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EQUITORSION GEODESIC MAPPINGS OF GENERALIZED RIEMANNIAN SPACES

Svetislav M. Minčić and **Mića S. Stanković**

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Abstract. We define an equitorsion geodesic mapping of two generalized Riemannian spaces and obtain some invariant geometric objects of this mapping, generalizing the Weil's tensor.

0. Introduction

A generalized Riemannian space GR_N in the sense of Eisenhart's definition [**1**] is a differentiable N-dimensional manifold, equipped with nonsymmetric basic tensor g_{ij}. Consider two N-dimensional generalized Riemannian spaces GR_N and $G\overline{R}_N$. Connexion coefficients of these spaces are generalized Cristoffel's symbols of second kind, respectively Γ^i_{jk} and $\overline{\Gamma}^i_{jk}$. Generally it is $\Gamma^i_{jk} \neq \Gamma^i_{kj}$.

One says that reciprocal one valued mapping $f : GR_N \to G\overline{R}_N$ is *geodesic* [**5**] (G-mapping), if geodesics of the space GR_N pass to geodesics of the space $G\overline{R}_N$. We can consider these spaces in the common by this mapping system of local coordinates. In the corresponding points $M(x)$ and $\overline{M}(x)$ we can put

$$\overline{\Gamma}^i_{jk}(x) = \Gamma^i_{jk}(x) + P^i_{jk}(x) \qquad (i, j, k = 1, ..., N), \tag{0.1}$$

where $P^i_{jk}(x)$ is *the deformation tensor* of the connection Γ of GR_N according to the mapping $f : GR_N \to G\overline{R}_N$.

A necessary and sufficient condition that the mapping $f : GR_N \to G\overline{R}_N$ be geodesic (see [**5**]) is that the deformation tensor P^i_{jk} from (0.1) at the mapping f has the form

$$P^i_{jk}(x) = \delta^i_j \, \psi_k(x) + \delta^i_k \, \psi_j(x) + \xi^i_{jk}(x), \tag{0.2}$$

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where

$$\psi_i(x) = \frac{1}{N+1}(\overline{\Gamma}^\alpha_{i\alpha}(x) - \Gamma^\alpha_{i\alpha}(x)),\ \xi^i_{jk}(x) = P^i_{\underset{\vee}{jk}} = \frac{1}{2}(P^i_{jk} - P^i_{kj}). \tag{0.3}$$

Notice that in GR_N we have

$$\Gamma^\alpha_{\underset{\vee}{i\alpha}} = 0, \tag{0.4}$$

(eq. (2.10) in [**5**]).

In a generalized Riemannian space one can define four kinds of covariant derivatives [**2, 3**]. For example, for a tensor a^i_j in GR_N we have

$$a^i_{j\underset{1}{|}m} = a^i_{j,m} + \Gamma^i_{\alpha m}a^\alpha_j - \Gamma^\alpha_{jm}a^i_\alpha, \quad a^i_{j\underset{2}{|}m} = a^i_{j,m} + \Gamma^i_{m\alpha}a^\alpha_j - \Gamma^\alpha_{mj}a^i_\alpha,$$
$$a^i_{j\underset{3}{|}m} = a^i_{j,m} + \Gamma^i_{\alpha m}a^\alpha_j - \Gamma^\alpha_{mj}a^i_\alpha, \quad a^i_{j\underset{4}{|}m} = a^i_{j,m} + \Gamma^i_{m\alpha}a^\alpha_j - \Gamma^\alpha_{jm}a^i_\alpha.$$

Denote by $\underset{\theta}{|}, \underset{\overline{\theta}}{|}$ a covariant derivative of the kind θ in GR_N and $G\overline{R}_N$ respectively.

In the case of the space GR_N we have five independent curvature tensors [**4**] (in [**4**] $\underset{5}{R}$ is denoted by $\underset{2}{\tilde{R}}$):

$$\begin{aligned}
\underset{1}{R}{}^i{}_{jmn} &= \Gamma^i_{jm,n} - \Gamma^i_{jn,m} + \Gamma^\alpha_{jm}\Gamma^i_{\alpha n} - \Gamma^\alpha_{jn}\Gamma^i_{\alpha m},\\
\underset{2}{R}{}^i{}_{jmn} &= \Gamma^i_{mj,n} - \Gamma^i_{nj,m} + \Gamma^\alpha_{mj}\Gamma^i_{n\alpha} - \Gamma^\alpha_{nj}\Gamma^i_{m\alpha},\\
\underset{3}{R}{}^i{}_{jmn} &= \Gamma^i_{jm,n} - \Gamma^i_{nj,m} + \Gamma^\alpha_{jm}\Gamma^i_{n\alpha} - \Gamma^\alpha_{nj}\Gamma^i_{\alpha m} + \Gamma^\alpha_{nm}(\Gamma^i_{\alpha j} - \Gamma^i_{j\alpha}),\\
\underset{4}{R}{}^i{}_{jmn} &= \Gamma^i_{jm,n} - \Gamma^i_{nj,m} + \Gamma^\alpha_{jm}\Gamma^i_{n\alpha} - \Gamma^\alpha_{nj}\Gamma^i_{\alpha m} + \Gamma^\alpha_{mn}(\Gamma^i_{\alpha j} - \Gamma^i_{j\alpha}),\\
\underset{5}{R}{}^i{}_{jmn} &= \frac{1}{2}(\Gamma^i_{jm,n} + \Gamma^i_{mj,n} - \Gamma^i_{jn,m} - \Gamma^i_{nj,m} + \Gamma^\alpha_{jm}\Gamma^i_{\alpha n} + \Gamma^\alpha_{mj}\Gamma^i_{n\alpha}\\
&\qquad - \Gamma^\alpha_{jn}\Gamma^i_{m\alpha} - \Gamma^\alpha_{nj}\Gamma^i_{\alpha m}).
\end{aligned}$$

By virtue of the geodesic mapping $GR_N \to G\overline{R}_N$ we obtain tensors $\underset{\theta}{\overline{R}}{}^i_{jmn}$ $(\theta = 1, ..., 5)$, where for example

$$\underset{1}{\overline{R}}{}^i_{jmn} = \overline{\Gamma}^i_{jm,n} - \overline{\Gamma}^i_{jn,m} + \overline{\Gamma}^\alpha_{jm}\overline{\Gamma}^i_{\alpha n} - \overline{\Gamma}^\alpha_{jn}\overline{\Gamma}^i_{\alpha m}. \tag{0.5}$$

In the case of geodesic mapping $f : A_N \to \overline{A}_N$ of the symmetric affine connection spaces A_N and $\overline{A}_N$ we have an invariant geometric object

(0.6)

$$W^i{}_{jmn} = R^i{}_{jmn} + \frac{2}{1+N}\delta^i_j\, R_{\underset{\vee}{mn}} + \frac{1}{N^2-1}\big[\delta^i_m\,(NR_{jn} + R_{nj}) - \delta^i_n\,(NR_{jm} + R_{mj})\big],$$

where $R^i{}_{jmn}$ is Riemann-Cristoffel's curvature tensor of the space A_N, and R_{jm} Richi's tensor. For Riemannian space (0.6) reduces to [**6**, p. 80]

$$W^i{}_{jmn} = R^i{}_{jmn} + \frac{1}{N-1}(\delta^i_m R_{jn} - \delta^i_n R_{jm}). \tag{0.6'}$$

The object $W^i{}_{jmn}$ is called Weil's tensor, or a tensor of projective curvature [**6**]. Having a geodesic mapping of two generalized Riemannian spaces, we can not find a generalization of Weil's tensor as an invariant of geodesic mapping in general case. For that reason we define a special geodesic mapping.

A mapping $f: GR_N \to G\overline{R}_N$ is *equitorsion geodesic mapping* (ETG mapping) if the torsion tensor of the spaces GR_N and $G\overline{R}_N$ are equival. Then from (0.1) and (0.2)

$$\xi^h_{ij}(x) = 0. \tag{0.7}$$

1. ET-projective parameter of the first kind

Using (0.1) and (0.5), we get a relation between the first kind curvature tensors of the spaces GR_N and $G\overline{R}_N$

$$\overline{R}{}_{\underset{1}{}}{}^i_{jmn} = \underset{1}{R}{}^i{}_{jmn} + P^i_{jm\underset{1}{|}n} - P^i_{jn\underset{1}{|}m} + P^\alpha_{jm}P^i_{\alpha n} - P^\alpha_{jn}P^i_{\alpha m} + 2\Gamma^\alpha_{\underset{\vee}{mn}}P^i_{j\alpha}.$$

Denoting $\underset{\theta}{\psi}_{mn} = \psi_{m\underset{\theta}{|}n} - \psi_m\psi_n \quad (\theta = 1,2)$ and substituting P with respect to (0.2), we obtain

$$\begin{aligned}\underset{1}{\overline{R}}{}^i_{jmn} = \underset{1}{R}{}^i{}_{jmn} &+ \delta^i_j\,(\underset{1}{\psi}_{mn} - \underset{1}{\psi}_{nm}) + \delta^i_m\,\underset{1}{\psi}_{jn} - \delta^i_n\,\underset{1}{\psi}_{jm} - \delta^i_m\,\xi^\alpha_{jn}\psi_\alpha\\ &+ \delta^i_n\,\xi^\alpha_{jm}\psi_\alpha + \xi^i_{jm\underset{1}{|}n} - \xi^i_{jn\underset{1}{|}m} + 2\psi_j\xi^i_{mn} + \xi^\alpha_{jm}\xi^i_{\alpha n} - \xi^\alpha_{jn}\xi^i_{\alpha m}\\ &+ 2\Gamma^i_{\underset{\vee}{mn}}\psi_j + 2\Gamma^\alpha_{\underset{\vee}{mn}}\psi_\alpha\delta^i_j + 2\Gamma^\alpha_{\underset{\vee}{mn}}\xi^i_{j\alpha}.\end{aligned} \tag{1.1}$$

From (0.7) and (1.1) we get

$$\underset{1}{\overline{R}}{}^i_{jmn} = \underset{1}{R}{}^i{}_{jmn} + \delta^i_j\,(\underset{1}{\psi}_{mn} - \underset{1}{\psi}_{nm}) + \delta^i_m\,\underset{1}{\psi}_{jn} - \delta^i_n\,\underset{1}{\psi}_{jm} + 2\Gamma^i_{\underset{\vee}{mn}}\psi_j + 2\Gamma^\alpha_{\underset{\vee}{mn}}\psi_\alpha\delta^i_j\,. \tag{1.2}$$

Contracting in (1.2) with respect to i and n and using (0.4), we obtain

$$\underset{1}{\overline{R}}_{jm} = \underset{1}{R}_{jm} - 2\underset{1}{\psi}_{\underset{\vee}{jm}} + (1-N)\underset{1}{\psi}_{jm} + 2\Gamma^\alpha_{\underset{\vee}{mj}}\psi_\alpha. \tag{1.3}$$

Here $\underset{1}{\overline{R}}_{jm}$ i $\underset{1}{R}_{jm}$ denote the first kind Richi tensors of the spaces $G\overline{R}_N$ and GR_N respectively.

From (1.3) we get

$$\underset{1}{\overline{R}}_{\underset{\vee}{jm}} = \underset{1}{R}_{\underset{\vee}{jm}} - (N+1)\underset{1}{\psi}_{\underset{\vee}{jm}} + 2\Gamma^{\alpha}_{\underset{\vee}{mj}}\psi_{\alpha}. \tag{1.4}$$

Substituting (0.3) in (1.4) we get

$$(1+N)\underset{1}{\psi}_{\underset{\vee}{jm}} = \underset{1}{R}_{\underset{\vee}{jm}} - \underset{1}{\overline{R}}_{\underset{\vee}{jm}} + \frac{2}{1+N}\Gamma^{\alpha}_{\underset{\vee}{mj}}(\overline{\Gamma}^{\beta}_{\alpha\beta} - \Gamma^{\beta}_{\alpha\beta}). \tag{1.5}$$

From (1.3), (0.3) and (1.5) we get

$$\begin{aligned}\underset{1}{\overline{R}}_{jm} = \underset{1}{R}_{jm} - \frac{1}{1+N}\Big[2\underset{1}{R}_{\underset{\vee}{jm}} - 2\underset{1}{\overline{R}}_{\underset{\vee}{jm}} + \frac{4}{1+N}\Gamma^{\alpha}_{\underset{\vee}{mj}}(\overline{\Gamma}^{\beta}_{\alpha\beta} - \Gamma^{\beta}_{\alpha\beta})\Big]\\ + (1-N)\underset{1}{\psi}_{jm} + \frac{2}{1+N}\Gamma^{\alpha}_{\underset{\vee}{mj}}(\overline{\Gamma}^{\beta}_{\alpha\beta} - \Gamma^{\beta}_{\alpha\beta}).\end{aligned}$$

Now it follows that

$$(1-N^2)\underset{1}{\psi}_{jm} = (N\underset{1}{\overline{R}}_{jm} + \underset{1}{\overline{R}}_{mj}) - (N\underset{1}{R}_{jm} + \underset{1}{R}_{mj}) + 2\Gamma^{\alpha}_{\underset{\vee}{mj}}(\overline{\Gamma}^{\beta}_{\alpha\beta} - \Gamma^{\beta}_{\alpha\beta})\frac{1-N}{1+N}. \tag{1.6}$$

Taking into account (0.3) and (1.6), we can write (1.2) in the form

$$\begin{aligned}&\underset{1}{\overline{R}}{}^{i}_{jmn} + \frac{2}{1+N}\delta^i_j\,\underset{1}{\overline{R}}_{\underset{\vee}{mn}} + \frac{1}{N^2-1}\big[\delta^i_m\,(N\underset{1}{\overline{R}}_{jn} + \underset{1}{\overline{R}}_{nj}) - \delta^i_n\,(N\underset{1}{\overline{R}}_{jm} + \underset{1}{\overline{R}}_{mj})\big]\\ &-\frac{2}{(1+N)^2}\overline{\Gamma}^{\beta}_{\alpha\beta}(2\delta^i_j\,\overline{\Gamma}^{\alpha}_{\underset{\vee}{nm}} + \delta^i_m\,\overline{\Gamma}^{\alpha}_{\underset{\vee}{nj}} - \delta^i_n\,\overline{\Gamma}^{\alpha}_{\underset{\vee}{mj}}) - \frac{2}{1+N}\overline{\Gamma}^{\beta}_{\alpha\beta}(\overline{\Gamma}^{i}_{\underset{\vee}{mn}}\delta^{\alpha}_j + \overline{\Gamma}^{\alpha}_{\underset{\vee}{mn}}\delta^i_j\,)\\ &= \underset{1}{R}{}^{i}{}_{jmn} + \frac{2}{1+N}\delta^i_j\,\underset{1}{R}_{\underset{\vee}{mn}} + \frac{1}{N^2-1}\big[\delta^i_m\,(N\underset{1}{R}_{jn} + \underset{1}{R}_{nj}) - \delta^i_n\,(N\underset{1}{R}_{jm} + \underset{1}{R}_{mj})\big]\\ &-\frac{2}{(1+N)^2}\Gamma^{\beta}_{\alpha\beta}(2\delta^i_j\,\Gamma^{\alpha}_{\underset{\vee}{nm}} + \delta^i_m\,\Gamma^{\alpha}_{\underset{\vee}{nj}} - \delta^i_n\,\Gamma^{\alpha}_{\underset{\vee}{mj}}) - \frac{2}{1+N}\Gamma^{\beta}_{\alpha\beta}(\Gamma^{i}_{\underset{\vee}{mn}}\delta^{\alpha}_j + \Gamma^{\alpha}_{\underset{\vee}{mn}}\delta^i_j\,).\end{aligned}$$

Therefore, the magnitude

$$\begin{aligned}\underset{1}{E}{}^{i}_{jmn} = \underset{1}{R}{}^{i}{}_{jmn} + \frac{2}{1+N}\delta^i_j\,\underset{1}{R}_{\underset{\vee}{mn}} + \frac{1}{N^2-1}\big[\delta^i_m\,(N\underset{1}{R}_{jn} + \underset{1}{R}_{nj}) - \delta^i_n\,(N\underset{1}{R}_{jm} + \underset{1}{R}_{mj})\big]\\ - \frac{2}{(1+N)^2}\Gamma^{\beta}_{\alpha\beta}(2\delta^i_j\,\Gamma^{\alpha}_{\underset{\vee}{nm}} + \delta^i_m\,\Gamma^{\alpha}_{\underset{\vee}{nj}} - \delta^i_n\,\Gamma^{\alpha}_{\underset{\vee}{mj}})\\ - \frac{2}{1+N}\Gamma^{\beta}_{\alpha\beta}(\Gamma^{i}_{\underset{\vee}{mn}}\delta^{\alpha}_j + \Gamma^{\alpha}_{\underset{\vee}{mn}}\delta^i_j\,),\end{aligned} \tag{1.7}$$

is invariant under an ETG mapping.

The magnitude (1.7) is not a tensor, and we call it *the ET-projective parameter of the first kind*.

2. ET-projective parameter of the second kind

For the second kind curvature tensors of the spaces GR_N and $G\overline{R}_N$ we get the relation

$$\underset{2}{\overline{R}}{}^i_{jmn} = \underset{2}{R}{}^i{}_{jmn} + P^i_{mj\underset{2}{|}n} - P^i_{nj\underset{2}{|}m} + P^\alpha_{mj}P^i_{n\alpha} - P^\alpha_{nj}P^i_{m\alpha} + 2\Gamma^\alpha_{\underset{\vee}{nm}}P^i_{\alpha j},$$

i.e., using (0.2) one obtains

$$\underset{2}{\overline{R}}{}^i_{jmn} = \underset{2}{R}{}^i{}_{jmn} + \delta^i_j\,(\underset{2}{\psi}_{mn} - \underset{2}{\psi}_{nm}) + \delta^i_m\,\underset{2}{\psi}_{jn} - \delta^i_n\,\underset{2}{\psi}_{jm} - \delta^i_m\,\xi^\alpha_{nj}\psi_\alpha + \delta^i_n\,\xi^\alpha_{mj}\psi_\alpha$$
$$+\,\xi^i_{mj\underset{2}{|}n} - \xi^i_{nj\underset{2}{|}m} + 2\psi_j\xi^i_{nm} + \xi^\alpha_{mj}\xi^i_{n\alpha} - \xi^\alpha_{nj}\xi^i_{m\alpha} + 2\Gamma^i_{\underset{\vee}{nm}}\psi_\alpha\delta^i_j + 2\Gamma^\alpha_{\underset{\vee}{nm}}\psi_j + 2\Gamma^\alpha_{\underset{\vee}{nm}}\xi^i_{\alpha j}.$$

Now, analogously to previous case, we get the invariant magnitude of the ETG mapping $f : GR_N \to G\overline{R}_N$

$$\underset{2}{E}{}^i{}_{jmn} = \underset{2}{R}{}^i{}_{jmn} + \frac{2}{1+N}\delta^i_j\,\underset{2}{R}{}_{\underset{\vee}{mn}} + \frac{1}{N^2-1}\left[\delta^i_m\,(N\underset{2}{R}{}_{jn} + \underset{2}{R}{}_{nj}) - \delta^i_n\,(N\underset{2}{R}{}_{jm} + \underset{2}{R}{}_{mj})\right]$$
$$-\,\frac{2}{(1+N)^2}\Gamma^\beta_{\alpha\beta}(2\delta^i_j\,\Gamma^\alpha_{\underset{\vee}{mn}} + \delta^i_m\,\Gamma^\alpha_{\underset{\vee}{jn}} - \delta^i_n\,\Gamma^\alpha_{\underset{\vee}{mj}}) - \frac{2}{1+N}\Gamma^\beta_{\alpha\beta}(\Gamma^i_{\underset{\vee}{nm}}\delta^\alpha_j + \Gamma^\alpha_{\underset{\vee}{nm}}\delta^i_j).$$

The magnitude $\underset{2}{E}{}^i{}_{jmn}$ is not a tensor and we call it *ET-projective parameter of the second kind*.

3. ET-projective parameter of the third kind

In the case of the third kind curvature tensors of the spaces GR_N and $G\overline{R}_N$ we get

$$\underset{3}{\overline{R}}{}^i_{jmn} = \underset{3}{R}{}^i{}_{jmn} + P^i_{jm\underset{2}{|}n} - P^i_{nj\underset{1}{|}m} + P^\alpha_{jm}P^i_{n\alpha} - P^\alpha_{nj}P^i_{\alpha m} + 2P^\alpha_{nm}\Gamma^i_{\underset{\vee}{\alpha j}} + 2P^\alpha_{nm}P^i_{\underset{\vee}{\alpha j}}$$

i.e., in virtue of (0.2)

$$\begin{aligned}\underset{3}{\overline{R}}{}^i_{jmn} &= \underset{3}{R}{}^i{}_{jmn} + \delta^i_j\,(\underset{2}{\psi}_{mn} - \underset{1}{\psi}_{nm}) + \delta^i_m\,\underset{2}{\psi}_{jn} - \delta^i_n\,\underset{1}{\psi}_{jm}\\ &+ \psi_\alpha(\delta^i_n\,\xi^\alpha_{jm} - \delta^i_m\,\xi^\alpha_{nj}) + \xi^i_{jm\underset{2}{|}n} - \xi^i_{nj\underset{1}{|}m} + \xi^\alpha_{jm}\xi^i_{n\alpha} - \xi^\alpha_{jn}\xi^i_{\alpha m}\\ &+ 2\psi_n(\Gamma^i_{\underset{\vee}{mj}} + \xi^i_{mj}) + 2\psi_m(\Gamma^i_{\underset{\vee}{nj}} + \xi^i_{nj}) + 2\xi^\alpha_{nm}(\Gamma^i_{\underset{\vee}{\alpha j}} + \xi^i_{\alpha j}).\end{aligned} \tag{3.1}$$

Also, it is satisfied

$$\underset{2}{\psi}_{mn} = \underset{1}{\psi}_{mn} + 2\Gamma^\alpha_{\underset{\vee}{mn}}\psi_\alpha. \tag{3.2}$$

From (3.1), (3.2) and (0.7) we get

$$
\begin{aligned}
\overline{\underset{3}{R}}{}^i_{jmn} = \underset{3}{R}{}^i{}_{jmn} &+ \delta^i_j\,(\underset{1}{\psi}_{mn} - \underset{1}{\psi}_{nm}) + \delta^i_m\,\underset{1}{\psi}_{jn} - \delta^i_n\,\underset{1}{\psi}_{jm} \\
&+ 2\psi_n\Gamma^i_{\underset{\vee}{mj}} + 2\psi_m\Gamma^i_{\underset{\vee}{nj}} + 2\delta^i_j\,\Gamma^\alpha_{\underset{\vee}{mn}}\psi_\alpha + 2\delta^i_m\,\Gamma^\alpha_{\underset{\vee}{jn}}\psi_\alpha .
\end{aligned}
\tag{3.3}
$$

Contracting (3.3) with respect to i and n, and alternating, we get

$$
(1+N)\underset{1}{\psi}_{\underset{\vee}{jm}} = 2\underset{3}{R}_{\underset{\vee}{jm}} - 2\overline{\underset{3}{R}}_{\underset{\vee}{jm}} + \frac{4}{1+N}(\overline{\Gamma}^\alpha_{\beta\alpha} - \Gamma^\alpha_{\beta\alpha})\Gamma^\beta_{\underset{\vee}{mj}},
\tag{3.4}
$$

From (0.3), (3.3) and (3.4) it follows

$$
(1-N^2)\underset{1}{\psi}_{jm} = (N\overline{\underset{3}{R}}_{jm} + \overline{\underset{3}{R}}_{mj}) - (N\underset{3}{R}_{jm} + \underset{3}{R}_{mj}) + 2\Gamma^\beta_{\underset{\vee}{mj}}(\overline{\Gamma}^\alpha_{\beta\alpha} - \Gamma^\alpha_{\beta\alpha})\frac{1-N}{1+N}.
\tag{3.5}
$$

Taking into account (0.3) and (3.5) we can write (3.3) in the form $\overline{\underset{3}{E}}{}^i{}_{jmn} = \underset{3}{E}{}^i{}_{jmn}$, where

$$
\begin{aligned}
\underset{3}{E}{}^i{}_{jmn} = \underset{3}{R}{}^i{}_{jmn} &+ \frac{2}{1+N}\delta^i_j\,\underset{3}{R}_{\underset{\vee}{mn}} + \frac{1}{N^2-1}\big[\delta^i_m\,(N\underset{3}{R}_{jn} + \underset{3}{R}_{nj}) - \delta^i_n\,(N\underset{3}{R}_{jm} + \underset{3}{R}_{mj})\big] \\
&- \frac{2}{(1+N)^2}\Gamma^\beta_{\alpha\beta}(2\delta^i_j\,\Gamma^\alpha_{\underset{\vee}{nm}} + \delta^i_m\,\Gamma^\alpha_{\underset{\vee}{nj}} - \delta^i_n\,\Gamma^\alpha_{\underset{\vee}{mj}}) \\
&- \frac{2}{1+N}(\Gamma^i_{\underset{\vee}{mj}}\Gamma^\alpha_{n\alpha} - \Gamma^i_{\underset{\vee}{nj}}\Gamma^\alpha_{m\alpha} - \delta^i_j\,\Gamma^\beta_{\underset{\vee}{mn}}\Gamma^\alpha_{\beta\alpha} - \delta^i_m\,\Gamma^\beta_{\underset{\vee}{jn}}\Gamma^\alpha_{\beta\alpha}).
\end{aligned}
$$

The magnitude $\underset{3}{E}{}^i{}_{jmn}$ is an invariant of the ETG mapping. We call it *ET-projective parameter of the third kind.*

4. ET-projective parameter of the fourth kind

For curvature tensors of the fourth kind we get

$$
\overline{\underset{4}{R}}{}^i_{jmn} = \underset{4}{R}{}^i{}_{jmn} + P^i_{jm\underset{2}{|}n} - P^i_{nj\underset{1}{|}m} + P^\alpha_{jm}P^i_{n\alpha} - P^\alpha_{nj}P^i_{\alpha m} + 2P^\alpha_{mn}\Gamma^i_{\underset{\vee}{\alpha j}} + 2P^\alpha_{mn}P^i_{\underset{\vee}{\alpha j}}
$$

i.e.,

$$
\begin{aligned}
\overline{\underset{4}{R}}{}^i_{jmn} = \underset{4}{R}{}^i{}_{jmn} &+ \delta^i_j\,(\underset{2}{\psi}_{mn} - \underset{1}{\psi}_{nm}) + \delta^i_m\,\underset{2}{\psi}_{jn} - \delta^i_n\,\underset{1}{\psi}_{jm} + \\
&+ \psi_\alpha(\delta^i_n\,\xi^\alpha_{jm} - \delta^i_m\,\xi^\alpha_{nj}) + \xi^i_{jm\underset{2}{|}n} - \xi^i_{nj\underset{1}{|}m} + \xi^\alpha_{jm}\xi^i_{n\alpha} - \xi^\alpha_{jn}\xi^i_{\alpha m} + \\
&+ 2\psi_n(\Gamma^i_{\underset{\vee}{mj}} + \xi^i_{mj}) + 2\psi_m(\Gamma^i_{\underset{\vee}{nj}} + \xi^i_{nj}) + 2\xi^\alpha_{mn}(\Gamma^i_{\underset{\vee}{\alpha j}} + \xi^i_{\alpha j}).
\end{aligned}
$$

In this case, analogously, we get an invariant magnitude of the ETG mapping in the form

$$\begin{aligned}\underset{4}{E}{}^i_{jmn} = \underset{4}{R}{}^i{}_{jmn} &+ \frac{2}{1+N}\delta^i_j \underset{4}{R}{}_{\underset{\vee}{mn}} + \frac{1}{N^2-1}\left[\delta^i_m \,(N\underset{4}{R}{}_{jn} + \underset{4}{R}{}_{nj}) - \delta^i_n \,(N\underset{4}{R}{}_{jm} + \underset{4}{R}{}_{mj})\right] \\ &- \frac{2}{(1+N)^2}\Gamma^\beta_{\alpha\beta}(2\delta^i_j \,\Gamma^\alpha_{\underset{\vee}{nm}} + \delta^i_m \,\Gamma^\alpha_{\underset{\vee}{nj}} - \delta^i_n \,\Gamma^\alpha_{\underset{\vee}{mj}}) \\ &- \frac{2}{1+N}(\Gamma^i_{\underset{\vee}{mj}}\Gamma^\alpha_{n\alpha} - \Gamma^i_{\underset{\vee}{nj}}\Gamma^\alpha_{m\alpha} - \delta^i_j \,\Gamma^\beta_{\underset{\vee}{mn}}\Gamma^\alpha_{\beta\alpha} - \delta^i_m \,\Gamma^\beta_{\underset{\vee}{jn}}\Gamma^\alpha_{\beta\alpha}).\end{aligned}$$

The magnitude $\underset{4}{E}{}^i{}_{jmn}$ is not a tensor and we call it *ET-projective parameter of the fourth kind* of the ETG mapping.

5. ET-projective curvature tensor

For the curvature tensors of the fifth kind of the spaces GR_N and $G\overline{R}_N$ we find the relation

$$\underset{5}{\overline{R}}{}^i_{jmn} =$$

$$\underset{5}{R}{}^i{}_{jmn} + \frac{1}{2}(P^i_{jm\underset{3}{|}n} - P^i_{jn\underset{4}{|}m} + P^i_{mj\underset{4}{|}n} - P^i_{nj\underset{3}{|}m} + P^\alpha_{jm}P^i_{\alpha n} - P^\alpha_{jn}P^i_{m\alpha} + P^\alpha_{mj}P^i_{n\alpha} - P^\alpha_{nj}P^i_{\alpha m})$$

i.e.,

$$\begin{aligned}(5.1)\quad \underset{5}{\overline{R}}{}^i_{jmn} = \underset{5}{R}{}^i{}_{jmn} \\ &+ \frac{1}{2}\delta^i_j \,(\underset{3}{\psi}{}_{mn} - \underset{4}{\psi}{}_{nm} + \underset{4}{\psi}{}_{mn} - \underset{3}{\psi}{}_{nm}) + \frac{1}{2}\delta^i_m \,(\underset{3}{\psi}{}_{jn} + \underset{4}{\psi}{}_{jn}) - \frac{1}{2}\delta^i_n \,(\underset{4}{\psi}{}_{jm} + \underset{3}{\psi}{}_{jm}) \\ &+ \frac{1}{2}(\xi^i_{jm\underset{3}{|}n} - \xi^i_{jn\underset{4}{|}m} + \xi^i_{mj\underset{4}{|}n} - \xi^i_{nj\underset{3}{|}m} + \xi^\alpha_{jm}\xi^i_{\alpha n} - \xi^\alpha_{jn}\xi^i_{m\alpha} + \xi^\alpha_{mj}\xi^i_{n\alpha} - \xi^\alpha_{nj}\xi^i_{\alpha m})\end{aligned}$$

Substituting (0.6) in (5.1) we get

$$\begin{aligned}\underset{5}{\overline{R}}{}^i_{jmn} = \underset{5}{R}{}^i{}_{jmn} &+ \frac{1}{2}(\underset{1}{\psi}{}_{mn} - \underset{2}{\psi}{}_{nm} + \underset{2}{\psi}{}_{mn} - \underset{1}{\psi}{}_{nm}) \\ &+ \frac{1}{2}\delta^i_m \,(\underset{1}{\psi}{}_{jn} + \underset{2}{\psi}{}_{jn}) - \frac{1}{2}\delta^i_n \,(\underset{1}{\psi}{}_{jm} + \underset{2}{\psi}{}_{jm}).\end{aligned}\tag{5.2}$$

Denoting $\underset{12}{\psi}{}_{jn} = \frac{1}{2}(\underset{1}{\psi}{}_{jn} + \underset{2}{\psi}{}_{jn})$, we can write (5.2) in the form

$$\underset{5}{\overline{R}}{}^i_{jmn} = \underset{5}{R}{}^i{}_{jmn} + \delta^i_j \,(\underset{12}{\psi}{}_{mn} - \underset{12}{\psi}{}_{nm}) + \delta^i_m \,\underset{12}{\psi}{}_{jn} - \delta^i_n \,\underset{12}{\psi}{}_{jm}.\tag{5.3}$$

Eliminating $\underset{12}{\psi}{}_{mn}$ from (5.3), analogously to previous cases, we get

$$\underset{5}{\overline{E}}{}^i{}_{jmn} = \underset{5}{E}{}^i{}_{jmn},$$

where we denoted

$$\underset{5}{E}{}^i{}_{jmn} = \underset{5}{R}{}^i{}_{jmn} + \frac{2}{1+N}\delta^i_j \underset{5}{R}{}_{\underset{\vee}{mn}} + \frac{1}{N^2-1}\big[\delta^i_m\,(N\underset{5}{R}{}_{jn} + \underset{5}{R}{}_{nj}) - \delta^i_n\,(N\underset{5}{R}{}_{jm} + \underset{5}{R}{}_{mj})\big].$$

The magnitude $\underset{5}{E}{}^i{}_{jmn}$ is an invariant of the ETG mapping. In contrast to the previous cases, when $\underset{\theta}{E}{}^i{}_{jmn}$ $(\theta = 1, ..., 4)$ are not tensors, the magnitude $\underset{5}{E}{}^i{}_{jmn}$ is a tensor. We call it *ET-projective curvature tensor*.

If $GR_N(G\overline{R}_N)$ reduces to $R_N(\overline{R}_N)$, then the magnitudes $\underset{\theta}{E}{}^i{}_{jmn}$ $(\theta = 1, ..., 5)$ reduce to Weil's tensor $(0.6')$.

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Matematički institut
Kneza Mihaila 2
11001 Beograd, p.p. 367
Yugoslavia

Filozofski fakultet
Ćirila i Metodija 2
18000 Niš
Yugoslavia