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ON A SUM OF DIVISORS PROBLEM

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Abstract. Several results concerning the set $\mathcal{S}$ are proved. This set was constructed by A. Granville as follows: Let $1 \in \mathcal{S}$, and for $n \in \mathbb{N}$ and $n > 1$ let $n \in \mathcal{S}$ if $\sum_{d|n, d<n, d\in\mathcal{S}} d \le n$. If equality holds then n is $\mathcal{S}$-perfect, and properties of $\mathcal{S}$-perfect are also discussed. Numerical data are given and some conjectures are formulated.

In December 1996 Andrew Granville proposed to us the following construction of the set $\mathcal{S}$ involving the sum of divisors of natural numbers. Let $1 \in \mathcal{S}$ and for $n \in \mathbb{N}$ and $n > 1$ let $n \in \mathcal{S}$ if

$$\sum_{d|n, d<n, d\in\mathcal{S}} d \le n. \tag{1}$$

Moreover if equality holds in (1) we shall say that n is *$\mathcal{S}$–perfect*.

Granville's problems in connection with $\mathcal{S}$ are to determine which numbers n belong to $\mathcal{S}$, and which are $\mathcal{S}$-perfect. Let henceforth p, q denote primes. If $\alpha \in \mathbb{N}$, then clearly $p^\alpha \in \mathcal{S}$ and $pq \in \mathcal{S}$. There are 17 $\mathcal{S}$-perfect numbers < 200000: 6, 24, 28, 96, 126, 224, 384, 496, 1536, 1792, 6144, 8128, 14336, 15872, 24576, 98304, 114688. Among these only 4 are perfect, namely 6, 28, 496 and 8128, which means that there are 13 $\mathcal{S}$-perfect numbers < 200000 which are not perfect: 24, 126, 224, 384, 496, 1536, 1792, 6144, 14336, 15872, 24576, 98304, 114688. Recall that a number n is *perfect* if $\sigma(n) = 2n$, where $\sigma(n) = \sum_{d|n} d$ is the sum of all positive divisors of n. It is, in general, difficult to determine which n belong to $\mathcal{S}$ and especially which ones are $\mathcal{S}$-perfect. This is not surprising, since the corresponding classical problem of determining the structure of perfect numbers is

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unsolved. In particular, it is not yet known whether there exists an odd perfect number, but it is known that any such number must be $\geq 10^{300}$ (see R.P. Brent [**2**]) and must have at least 8 distinct prime factors (see P. Hagis [**6**]). However in dealing with perfect numbers one can exploit the fact that $\sigma(n)$ is multiplicative ($\sigma(mn) = \sigma(m)\sigma(n)$ if $(m,n) = 1$), whereas the problems involving $\mathcal{S}$-perfect numbers cannot be linked to the distribution of values of a multiplicative function in any obvious way. Traditionally numbers n are called *deficient* if $\sigma(n) < 2n$ and *abundant* if $\sigma(n) > 2n$. Trivially deficient and perfect numbers are in $\mathcal{S}$. Also it is an easy matter to check that the smallest odd number not in $\mathcal{S}$ is 2835, and that the smallest two consecutive integers not in $\mathcal{S}$ are 5984 and 5985. Moreover, as we shall see in Theorem 5 below, the smallest three consecutive integers not in $\mathcal{S}$ are 171078830, 171078831 and 171078832.

We can prove

THEOREM 1. *If $p \geq 3$ is a given prime and $r \geq 2$ is the unique integer such that*

$$2^r - 1 \leq p < 2^{r+1} - 1, \tag{2}$$

then

$$2^a \cdot p \in \mathcal{S} \iff a \not\equiv 0 \pmod{r}.$$

Proof 1. Since $p \geq 3$ is a prime, the condition (2) may be written as

$$2^r - 1 \leq p < 2(2^r - 1). \tag{3}$$

Thus $2^a p \notin \mathcal{S}$ is equivalent to $r \mid a, a > 0$. The proof is by induction on $a \geq 0$. For $a = 0$, trivially $p \in \mathcal{S}$. So assume the result for $0, 1, \ldots, a-1$ and write

$$a = \alpha r + \beta, \quad 1 \leq \beta \leq r, \quad \alpha \geq 0.$$

By the induction hypothesis we have

$$\begin{aligned}\sum_{\substack{d \mid 2^a p \\ d \in \mathcal{S}, d < 2^a p}} d &= 1 + 2 + \ldots + 2^a + \sum_{\substack{j=0 \\ j=0 \text{ or } r \nmid j}}^{a-1} 2^j p \\ &= 2^{a+1} - 1 + p(2^a - \sum_{j=0, r \mid j}^{a-1} 2^j) = 2^a p + 2^{a+1} - 1 - p \cdot \frac{2^{(\alpha+1)r} - 1}{2^r - 1}.\end{aligned}$$

Thus

$$2^a p - \sum_{\substack{d \mid 2^a p \\ d \in \mathcal{S}, d < 2^a p}} d = p \cdot \frac{2^{(\alpha+1)r} - 1}{2^r - 1} - (2^{a+1} - 1). \tag{4}$$

If $\beta < r$ then the left-hand side of (4) is (since $p \ge 2^r - 1$ by (3))

$$\ge (2^{(\alpha+1)r} - 1) - (2^{a+1} - 1) = 2^{\alpha r + r} - 2^{\alpha r + \beta + 1} \ge 0.$$

If $\beta = r$ then the left-hand side of (4) is (since $p < 2(2^r - 1)$ by (3))

$$< 2(2^{(\alpha+1)r} - 1) - (2^{a+1} - 1) = (2^{(\alpha+1)r+1} - 2) - (2^{(\alpha+1)r+1} - 1) = -1 < 0.$$

This proves Theorem 1.

In what concerns the structure of $\mathcal{S}$-perfect numbers we can prove

THEOREM 2. *An ordinary perfect number is also an $\mathcal{S}$-perfect number. The number $n = 2^m p$ $(p > 2)$ is an $\mathcal{S}$-perfect number if and only if $p = 2^r - 1$ (a Mersenne prime) and $m = kr - 1$ $(k \in \mathbb{N})$. Moreover if $n = 2 \cdot 3^2 \cdot p$ is an $\mathcal{S}$-perfect number, then $p = 7$, i.e. $n = 126$.*

Proof. Note that $d \mid n$ implies (by multiplicativity of $\sigma(n)$) that $\sigma(d)/d \le \sigma(n)/n$, and the inequality is strict if $d < n$. Hence $\sigma(d) - d \le d$ for all divisors d of n if n is perfect (i.e. $\sigma(n) = 2n$) and

$$\sum_{\delta \mid d, \delta < d, \delta \in \mathcal{S}} \delta \;\le\; \sigma(d) - d \;\le\; d,$$

so that $d \in \mathcal{S}$. Therefore if n is perfect

$$\sum_{d \mid n, d < n, d \in \mathcal{S}} d = \sum_{d \mid n, d < n} d = \sigma(n) - n = n,$$

hence n is $\mathcal{S}$-perfect.

Now let r be as in (2) and suppose that $m \le r - 1$, where $n = 2^m p$ is $\mathcal{S}$-perfect. Then by Theorem 1 we have $2^a p \in \mathcal{S}$ for $a = 1, \ldots, m$. Thus

$$\begin{aligned} 2^m p &= \sum_{d \mid n, d < n, d \in S} d = 1 + 2 + \ldots + 2^m + p(1 + 2 + \ldots + 2^{m-1}) \\ &= 2^{m+1} - 1 + p(2^m - 1). \end{aligned}$$

Hence $p = 2^{m+1} - 1$, which in view of (2) gives

$$2^r - 1 \le 2^{m+1} - 1, \quad m \ge r - 1,$$

which combined with $m \le r - 1$ gives $m = r - 1$. Therefore $n = 2^{r-1} p$, where $p = 2^r - 1$ is a so-called *Mersenne prime.* If $m = kr + \ell$, $k \ge 1$, $1 \le \ell \le r - 1$ and $2^m p$ is $\mathcal{S}$-perfect, then by using Theorem 1 again we shall obtain

$$\begin{aligned} 2^m p &= 2^{m+1} - 1 + p(2^m - 1 - 2^r - \ldots - 2^{kr}), \\ 2^{kr+\ell} p &= 2^{kr+\ell+1} - 1 + p\left(2^{kr+\ell} - \frac{2^{(k+1)r} - 1}{2^r - 1}\right). \end{aligned}$$

Using this and (2) we have

$$2^{(k+1)r} - 1 \le p\frac{2^{(k+1)r}-1}{2^r-1} = 2^{kr+\ell+1} - 1,$$

giving $r \le \ell + 1$, which combined with $1 \le \ell \le r-1$ gives $\ell = r-1$, as asserted. Hence combining the two above cases we conclude that $m = kr-1$, $k \in \mathbb{N}$. Conversely if $n = 2^{kr-1}p$, $p = 2^r - 1$, $k \in \mathbb{N}$, then by Theorem 1 we have

$$\begin{aligned}\sum_{d|n,d<n,d\in\mathcal{S}} d &= 1 + 2 + \cdots + 2^{kr-1}\\ &\quad + p(1 + 2 + \cdots + 2^{kr-2} - 2^r - \cdots - 2^{(k-1)r})\\ &= 2^{kr} - 1 + p\big(2^{kr-1} - 1 + 1 - \frac{2^{kr}-1}{2^r-1}\big) = 2^{kr} - 1 + n - (2^{kr}-1) = n,\end{aligned}$$

so that n is $\mathcal{S}$-perfect, as asserted.

For the last part of Theorem 2 note first that $2\cdot 3^2$ is not in $\mathcal{S}$ and that for $p > 2$ the number 3^2p is in $\mathcal{S}$, while $2\cdot 3\cdot p$ is not. Let $n = 2\cdot 3^2\cdot p$. For $p = 2, 3$ it is easily checked that n is not $\mathcal{S}$-perfect. If $p > 3$ then n is $\mathcal{S}$-perfect if and only if

$$\sigma(n) - 2\cdot 3^2 - 2\cdot 3\cdot p = 2n,$$

giving $39(p+1) - 18 - 6p = 36p$, $p = 7$, $n = 126$.

Perhaps 126 is the only even $\mathcal{S}$-perfect number that is not of the form $2^m p$, although this seems difficult to prove. It is well-known that all even perfect numbers are of the form $n = 2^{r-1}p$, $p = 2^r - 1$ (see e.g. [**8**]). Thus Theorem 2 tells us that there exist many more even $\mathcal{S}$-perfect numbers than ordinary even perfect numbers. We have made a numerical check for numbers up to 10^6, and have not been able to find any $\mathcal{S}$-perfect number which is not of the form given by Theorem 2. Very likely there are very few $\mathcal{S}$-perfect numbers not given by Theorem 2 (if any). In particular, we have not found any odd $\mathcal{S}$-perfect number, but similarly to the case of ordinary odd perfect numbers, we cannot prove that they do not exist. We can, however, use the characterization of even S-perfect numbers given by Theorem 2 to prove a quantitative result. At present it is not known whether there are infinitely many Mersenne primes, but it is known that there are at least 37 primes q such that $2^q - 1$ is prime (the 35th, 36th and 37th Mersenne primes were all found using a program written by G.F. Woltman and offered on the Internet – Web site `www.mersenne.org` – to those PC owners willing to use their own computers to run Woltman's program and thus find new Mersenne primes; the largest one was actually found on January 29, 1998, by R. Clarkson, a 19-year old student). This is the set

$$\begin{aligned}\mathcal{M} := \Big\{&2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281,\\ &3217, 4253, 4423, 9689, 9941, 11213, 19937, 21701, 23209, 44497,\\ &86243, 110503, 132049, 216091, 756839, 859433, 1257787,\\ &1398269, 2976221, 3021377\Big\}.\end{aligned}$$

Then we have

THEOREM 3. *If $P(x)$ denotes the number of $\mathcal{S}$-perfect numbers not exceeding x, then*

$$P(x) \ \geq \ \frac{c}{\log 2}\log x + O(1), \qquad c = \sum_{q\in\mathcal{M}} \frac{1}{q} = 1.448\dots. \tag{5}$$

Proof. From Theorem 2 we obtain

$$P(x) \geq \sum_{q\in\mathcal{M}} \sum_{k\geq 1} \sum_{2^{kq-1}(2^q-1)\leq x} 1.$$

The condition $2^{kq-1}(2^q-1) \leq x$ is equivalent to

$$k \leq \frac{1}{q}\frac{\log x}{\log 2} + \frac{1}{q} - \frac{\log(2^q-1)}{q\log 2} = \frac{1}{q}\frac{\log x}{\log 2} + O(1).$$

Therefore it follows that

$$P(x) \geq \sum_{q\in\mathcal{M}} \sum_{k\leq \frac{1}{q}\frac{\log x}{\log 2}+O(1)} 1 = \frac{c}{\log 2}\log x + O(1),$$

where $c = \sum_{q\in\mathcal{M}} \frac{1}{q}$. Thus $\frac{c}{\log 2} = 2.089\dots$, so that in particular from (5) it follows that

$$P(x) \ > \ 2\log x \qquad (x \geq x_0 > 0).$$

Let $\mathcal{P}$ denote the set of all primes and let

$$\mathcal{M}_0 := \{q \in \mathcal{P} : 2^q - 1 \in \mathcal{P}\}, \qquad c_1 := \sum_{q\in\mathcal{M}_0} \frac{1}{q}.$$

It is plausible to surmise that the primes in $\mathcal{M}_0$ are scarce enough so that the series defining c_1 is convergent. In fact, according to a heuristic argument developed by S.S. Wagstaff [**9**] (who improved an earlier argument of Gillies), the number of Mersenne primes not exceeding x is about $\frac{e^\gamma}{\log 2}\log\log x$. It follows that the number of primes $p \leq x, p \in \mathcal{M}$, is certainly $< c_0 \log x$ for some positive constant c_0, in which case the series defining c_1 is indeed convergent. If this is so then it seems to us plausible to conjecture that

$$P(x) \ = \ \left(\frac{c_1}{\log 2} + o(1)\right)\log x \qquad (x \to \infty).$$

Another type of problem is to investigate the size of $\mathcal{S}$. Here the "size of $\mathcal{S}$" can be understood in terms of the corresponding densities

$$\underline{\delta} = \liminf_{x\to\infty} \frac{1}{x} \sum_{n\leq x, n\in\mathcal{S}} 1, \quad \overline{\delta} = \limsup_{x\to\infty} \frac{1}{x} \sum_{n\leq x, n\in\mathcal{S}} 1, \quad \delta = \lim_{x\to\infty} \frac{1}{x} \sum_{n\leq x, n\in\mathcal{S}} 1,$$

which represent the *lower density, upper density* and the *density* of $\mathcal{S}$, respectively.

Observe that $\overline{\delta} \le 17/18$. To prove this note that, for $m > 1$, at least one of the numbers $m, 2m, 3m, 6m$ is not in $\mathcal{S}$. Namely if all four are in $\mathcal{S}$, then $1, m, 2m, 3m$ are allowable summands for $6m$, but as $1+m+2m+3m > 6m$ we get that $6m \notin \mathcal{S}$, which is a contradiction. More generally, if a is any non-deficient number (meaning that $\sigma(a) \ge 2a$), then for any $m > 1$ at least one member of the set $\{\, dm : d \mid a \,\}$ is not in $\mathcal{S}$. Now consider numbers $m = 6k \pm 1$ from $[1, x/6]$. Then the numbers $m, 2m, 3m, 6m$ are all distinct, lie in $[1, x]$ and at least one of them is not in $\mathcal{S}$. Hence there are $\ge \frac{x}{18} + o(x)$ numbers in $[1, x]$ which are not in $\mathcal{S}$, and this gives $\overline{\delta} \le 17/18$.

Suppose that the density of $\mathcal{S}$

$$\delta = \lim_{x\to\infty} \frac{1}{x} \sum_{n\le x, n\in\mathcal{S}} 1 \tag{6}$$

exists. By elaborating the above argument one can in fact give an explicit upper bound for δ. This is contained in the following

THEOREM 4. *If the density δ of $\mathcal{S}$ exists, then $\delta \le 11/12$.*

Proof. By the definition (6) of δ there are $\delta x + o(x)$ numbers $m \le x$ from $\mathcal{S}$ in $[1, x]$ as $x \to \infty$. Thus there are at least $\delta x + o(x)$ integers (of the form $2m, 3m$ or $6m$) not in $\mathcal{S}$ lying in $[1, 2x], [1, 3x]$ or in $[1, 6x]$. Let $\alpha, \beta, \gamma \ge 0$ satisfy $\alpha + \beta + \gamma = 1$. Then for any fixed $\varepsilon > 0$ and $x \ge x_0(\varepsilon)$ the interval $[1, 2x]$ contains at least $(\alpha\delta - \varepsilon)x$ integers not in $\mathcal{S}$, $[1, 3x]$ contains at least $(\beta\delta - \varepsilon)x$ integers not in $\mathcal{S}$, and $[1, 6x]$ contains at least $(\gamma\delta - \varepsilon)x$ integers not in $\mathcal{S}$. In the first case we have

$$2x + O(1) = \sum_{n\le 2x} 1 = \sum_{n\le 2x, n\in\mathcal{S}} 1 + \sum_{n\le 2x, n\notin\mathcal{S}} 1 \ge 2\delta x + o(x) + (\alpha\delta - \varepsilon)x,$$

hence if we divide by x and let $x \to \infty$ we shall obtain, since ε may be arbitrarily small,

$$2 \ \ge\ 2\delta + \alpha\delta,$$

and from the remaining cases we likewise have

$$3 \ \ge\ 3\delta + \beta\delta, \qquad 6 \ \ge\ 6\delta + \gamma\delta.$$

Adding the above inequalities it follows that

$$11 \ \ge\ 11\delta + (\alpha + \beta + \gamma)\delta = 12\delta, \qquad \delta \le 11/12.$$

Note that the same upper bound follows unconditionally for the lower density of $\mathcal{S}$ (since 6 is the smallest non-deficient number > 1). We also remark that $\mathcal{S}$ trivially contains deficient numbers which are known to have asymptotic density $> 3/4$ (see p. 189 of [**3**]), hence it follows that $\underline{\delta} > 3/4$. Numerical data suggest that the density δ of $\mathcal{S}$ exists and is near $\delta = 0.846\ldots$. It is nevertheless unconditionally true that there exists a sequence of numbers of positive density and belonging to $\mathcal{S}$, none of which are deficient numbers. Namely it follows that $40m \in \mathcal{S}$ if $\sigma(m)/m < 10/9 - 40/m$, and then one has only to apply (8). Thus suppose that $\sigma(m)/m < 10/9 - 40/m$. We have that $\sigma(d)/d \le \sigma(n)/n$ if $d|n$, so that $d \in \mathcal{S}$ if $d|n$ and n is deficient. If $p(m)$ is the smallest prime factor of m then $\sigma(m)/m < 10/9$ implies that $p(m) > 5$. Thus $(20, m) = 1$ and if $d|m$, then we have $2d$, $4d$, $5d \in \mathcal{S}$. We want to show that

$$\sum(m) \ \le \ 40m,$$

$$\sum(m) := \sum_{d|40m, d<40m, d\in\mathcal{S}} d = \sum_{k=0}^{3} 2^k \sum_{\ell=0}^{1} 5^\ell \sum_{\delta|m, 2^k5^\ell\delta\in\mathcal{S}} \delta - 40m.$$

But $40d < 20d + 10d + 5d + 4d + 2d$, so at least one of the numbers $10d, 20d, 40d$ is not in $\mathcal{S}$, since $2d, 4d, 5d \in \mathcal{S}$. Suppose that $10d \notin \mathcal{S}$. Then we use $40d = (20+8+5+4+2+1)d \ \ (d > 1)$, and it follows that at least one of the numbers $8d, 20d, 40d \notin \mathcal{S}$. The conclusion is, after analysing all the cases, that

$$\begin{aligned} \sum(m) &\le \sigma(40)\sigma(m) - (10+8)\sum_{d|m, d>1} d - 40m \\ &= 90\sigma(m) - 18\sigma(m) - 40m + 40 \le 40m \end{aligned}$$

for $\sigma(m)/m < 10/9 - 40/m$. In a similar way one can prove that, if $1 < \sigma(m)/m < 4/3$, then $6m \notin \mathcal{S}$. A consequence of this is that both $12m$ and $18m$ are in $\mathcal{S}$ if $1 < \sigma(m)/m \le 15/14$. Since $12m$ and $18m$ are abundant, this is then another way to obtain that there is a set of abundant numbers with positive density which are all in $\mathcal{S}$. One can find more inequalities of the above type which show that certain numbers are in $\mathcal{S}$ or not in $\mathcal{S}$. By elaborating the above arguments the upper bound for $\overline{\delta}$ and the lower bound for $\underline{\delta}$ can be improved, but it seems rather difficult to obtain in this way that $\overline{\delta} = \underline{\delta}$, namely that δ exists.

A concept related somewhat to the set $\mathcal{S}$ is that of *weird* numbers. An integer $n \ge 1$ is called weird by S.J. Benkoski and P. Erdős [**1**] if n is abundant, but n is not the distinct sum of some of its proper divisors. There are 24 weird numbers less than one million, the smallest of which is 70. Benkoski and Erdős [**1**] proved that the density of weird numbers is positive, and obtained also several related results. However it appears that their methods of proof cannot be of help in proving that the density of $\mathcal{S}$ exists.

Although we could not prove that the density of $\mathcal{S}$ exists, it seems that in the range that we have investigated (up to 300000) the elements of $\mathcal{S}$ are quite evenly distributed. This is suggested by the following table, in which $\mathcal{S}(x)$ stands for the number of elements of $\mathcal{S}$ not exceeding x, and $\mathcal{S}(x)/x$ is calculated to the fourth decimal place.

i	$x_i = 10000i$	$\mathcal{S}(x_i)$	$\mathcal{S}(x_i) - \mathcal{S}(x_{i-1})$	$\mathcal{S}(x_i)/x_i$
1	10000	8433	8433	0.8433
2	20000	16962	8529	0.8481
3	30000	25421	8459	0.8474
4	40000	33857	8436	0.8464
5	50000	42319	8462	0.8464
6	60000	50759	8440	0.8460
7	70000	59248	8489	0.8464
8	80000	67712	8464	0.8464
9	90000	76175	8463	0.8464
10	100000	84664	8489	0.8466
11	110000	93151	8487	0.8468
12	120000	101630	8479	0.8469
13	130000	110111	8481	0.8470
14	140000	118583	8472	0.8470
15	150000	127063	8480	0.8471
16	160000	135557	8494	0.8472
17	170000	144042	8485	0.8473
18	180000	152538	8496	0.8474
19	190000	160992	8454	0.8473
20	200000	169444	8452	0.8472
21	210000	177901	8457	0.8471
22	220000	186366	8465	0.8471
23	230000	194824	8458	0.8470
24	240000	203283	8459	0.8470
25	250000	211722	8439	0.8469
26	260000	220196	8474	0.8469
27	270000	228654	8458	0.8469
28	280000	237096	8442	0.8468
29	290000	245530	8434	0.8467
30	300000	253975	8445	0.8466.

We can generalize Granville's construction of the set $\mathcal{S}$ in the following way. Fix $\alpha > 0$ and define the set $\mathcal{S}_\alpha$ recursively: Let $1 \in \mathcal{S}_\alpha$ and for $n > 1$ let $n \in \mathcal{S}_\alpha$ if

$$s_\alpha(n) := \sum_{d|n, d<n, d\in\mathcal{S}_\alpha} d \ \le \ \alpha n.$$

Further we shall say that n is $\mathcal{S}_\alpha$-perfect if $s_\alpha(n) = n$. Let

$$\mathcal{S}_\alpha(x) := \sum_{n\le x, n\in\mathcal{S}_\alpha} 1.$$

We can show that

$$\mathcal{S}_\alpha(x) \gg_\alpha x. \tag{7}$$

Namely by Theorem 5.6 of P. Elliott [**3**] we have, uniformly for $z > 0$,

$$\sum_{n \le x, \sigma(n) \le zn} 1 = h(z)x + O\left(\frac{x \log\log x}{\log x \log\log\log x}\right) \tag{8}$$

for a suitable distribution function $h(z)$ (> 0). But if $\sigma(n) \le (\alpha+1)n$, then

$$\sum_{d|n, d<n, d\in\mathcal{S}_\alpha} d \le \sum_{d|n, d<n} d = \sigma(n) - n \le \alpha n,$$

hence n is counted by the sum in (8) with $z = \alpha + 1$ and (7) follows from (8). We state now the following

CONJECTURE. There is a constant $C_\alpha > 0$ such that

$$\mathcal{S}_\alpha(x) \sim C_\alpha x \qquad (x \to \infty). \tag{9}$$

In other words the density of the set $\mathcal{S}_\alpha$ should exist for any given $\alpha > 0$.

Problem 1. If (9) holds, for which $\alpha > 0$ is $C_\alpha > h(\alpha+1)$?

It is obvious that $\mathcal{S}_\alpha = \mathcal{S}$ if $\alpha = 1$, that is, Granville's construction is the case $\alpha = 1$ of the general one. Since we have shown that (in the case of $\mathcal{S}$) $\underline{\delta} > h(2)$, the answer to the above problem is affirmative in case $\alpha = 1$ if the density of $\mathcal{S}$ exists. The distribution of $\mathcal{S}_\alpha$-perfect numbers seems hard, and the set of $\mathcal{S}_\alpha$-perfect numbers must be very "thin", although we have no ideas how to obtain any quantitative result in this direction, even when $\alpha = 1$. Even proving the assertion that their density is zero, which seems very plausible, seems difficult. Namely it is well-known that the density of ordinary perfect numbers is zero (see e.g. P. Erdős [**4**], [**5**]). Actually in [**4**] Erdős proves that, for any $\alpha > 1$, one has uniformly

$$\sum_{\sigma(n)/n=\alpha} \frac{1}{n} \ll 1.$$

This result (for $\alpha = 2$) is obviously stronger than the statement that perfect numbers have density zero. In analogy with the above bound of Erdős we propose

Problem 2. Is it true that for any given $\alpha > 0$ we have uniformly

$$\sum_{s(n)/n=\alpha} \frac{1}{n} \ll 1, \quad s(n) := \sum_{d|n, d<n, d\in\mathcal{S}} d?$$

In fact it is known that the number of $n \le x$ for which $\sigma(n)/n = \alpha$ is $\ll_\varepsilon x^\varepsilon$ uniformly for all α (see [**7**], [**10**]), and very likely this bound holds also for the number of $n \le x$ for which $s(n)/n = \alpha$.

Let $\mathcal{S}_\alpha^*$ denote the set of $\mathcal{S}_\alpha$-perfect numbers. $\mathcal{S}_\alpha^*$ can be non-empty only if α is a rational number. Determining for which given rational α the set $\mathcal{S}_\alpha^*$ is non-empty is difficult, and even more difficult is to determine whether $\mathcal{S}_\alpha^*$ is infinite for a given α, if it is known that it is non-empty. Since $\mathcal{S}_1^*$ coincides with the set of $\mathcal{S}$-perfect numbers, Theorem 3 shows that $\mathcal{S}_1^*$ is infinite, and this fact is given in a quantitative form by (5). We shall prove now the infinitude of $\mathcal{S}_{\frac{1}{2}}^*$ and $\mathcal{S}_{\frac{3}{2}}^*$, moreover the proof will show that bounds analogous to (5) hold for the counting functions of $\mathcal{S}_\alpha$-perfect numbers when $\alpha = \frac{1}{2}$ or $\alpha = \frac{3}{2}$. The result is

THEOREM 5. *For $k = 0, 1, 2, \ldots$ we have*

$$4 \cdot 3^{2k+1} \in \mathcal{S}_{\frac{1}{2}}^*, \quad 2^3 \cdot 3 \cdot 5^k \in \mathcal{S}_{\frac{3}{2}}^*. \tag{10}$$

Proof. The first inclusion in (10) will follow from

$$4 \cdot 3^{2k} \notin \mathcal{S}_{\frac{1}{2}}^*, \quad 4 \cdot 3^{2k+1} \in \mathcal{S}_{\frac{1}{2}}^* \quad (k = 0, 1, 2, \ldots); \tag{11}$$

note that also $2 \in \mathcal{S}_{\frac{1}{2}}^*$. We prove (11) by induction. Clearly $4 \notin \mathcal{S}_{\frac{1}{2}}$, so that the first assertion in (11) is true for $k = 0$. Obviously $4 \cdot 3 \in \mathcal{S}_{\frac{1}{2}}^*, 3^\alpha \in \mathcal{S}_{\frac{1}{2}}$ and $2 \cdot 3^\alpha \notin \mathcal{S}_{\frac{1}{2}}$ for $\alpha = 1, 2, \ldots$. Suppose now that (11) is true for all non-negative integers less than k. Let $s_\alpha(n) = \sum_{d|n, d<n, d\in\mathcal{S}_\alpha} d$ as before. Then we obtain

$$\begin{aligned} s_{\frac{1}{2}}(4 \cdot 3^{2k+1}) &= 1 + 3 + 3^2 + \ldots + 3^{2k+1} + 2 + 4 \cdot 3 + 4 \cdot 3^3 + \ldots + 4 \cdot 3^{2k-1} \\ &= \frac{3^{2k+2} - 1}{2} + 2 + 4 \cdot 3 \frac{9^k - 1}{9 - 1} = 2 \cdot 3 \cdot 9^k = 2 \cdot 3^{2k+1}, \end{aligned}$$

yielding $4 \cdot 3^{2k+1} \in \mathcal{S}_{\frac{1}{2}}^*$. On the other hand

$$\begin{aligned} s_{\frac{1}{2}}(4 \cdot 3^{2k}) &= 1 + 3 + 3^2 + \ldots + 3^{2k} + 2 + 4 \cdot 3 + 4 \cdot 3^3 + \ldots + 4 \cdot 3^{2k-1} \\ &= \frac{3^{2k+1} - 1}{2} + 2 + 4 \cdot 3 \cdot \frac{9^k - 1}{9 - 1} = 3 \cdot 9^k > \frac{4 \cdot 3^{2k}}{2}, \end{aligned}$$

which implies that $4 \cdot 3^{2k} \notin \mathcal{S}_{\frac{1}{2}}^*$.

To prove the second inclusion in (10) note that $\sigma(n) \le \frac{5}{2}n$ implies that $n \in \mathcal{S}_{\frac{3}{2}}$. Thus we can readily check that for $\alpha = 0, 1, 2, \ldots$ the following numbers are in $\mathcal{S}_{\frac{3}{2}}$:

$$5^\alpha, \quad 2 \cdot 5^\alpha, \quad 3 \cdot 5^\alpha, \quad 2^2 \cdot 5^\alpha, \quad 2 \cdot 3 \cdot 5^\alpha, \quad 2^3 \cdot 5^\alpha, \quad 2^2 \cdot 3.$$

However, $2^2 \cdot 3 \cdot 5^\alpha \notin \mathcal{S}_{\frac{3}{2}}$ for $\alpha \in \mathbb{N}$. This can be seen by induction, the case $\alpha = 1$ being easy. If the assertion is true for natural numbers less than α, then

$$\begin{aligned} s_{\frac{3}{2}}(2^2 \cdot 3 \cdot 5^\alpha) &= \sigma(2^2 \cdot 3 \cdot 5^\alpha) - 2^2 \cdot 3(5 + \ldots + 5^\alpha) \\ &= 7 \cdot 4 \cdot \frac{5^{\alpha+1} - 1}{4} - 2^2 \cdot 3 \cdot 5 \cdot \frac{5^\alpha - 1}{4} \\ &= 20 \cdot 5^\alpha + 8 > \frac{3}{2} \cdot 2^2 \cdot 3 \cdot 5^\alpha, \end{aligned}$$

hence $2^2 \cdot 3 \cdot 5^\alpha \notin \mathcal{S}_{\frac{3}{2}}$. On the other hand $2^3 \cdot 3 \cdot 5 \in \mathcal{S}^*_{\frac{3}{2}}$. Therefore by induction we have, supposing that $2^3 \cdot 3 \cdot 5^\alpha \in \mathcal{S}^*_{\frac{3}{2}}$ for $1 \le \alpha < k$,

$$\begin{aligned} s_{\frac{3}{2}}(2^3 \cdot 3 \cdot 5^k) &= \sigma(2^3 \cdot 3 \cdot 5^k) - 2^3 \cdot 3 \cdot 5^k - 2^2 \cdot 3(5 + 5^2 + \ldots + 5^k) \\ &= 36 \cdot 5^k = \frac{3}{2} \cdot 2^3 \cdot 3 \cdot 5^k, \end{aligned}$$

implying that $2^3 \cdot 3 \cdot 5^k \in \mathcal{S}^*_{\frac{3}{2}}$. This completes the proof of Theorem 5.

Because of the recursive nature of the set $\mathcal{S}$, it is generally difficult to establish if a given integer belongs to $\mathcal{S}$ or not. Nevertheless, as we mentioned in our opening remarks, we have found the first three consecutive integers which are not in $\mathcal{S}$. This is the object of our last theorem and it serves as an introduction to our third problem.

Theorem 6. *The smallest three consecutive integers not in $\mathcal{S}$ are* 171078830, 171078831 *and* 171078832.

Proof. Clearly any three consecutive integers not in $\mathcal{S}$ are necessarily abundant. But using a computer one can establish that the smallest abundant integer n such that both $n+1$ and $n+2$ are also abundant is 171078830. For these three numbers we have

$$\begin{aligned} &n = 171078830 = 2 \cdot 5 \cdot 13 \cdot 23 \cdot 29 \cdot 1973; \quad \sigma(n) - 2n = 16004900; \\ &n+1 = 171078831 = 3^3 \cdot 7 \cdot 11 \cdot 19 \cdot 61 \cdot 71; \quad \sigma(n+1) - 2(n+1) = 677538; \\ &n+2 = 171078832 = 2^4 \cdot 31 \cdot 344917; \quad \sigma(n+2) - 2(n+2) = 992. \end{aligned}$$

Among the 63 proper divisors of n, only 5 are abundant, namely 2990, 3770, 86710, 5899270 and 7438210. The other 58 are therefore all in $\mathcal{S}$. Since the sum of these 5 numbers is 13430950, which is smaller that 16004900, this proves that $n \notin \mathcal{S}$. All of the 127 proper divisors of $n+1$ being deficient, they are all in $\mathcal{S}$, and their sum exceeds $n+1$, which proves that $n+1 \notin \mathcal{S}$. Finally, among the 19 proper divisors of $n+2$, only one, namely 496, is abundant. The other 18 are therefore all in $\mathcal{S}$ and their sum exceeds $n+2$ by $992 - 496 = 496$, which proves that $n+2 \notin \mathcal{S}$. This ends the proof of Theorem 6.

Using the Chinese Remainder Theorem and the fact that $\prod_p (1 + \frac{1}{p})$ diverges, one can easily show that for each integer $k \ge 2$, there exist infinitely many k-tuples $(n, n+1, \ldots, n+k-1)$ made of non-deficient numbers. In particular, if $a_1 < a_2 < \ldots$ stands for the sequence of deficient numbers, then $\limsup_{r\to\infty}(a_{r+1} - a_r) = +\infty$. Note that if follows trivially from the fact that the density of deficient numbers exceeds $1/2$ that $\liminf_{r\to\infty}(a_{r+1} - a_r) = 1$. Similarly if $s_1 < s_2 < \ldots$ stand for the elements of $\mathcal{S}$, then since the lower density of $\mathcal{S}$ is $> 1/2$, then $\liminf_{r\to\infty}(s_{r+1} - s_r) = 1$. The following problem is about the value of $\limsup_{r\to\infty}(s_{r+1} - s_r)$.

Problem 3. Let $s_1 < s_2 < \dots$ stand for the elements of $\mathcal{S}$. Prove that

$$\limsup_{r\to\infty} (s_{r+1} - s_r) = +\infty.$$

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