}$$

where

$$U_1(z') = \int U(z', x'')dx'' = \int U(z', x'' + iy'')\phi x'', \quad |\operatorname{Im} z'|^2 + |y''|^2 < 1$$

is an analytic function when $|\operatorname{Im} z'| < 1$ which is bounded by $Ce^{\tilde{M}(\frac{k}{1-|y|})}$ for some $k > 0$ and $C > 0$ in (M_p)-case (respectively, for every $k > 0$ there is C in $\{M_p\}$-case). Now, by similar arguments as in the proof of Theorem 8.5.4 in [**9**], we finish the proof by using a corresponding result of [**17**].

THEOREM 6. *Assume* $K \in \mathcal{D}'(X \times Y)$, *where* $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$ *are open, and* $\pi_1 : \operatorname{supp} K \to X$ *is proper* (π_1 *is the first projection*). *If* $u \in \mathcal{E}^*(Y)$, *then*

$$SS_* \mathcal{K}u \subset \{(x, \xi);\ \ (x, y, \xi, 0) \in SS_* K \textit{ for some } y \in \operatorname{supp} u\},$$

where $\mathcal{K}$ is the linear operator with the kernel K. The same holds for $SS\mathcal{K}u$.

THEOREM 7. *If $u \in \mathcal{E}'^*$ and $SS_*u \cap SS'_*(K)_Y = \emptyset$, then*

$$SS_*\mathcal{K}u \subset SS_*(K)_X \cup (SS'_*(K) \circ SS_*u),$$

where

$$SS'_*(K) = \{(x, y, \xi, \eta);\ \ (x, y, \xi, -\eta) \in SS_*K\},$$

$$SS_*(K)_X = \{(x, \xi);\ \ (x, y, \xi, 0) \in SS_*K \ \textit{ for some } \ y \in Y\},$$

$$SS'_*(K)_Y = \{(y, \eta);\ \ (x, y, 0, -\eta) \in SS_*K \ \textit{ for some } \ x \in X\}.$$

The same holds for $SS\mathcal{K}u$.

Now, one can easily prove the following assertion.

THEOREM 8. *If $u \in \mathcal{D}'^*$ and $v \in \mathcal{E}'^*$, then*

$$SS_*(u * v) \subset \{(x + y, \xi);\ (x, \xi) \in SS_*u, (y, \xi) \in SS_*v\}.$$

The same holds for $SS(u * v)$.

Remark. Theorem 8 has applications in the theory of pseudo-differential operators. For example, for a pseudo-differential operator P it follows that $SS_*Pu \subset SS_*u$, $u \in \mathcal{D}'^*$.

If $P(D)$ is an ultradifferential operator with constant coefficients of $*$-class [**12**], then Theorem 8 implies $SS_*P(D)u \subset SS_*u$, $u \in \mathcal{D}'$. The same theorem implies that the Poisson transform of a $u \in \mathcal{D}'$, if it exists, does not enlarge the SS_*u because $SS\mathcal{P}$ is empty (cf. [**18**]).

5. Quotient sheaves of microfunctions and flabbiness

The results of this paragraph are from [**6**] and [**7**].

Let $\pi : \sqrt{-}S^*\mathbb{R}^n \to \mathbb{R}^n$ be the canonical projection. The sheaf of microfunctions on $\sqrt{-1}S^*\mathbb{R}^n$ is the associated sheaf of the presheaf

$$\sqrt{-1}S^*\mathbb{R}^n \supset \Omega \to \Gamma(\pi(\Omega); \mathcal{B})/\{u \in \Gamma(\pi(\Omega); \mathcal{B}) | SSu \cap \Omega = \emptyset\}.$$

This sheaf is denoted by $\mathcal{C}$. It enjoys the exact sequence

$$0 \to \mathcal{A} \to \mathcal{B} \to \pi_*\mathcal{C} \to 0, \qquad [\mathbf{21}]$$

where $\mathcal{A}$ denotes the sheaf of real analytic functions on $\mathbb{R}^n$. Moreover, there exists a canonical surjective spectrum map

$$Sp : \pi^{-1}\mathcal{B} \to \mathcal{C}.$$

Then, for $u \in \mathcal{B}$, $SS(u) = \mathrm{supp}(Sp(u))$. The injection

$$\mathcal{D}'^{*} \hookrightarrow \mathcal{B} \text{ (respectively, } \mathcal{E}^{*} \hookrightarrow \mathcal{B})$$

induces a sheaf homomorphism

$$\pi^{-1}\mathcal{D}'^{*} \to \mathcal{C} \text{ (respectively } \pi^{-1}\mathcal{E}^{*} \to \mathcal{C}).$$

We define a subsheaf $\mathcal{C}^{*}$ (respectively, $\mathcal{C}^{d,*}$) of $\mathcal{C}$ as the image of the above morphism and call it the sheaf of microfunctions of class $*$ (respectively $d,*$). Furthermore, we have a canonical exact sequence

$$0 \to \mathcal{A} \to \mathcal{D}'^{*} \to \pi_{*}\mathcal{C}^{*} \to 0$$
$$(\text{resp. } 0 \to \mathcal{A} \to \mathcal{E}'^{*} \to \pi_{*}\mathcal{C}^{d,*} \to 0).$$

We have given in Section 1 an order to the set of sequences satisfying conditions (M.1), (M.2) and (M.3)'. If $\dagger \leq *$, we have canonical injections

$$\mathcal{C}^{d,\dagger} \hookrightarrow \mathcal{C}^{d,*} \hookrightarrow \mathcal{C}^{*} \hookrightarrow \mathcal{C}^{\dagger} \hookrightarrow \mathcal{C}.$$

From now on we use the notation $\mathcal{C}^{1} = \mathcal{C}$ and $1 \leq *$ for any $*$.

We define sheaves $\mathcal{C}^{\dagger,*}, \mathcal{C}^{\dagger/*}, \mathcal{C}^{d,\dagger,*}$ on $\sqrt{-1}S^{*}\mathbb{R}^{n}$ by the following exact sequences:

i) $0 \to \mathcal{C}^{d,*} \to \mathcal{C}^{\dagger} \to \mathcal{C}^{\dagger,*} \to 0$, when $\dagger \leq *$,

ii) $0 \to \mathcal{C}^{*} \to \mathcal{C}^{\dagger} \to \mathcal{C}^{\dagger/*} \to 0$, when $\dagger \leq *$,

iii) $0 \to \mathcal{C}^{d,*} \to \mathcal{C}^{d,\dagger} \to \mathcal{C}^{d,\dagger,*} \to 0$, when $* \leq \dagger$.

The canonical surjective spectrum map Sp induces the following surjective spectrum maps:

$$Sp^{1,*} : \pi^{-1}\mathcal{B} \to \mathcal{C}^{1,*},$$
$$Sp^{1/*} : \pi^{-1}\mathcal{B} \to \mathcal{C}^{1/*}.$$

Let $u \in \mathcal{B}$. We can also define the singular spectrum of class $*$, $SS_{*}(u)$, and that of class $1/*, SS^{*}(u)$, in the following way:

$$SS_{*}u = \mathrm{supp}(Sp^{1,*}u),$$
$$SS^{*}u = \mathrm{supp}(Sp^{1/*}u).$$

Let $\mathcal{F}$ be a sheaf of Abelian groups on a topological space X. Recall [**1**], $\mathcal{F}$ is supple if for any open set Ω of X, any closed Z, Z_1, Z_2 of Ω such that $Z = Z_1 \cup Z_2$, and any section $u \in \Gamma_Z(\Omega, \mathcal{F})$, there exists $u_i \in \Gamma_{Z_i}(\Omega, \mathcal{F})$ $(i = 1, 2)$ such that $u = u_1 + u_2$. Clearly, if a sheaf is flabby, then it is supple.

THEOREM 9. [**6**] a) *The quotient sheaf* $\mathcal{D}'^*/\mathcal{E}^*$ *is supple.*

b) *The sheaves* $\mathcal{C}^{1,*}$ *and* $\mathcal{C}^{1/*}$ *are flabby and the following sequences are exact*

$$0 \to \mathcal{E}^* \to \mathcal{B} \to \pi_*\mathcal{C}^{1,*} \to 0,$$
$$0 \to \mathcal{D}'^* \to \mathcal{B} \to \pi_*\mathcal{C}^{1/*} \to 0.$$

c) *Let* $1 \leq \dagger \leq *$. *The sheaves* $\mathcal{C}^{\dagger,*}$ *and* $\mathcal{C}^{\dagger/*}$ *are supple and the sequences*

$$0 \to \mathcal{E}^* \to \mathcal{D}'^\dagger \to \pi_*\mathcal{C}^{\dagger,*} \to 0,$$
$$0 \to \mathcal{D}'^* \to \mathcal{D}'^\dagger \to \pi_*\mathcal{C}^{\dagger/*} \to 0,$$

are exact.

d) Let $1 \leq * \leq \dagger$. *The sheaf* $\mathcal{C}^{d,\dagger,*}$ *is supple and the sequence*

$$0 \to \mathcal{D}^* \to \mathcal{E}^\dagger \to \pi_*\mathcal{C}^{d,\dagger,*} \to 0$$

is exact.

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(Received 11 06 1998)