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NORMAL FLOWS AND HARMONIC MANIFOLDS

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Abstract. We prove that a 2-stein space equipped with a non-vanishing vector field ξ such that the ξ-sectional curvature is pointwise constant is a space of constant sectional curvature. From this it then follows that a harmonic space equipped with a unit Killing vector field such that its flow is normal, has constant sectional curvature.

Introduction

A Riemannian manifold (M,g) such that every small geodesic sphere is a constant mean curvature hypersurface is called a harmonic space [**2**], [**3**], [**10**], [**14**]. Riemannian manifolds which are locally isometric to a two-point homogeneous space are trivial examples and the fundamental conjecture of Lichnerowicz stated that the converse holds. As shown by Z. I. Szabó [**11**], [**12**], this conjecture holds for compact (M,g) with finite fundamental group and for complete (M,g) with non-negative scalar curvature. However the conjecture fails to be true in general, even for complete (M,g). This was shown in [**4**] where the authors provided an infinite number of non-symmetric solvable Lie groups equipped with a harmonic metric. Such spaces are now called Damek-Ricci spaces. We refer to [**1**] for a detailed study of the rich geometry of these DR-spaces and for further references.

This remarkable result makes the study of harmonic spaces much more interesting. The global and local classification of harmonic spaces is far for being achieved and since the only known examples are all locally homogeneous, it is an intriguing question whether each harmonic space has to be locally homogeneous or not. We note that this question is still open, even inside the class of Kähler or quaternionic Kähler manifolds. On the other hand, and as is shown in [**5**], Sasakian

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harmonic manifolds are spaces of constant curvature 1. (See also [**5**] for some extensions of this result to more general classes of almost contact metric manifolds.)

Sasakian manifolds are endowed with a unit Killing vector field. In a series of papers, M. C. González-Dávila and the authors have studied the geometry of Riemannian manifolds equipped with such a vector field, generalizing in this way many aspects of Sasakian geometry to what they called flow geometry. See for example [**6**], [**7**]. In this paper we continue this work and prove that any harmonic (M,g) equipped with a unit Killing vector field whose flow is normal (see Section 3 for the definition) is a space of non-negative constant sectional curvature. To derive this result we will first prove the more or less immediate result which states that a 2-stein Riemannian manifold (see Section 2) of dimension > 2 and equipped with a non-vanishing vector field ξ such that the sectional curvature of all two-planes containing ξ is pointwise constant, must have constant sectional curvature.

2-stein spaces and harmonicity

Let (M,g) be an n-dimensional, connected, smooth Riemannian manifold and denote by ∇ its Levi Civita connection. Further, let R, ρ and τ be its associated curvature tensor, Ricci tensor and scalar curvature, respectively.

An Einstein manifold, that is, $\rho = \lambda g$, $\lambda = \dfrac{\tau}{n}$, is said to be a 2-*stein space* if

$$\sum_{a,b=1}^{n} R^2_{xaxb} = \mu g(x,x)^2 \tag{2.1}$$

for any tangent vector x at m and all $m \in M$. Here, $R_{xaxb} = g(R_{xa}x, b)$ and $\{e_a, a = 1, \dots, n\}$ is an arbitrary orthonormal basis of the tangent space $T_m M$. In this case we have

$$\mu = \frac{1}{n(n+2)}\Big(\frac{3}{2}\|R\|^2 + \|\rho\|^2\Big) \tag{2.2}$$

(see for example [**2**], [**3**]).

As mentioned in the Introduction, an (M,g) is said to be a *harmonic manifold* if all geodesic spheres of sufficiently small radius are constant mean curvature hypersurfaces. Any harmonic manifold is a 2-stein space [**2**], [**3**].

Now, we prove

THEOREM 2.1. *Let* (M,g), $\dim M \geq 3$, *be a* 2*-stein space equipped with a non-vanishing vector field* ξ *such that the sectional curvature of the two-planes containing* ξ *is pointwise constant. Then* (M,g) *is a space of constant curvature.*

Proof. Since (M,g) is Einsteinian, we have

$$\rho(\xi,\xi) = \frac{\tau}{n} g(\xi,\xi) = \sum_{a=1}^{n} R_{\xi a \xi a} = (n-1)c g(\xi,\xi)$$

where c is the pointwise and hence, globally constant ξ-sectional curvature. So, we get

$$\|\rho\|^2 = \frac{\tau^2}{n} = n(n-1)^2c^2. \tag{2.3}$$

Then it follows at once from (2.1), (2.2) and (2.3), by putting $x = \xi$, that $\|R\|^2 = \frac{2}{n-1}\|\rho\|^2$ and, as is well-known, this yields that (M, g) is a space of constant curvature. This curvature is equal to c.

Since a 2-dimensional harmonic space has constant curvature (see, for example, [**2**], [**3**], [**15**]), we get at once

THEOREM 2.1. *A harmonic space equipped with a non-vanishing vector field ξ such that the ξ-sectional curvature is pointwise constant, is a space of constant curvature.*

Normal flows and harmonicity

Now we turn to the consideration of Riemannian manifolds equipped with a normal flow and prove our main result. We first collect some basic material and refer to [**6**], [**7**], [**8**] for more details.

Let (M, g) be as in Section 2 and note that we take R with the sign convention

$$R_{UV} = \nabla_{[U,V]} - [\nabla_U, \nabla_V]$$

for all $U, V \in \mathfrak{X}(M)$, the Lie algebra of smooth vector fields on M. Further, let (M, g) be equipped with an *isometric* flow [**13**] $\mathfrak{F}_\xi$ generated by a unit Killing vector field ξ. Vectors orthogonal to ξ are called *horizontal* vectors.

Next, put $HU = -\nabla_U \xi$ and $h(U, V) = g(HU, V)$ for all $U, V \in \mathfrak{X}(M)$. Since ξ is a Killing vector field, it follows that h is skew-symmetric and moreover, $h = -d\eta$ where η is the metric dual one-form of ξ. Further, we have

$$R(X, \xi, Y, \xi) = g(HX, HY) = g(-H^2X, Y). \tag{3.1}$$

This implies that the ξ-sectional curvature $K(X, \xi)$ is non-negative and since $H\xi = 0$, $K(X, \xi) = 0$ for all horizontal X if and only if $h = 0$, that is, the horizontal distribution is integrable. In that case, (M, g) is locally a Riemannian product of an $(n-1)$-dimensional space and a line. Moreover, $K(X, \xi) > 0$ for all horizontal X if and only if H is of maximal rank $n-1$ or equivalently, η is a contact form.

An isometric flow $\mathfrak{F}_\xi$ determines locally a Riemannian submersion. In fact, for each point m in M, let $\mathcal{U}$ be a small open neighborhood of m such that ξ is regular on $\mathcal{U}$. Then the mapping $\pi : \mathcal{U} \to \tilde{\mathcal{U}} = \mathcal{U}/\xi$ is a submersion. Further, let $\tilde{g}$ denote the induced metric on $\tilde{\mathcal{U}}$ given by

$$(\tilde{g}(\tilde{X}, \tilde{Y}))^* = g(\tilde{X}^*, \tilde{Y}^*)$$

for $\tilde{X}, \tilde{Y} \in \mathfrak{X}(\tilde{\mathcal{U}})$ and where $\tilde{X}^*, \tilde{Y}^*$ denote the horizontal lifts of $\tilde{X}, \tilde{Y}$ with respect to the distribution on $\mathcal{U}$ determined by $\eta = 0$. Then $\pi : (\mathcal{U}, g_{|\mathcal{U}}) \to (\tilde{\mathcal{U}}, \tilde{g})$ is a Riemannian submersion. The Levi Civita connections ∇, $\tilde{\nabla}$ of g, $\tilde{g}$, respectively, are related by

$$\nabla_{\tilde{X}^*} \tilde{Y}^* = (\tilde{\nabla}_{\tilde{X}} \tilde{Y})^* + h(\tilde{X}^*, \tilde{Y}^*)\xi \tag{3.2}$$

for all $\tilde{X}, \tilde{Y} \in \mathfrak{X}(\tilde{\mathcal{U}})$ and the Riemannian curvature tensor $\tilde{R}$ of $\tilde{\nabla}$ is given by

$$\begin{aligned}(\tilde{R}_{\tilde{X}\tilde{Y}}\tilde{Z})^* = R_{\tilde{X}^*\tilde{Y}^*}\tilde{Z}^* &+ 2h(\tilde{X}^*, \tilde{Y}^*)H\tilde{Z}^* \\ &+ \Big\{(\nabla_{\tilde{X}^*} h)(\tilde{Y}^*, \tilde{Z}^*) - (\nabla_{\tilde{Y}^*} h)(\tilde{X}^*, \tilde{Z}^*)\Big\}\xi \\ &+ h(\tilde{X}^*, \tilde{Z}^*)H\tilde{Y}^* - h(\tilde{Y}^*, \tilde{Z}^*)H\tilde{X}^*\end{aligned} \tag{3.3}$$

for all $\tilde{X}, \tilde{Y}, \tilde{Z} \in \mathfrak{X}(\tilde{\mathcal{U}})$. From this we then get

$$(\tilde{\rho}(\tilde{X}, \tilde{Y}))^* = \rho(\tilde{X}^*, \tilde{Y}^*) + 2g(H\tilde{X}^*, H\tilde{Y}^*), \tag{3.4}$$

$$\tilde{\tau}^* = \tau + \rho(\xi, \xi). \tag{3.5}$$

Now, the flow $\mathfrak{F}_\xi$ is said to be *normal* if for all horizontal X, Y, the transformations R_{XY} leave the horizontal subspaces of the flow $\mathfrak{F}_\xi$ invariant or equivalently, $R(X, Y, X, \xi) = 0$. Then, from (3.1), we get

$$(\nabla_U H)V = g(HU, HV)\xi + \eta(V)H^2 U \tag{3.6}$$

for all $U, V \in \mathfrak{X}(M)$ and in this case R satisfies

$$\begin{aligned}R_{UV}\xi &= \eta(V)H^2U - \eta(U)H^2V, \\ R_{U\xi}V &= g(HU, HV)\xi + \eta(V)H^2U.\end{aligned} \tag{3.7}$$

This yields $\rho(X, \xi) = 0$ for each horizontal X and moreover, (3.1) and (3.6) yield that $\rho(\xi, \xi)$ is a non-negative constant.

Next, for a normal flow $\mathfrak{F}_\xi$, (3.3) reduces to

$$\begin{aligned}(\tilde{R}_{\tilde{X}\tilde{Y}}\tilde{Z})^* = R_{\tilde{X}^*\tilde{Y}^*}\tilde{Z}^* &- g(H\tilde{Y}^*, \tilde{Z}^*)H\tilde{X}^* \\ &+ g(H\tilde{X}^*, \tilde{Z}^*)H\tilde{Y}^* + 2g(H\tilde{X}^*, \tilde{Y}^*)H\tilde{Z}^*\end{aligned} \tag{3.8}$$

for $\tilde{X}, \tilde{Y}, \tilde{Z} \in \mathfrak{X}(\tilde{\mathcal{U}})$ and from (3.4) we get

$$\Big((\tilde{\nabla}_{\tilde{X}}\tilde{\rho})(\tilde{Y}, \tilde{Z})\Big)^* = (\nabla_{\tilde{X}^*}\rho)(\tilde{Y}^*, \tilde{Z}^*). \tag{3.9}$$

Finally, let $\tilde{H}$ be the $(1,1)$-tensor field on $\tilde{\mathcal{U}}$ defined by

$$\tilde{H}\tilde{X} = \pi_* H\tilde{X}^*.$$

Then $\tilde{H}$ is skew-symmetric and it follows that $\mathfrak{F}_\xi$ is normal if and only $\tilde{\nabla}\tilde{H} = 0$. Further, on $\tilde{\mathcal{U}}$ we have in that case [**8**]:

$$\tilde{R}_{\tilde{H}\tilde{X}\tilde{Y}} = \tilde{R}_{\tilde{H}\tilde{Y}\tilde{X}}. \tag{3.10}$$

Now, we state and prove our main results. We always suppose $\dim M \geq 3$.

Theorem 3.1 *Let (M,g) be a Riemannian manifold equipped with a normal flow. If (M,g) is a 2-stein space, then it is a space of (non-negative) constant sectional curvature.*

From this result we then get at once

Corollary 3.1. *A harmonic manifold which is equipped with a normal flow is a space of (non-negative) constant sectional curvature.*

To prove Theorem t3.1 we first consider

Lemma 3.1. *Let (M,g) be an Einstein manifold equipped with a normal flow $\mathfrak{F}_\xi$ and let $\pi : \mathcal{U} \to \tilde{\mathcal{U}} = \mathcal{U}/\xi$ be a local Riemannian submersion determined by $\mathfrak{F}_\xi$. If $\tilde{\mathcal{U}}$ is locally irreducible, then $\tilde{\mathcal{U}}$ is an Einstein manifold and the ξ-sectional curvature is constant on $\mathcal{U}$.*

Proof. Since ρ is parallel, it follows from (3.9) that $\tilde{\mathcal{U}}$ has parallel Ricci tensor and because of the local irreducibility, it is an Einstein space. Hence, (3.4) yields

$$2g(H^2\tilde{X}^*,\tilde{Y}^*) = \Big(\frac{\tau}{n} - \frac{\tilde{\tau}}{n-1}\Big)g(\tilde{X}^*,\tilde{Y}^*)$$

for all $\tilde{X},\tilde{Y} \in \mathfrak{X}(\tilde{\mathcal{U}})$. So $H^2 = c^2(-I + \eta\otimes\xi)$ on $\mathcal{U}$ and from (3.5) we get $c^2 = \tau/n(n-1) = \text{const}$. Hence, the ξ-sectional curvature is constant and equal to c^2. □

Now, we proceed with the

Proof of Theorem 3.1. We shall prove that the 2-stein space (M,g), equipped with a normal flow $\mathfrak{F}_\xi$, has pointwise constant ξ-sectional curvature. Then the result follows at once from Theorem 2.1.

So, assume the contrary, that is, suppose that the ξ-sectional curvature is not pointwise constant. Then there exists a point m in M such that the ξ-sectional curvature at m is not constant. In this case it follows from Lemma l3.1 that there exists a small open neighborhood $\mathcal{U}$ of m such that $\tilde{\mathcal{U}} = \mathcal{U}/\xi$ is reducible and we may write $\tilde{\mathcal{U}} = \tilde{\mathcal{U}}_1 \times \cdots \times \tilde{\mathcal{U}}_r$ where $\tilde{\mathcal{U}}_i$ is an Einstein space for each $i = 1,\ldots,r$. Put $\dim\tilde{\mathcal{U}}_i = n_i$ and denote by $\tilde{\tau}_i$, $i = 1,\ldots,r$, the scalar curvature of $\tilde{\mathcal{U}}_i$. Then

$\sum\limits_{i=1}^{r} n_i = n-1$ and $\sum\limits_{i=1}^{r} \tilde{\tau}_i = \tilde{\tau}$. Moreover, we may assume that $\dfrac{\tilde{\tau}_i}{n_i} \neq \dfrac{\tilde{\tau}_j}{n_j}$ for $i \neq j$. Applying (3.4) again, we get

$$2g(H^2\tilde{X}^*, \tilde{Y}^*) = \sum_{i=1}^{r} \Big(\frac{\tau}{n} - \frac{\tilde{\tau}_i}{n_i}\Big) g(\tilde{X}_i^*, \tilde{Y}_i^*)$$

for all $\tilde{X} = \sum\limits_{i=1}^{r} \tilde{X}_i$ and $\tilde{Y} = \sum\limits_{i=1}^{r} \tilde{Y}_i$ of $\mathfrak{X}(\tilde{\mathcal{U}})$. Hence, $H^2\tilde{X}_i^* = -c_i^2 \tilde{X}_i^*$ for $i = 1, \ldots, r$ and where c_i^2 is the ξ-sectional curvature $K(\tilde{X}_i^*, \xi)$ given by

$$2c_i^2 = \frac{\tilde{\tau}_i}{n_i} - \frac{\tau}{n} \tag{3.11}$$

and, because of our assumption, we have $c_i^2 \neq c_j^2$, $i \neq j$.

Next, we note that (3.1) implies

$$\operatorname{tr} H^2 = -\frac{\tau}{n} \tag{3.12}$$

and with (2.1), (2.2) we also obtain

$$\operatorname{tr} H^4 = \mu = \frac{1}{n(n+2)} \Big\{ \frac{3}{2} \|R\|^2 + \|\rho\|^2 \Big\}. \tag{3.13}$$

Further, let u be an arbitrary unit *horizontal* vector at $m \in \mathcal{U}$ and denote its projection on $\tilde{\mathcal{U}}$ also by u. Let $\{e_i, i = 1, \ldots, n\}$ be an orthonormal basis of $T_m M$ such that $e_n = \xi$. From (3.8) we get

$$\tilde{R}_{uaub} = R_{uaub} + 3g(Hu, e_a) g(Hu, e_b)$$

for $a, b \in \{1, \ldots, n-1\}$. Hence, we have

$$\mu = \sum_{a,b=1}^{n-1} \tilde{R}_{uaub}^2 - 6\tilde{R}_{u\tilde{H}uu\tilde{H}u} + 10\|\tilde{H}u\|^4. \tag{3.14}$$

Now, let $\tilde{m} = \pi(m) = (\tilde{m}_1, \ldots, \tilde{m}_r) \in \tilde{\mathcal{U}}_1 \times \cdots \times \tilde{\mathcal{U}}_r$ and let $v = v_i + v_j \in T_{\tilde{m}_i}\tilde{\mathcal{U}}_i \oplus T_{\tilde{m}_j}\tilde{\mathcal{U}}_j$, $\|v\| = 1$ and where $i, j \in \{1, \ldots, r\}$, $i \neq j$. Then we have

$$\|v_i\|^2 + \|v_j\|^2 = 1, \qquad \|\tilde{H}v\|^2 = c_i^2 \|v_i\|^2 + c_j^2 \|v_j\|^2.$$

Since the expression in (3.14) is independent of u, we take $u = \dfrac{v_i}{\|v_i\|}$ and $u = \dfrac{v_j}{\|v_j\|}$ in (3.14) and take into account that $\tilde{H}v_l$ is tangent to $\tilde{\mathcal{U}}_l$ for $l = i, j$. Summing up the obtained expressions, we get

$$\begin{aligned} \mu\Big(\|v_i\|^4 + \|v_j\|^4\Big) &= \sum_{a,b=1}^{n-1} \tilde{R}_{vavb}^2 - 6\tilde{R}_{v\tilde{H}vv\tilde{H}v} + 10\Big(\|\tilde{H}v_i\|^4 + \|\tilde{H}v_j\|^4\Big) \\ &= \mu - 20 c_i^2 c_j^2 \|v_i\|^2 \|v_j\|^2. \end{aligned}$$

Hence, we obtain

$$\mu = 10c_i^2 c_j^2. \tag{3.15}$$

From this it follows, since the ξ-sectional curvature is not constant at m, that $\tilde{\mathcal{U}}$ has exactly two factors.

So, put $\tilde{\mathcal{U}} = \tilde{\mathcal{U}_1} \times \tilde{\mathcal{U}_2}$ and let $\tilde{R}_1$, respectively $\tilde{R}_2$, denote the Riemann curvature tensor of $\tilde{\mathcal{U}_1}$, respectively $\tilde{\mathcal{U}_2}$. At $\tilde{m} = \pi(m) \in \tilde{\mathcal{U}}$ we choose an orthonormal basis $\{e_i, i = 1, \ldots, n-1\}$ such that $e_1, \ldots, e_{n_1}$ span $T_{\tilde{m}}\tilde{\mathcal{U}_1}$ and $e_{n_1+1}, \ldots, e_{n-1}$ span $T_{\tilde{m}}\tilde{\mathcal{U}_2}$. Now, let u_1 be a unit vector of $T_{\tilde{m}}\tilde{\mathcal{U}_1}$. Then we have [**3**], [**9**]

$$\int_{S^{n_1-1}(1)} \sum_{a,b=1}^{n_1} \tilde{R}^2_{1u_1au_1b} du_1 = \frac{C_{n_1-1}}{n_1(n_1+2)} \Big(\frac{3}{2} \|\tilde{R}_1\|^2 + \frac{\tilde{\tau}_1^2}{n_1} \Big),$$

$$\int_{S^{n_1-1}(1)} \tilde{R}_{1u_1\tilde{H}u_1u_1\tilde{H}u_1} du_1 = \frac{C_{n_1-1}}{n_1(n_1+2)} \sum_{a,b=1}^{n_1} \Big(\tilde{R}_{a\tilde{H}ab\tilde{H}b} + \tilde{R}_{a\tilde{H}ba\tilde{H}b} + \tilde{R}_{a\tilde{H}bb\tilde{H}a} \Big),$$

$$\begin{aligned} \int_{S^{n_1-1}(1)} \|\tilde{H}u_1\|^4 du_1 = \frac{C_{n_1-1}}{n_1(n_1+2)} \sum_{a,b=1}^{n_1} \Big\{ & \tilde{g}(\tilde{H}e_a, \tilde{H}e_a)\tilde{g}(\tilde{H}e_b, \tilde{H}e_b) \\ & + 2\tilde{g}(\tilde{H}e_a, \tilde{H}e_b)\tilde{g}(\tilde{H}e_a, \tilde{H}e_b) \Big\} \end{aligned}$$

where C_{n_1-1} denotes the volume of the unit sphere $S^{n_1-1}(1)$ in $\mathbb{R}^{n_1}$. Using the first Bianchi identity and (3.10) we then get

$$\int_{S^{n_1-1}(1)} \tilde{R}_{1u_1\tilde{H}u_1u_1\tilde{H}u_1} du_1 = \frac{4C_{n_1-1}}{n_1(n_1+2)} c_1^2 \tilde{\tau}_1$$

and further, we have

$$\int_{S^{n_1-1}(1)} \|\tilde{H}u_1\|^4 du_1 = \frac{C_{n_1-1}}{n_1(n_1+2)} \Big\{ (\operatorname{tr} \tilde{H}_1^2)^2 + 2 \operatorname{tr} \tilde{H}_1^4 \Big\} = c_1^4 C_{n_1-1}.$$

So, the integration of (3.14) over $S^{n_1-1}(1)$, taking $u = u_1$, gives

$$n_1(n_1+2)\mu = \frac{3}{2} \|\tilde{R}_1\|^2 + \frac{\tilde{\tau}_1^2}{n_1} - 24c_1^2\tilde{\tau}_1 + 10c_1^4 n_1(n_1+2)$$

and, doing the same for a unit vector $u_2 \in T_{\tilde{m}}\tilde{\mathcal{U}_2}$, we obtain

$$n_2(n_2+2)\mu = \frac{3}{2} \|\tilde{R}_2\|^2 + \frac{\tilde{\tau}_2^2}{n_2} - 24c_2^2\tilde{\tau}_2 + 10c_2^4 n_2(n_2+2).$$

Now, summing up these last two relations gives, using (3.12) and (3.15),

$$\frac{3}{2} \|\tilde{R}\|^2 + \frac{\tilde{\tau}_1^2}{n_1} + \frac{\tilde{\tau}_2^2}{n_2} + 10 \Big(\frac{\tau}{n}\Big)^2 - 24(c_1^2\tilde{\tau}_1 + c_2^2\tilde{\tau}_2) = (n^2 - 21)\mu.$$

Further, from (3.5) and (3.11), we get

$$\frac{\tilde{\tau}_1^2}{n_1} + \frac{\tilde{\tau}_2^2}{n_2} = 2(c_1^2\tilde{\tau}_1 + c_2^2\tilde{\tau}_2) + (n+1)\Big(\frac{\tau}{n}\Big)^2$$

and so,

$$\frac{3}{2}\|\tilde{R}\|^2 + (n+11)\Big(\frac{\tau}{n}\Big)^2 - 22(c_1^2\tilde{\tau}_1 + c_2^2\tilde{\tau}_2) = (n^2-21)\mu. \tag{3.16}$$

Finally, we express $\|\tilde{R}\|^2$ in terms of $\|R\|^2$. First, note that

$$\|R\|^2 = \sum_{\alpha,\beta,\gamma,\delta=1}^{n-1} R^2_{\alpha\beta\gamma\delta} + 4\mu.$$

Using (3.8) we obtain

$$\sum_{\alpha,\beta,\gamma,\delta=1}^{n-1} R^2_{\alpha\beta\gamma\delta} = \|\tilde{R}\|^2 + 6(\operatorname{tr} H^2)^2 + 6\operatorname{tr} H^4 - 4\sum_{\alpha,\beta,\gamma,\delta=1}^{n-1}\Big(\tilde{R}_{\alpha\beta\tilde{H}\alpha\tilde{H}\beta} + \tilde{R}_{\alpha\tilde{H}\alpha\beta\tilde{H}\beta}\Big)$$

which, by applying the first Bianchi identity and (3.10), becomes

$$\sum_{\alpha,\beta,\gamma,\delta=1}^{n-1} R^2_{\alpha\beta\gamma\delta} = \|\tilde{R}\|^2 + 6(\operatorname{tr} H^2)^2 + 6\operatorname{tr} H^4 - 12\sum_{\alpha=1}^{n-1}\tilde{\rho}_{\tilde{H}\alpha\tilde{H}\alpha}.$$

So, taking into account (3.12) and (3.13), we have

$$\|\tilde{R}\|^2 = \|R\|^2 + 12(c_1^2\tilde{\tau}_1 + c_2^2\tilde{\tau}_2) - 6\Big(\frac{\tau}{n}\Big)^2 - 10\mu.$$

Form this we see that (3.16) may be written as

$$(n+3)\mu + \Big(\frac{\tau}{n}\Big)^2 - 2(c_1^2\tilde{\tau}_1 + c_2^2\tilde{\tau}_2) = 0$$

or, using (3.12), (3.13), as

$$(n+3)(n_1c_1^4 + n_2c_2^4) + (n_1c_1^2 + n_2c_2^2)^2 - 2(c_1^2\tilde{\tau}_1 + c_2^2\tilde{\tau}_2) = 0. \tag{3.17}$$

Further, (3.11) and (3.12) yield

$$\frac{\tilde{\tau}_1}{n_1} = (n_1+2)c_1^2 + n_2c_2^2, \qquad \frac{\tilde{\tau}_2}{n_2} = n_1c_1^2 + (n_2+2)c_2^2$$

and with this, (3.17) yields

$$n_1 n_2 (c_1^2 - c_2^2)^2 = 0.$$

Hence, it follows that $c_1^2 = c_2^2$ which contradicts the hypothesis that the ξ-sectional curvature is not constant at m.

This completes the proof of the theorem. □

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