PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 61 (75), 1997, 61–72

ON NECESSARY CONDITIONS IN THE CALCULUS OF VARIATIONS

Vladimir Janković

Communicated by Gradimir Milovanović

Abstract. We consider weak and strong local solutions of the general isoperimetric problem. That problem differs from the classical calculus of variations in the fact that among constraints both constraints of the equality and of the inequality type appear. Necessary conditions (for both types of local solutions) are obtained, with no assumptions on integrand's phase variable. In the case of the simplest problem of calculus of variations necessary condition for $\hat{x}(\cdot)$ to be the weak local solution reduces here to the following equation

$$\frac{d}{dt}[\hat{L}_{\dot{x}}(t)\dot{\hat{x}}(t)(t) - \hat{L}(t)] = \hat{L}_t(t), \quad t \in [t_0, t_1],$$

and necessary condition for $\hat{x}(\cdot)$ to be the strong local solution reduces here to the above differential equation together with the Weierstrass inequality.

1. Formulation of the problems and statements of theorems

1.1. General isoperimetric problem with weak local extreme. We shall assume that $[t_0, t_1]$ is a closed interval of the real line, x_0 and x_1 are two points from R^n, V is an open set in $R \times R^n \times R^n$ and $L_i(t, x, \dot{x})$, $i = 0, 1, \dots, m$, are continuous integrands defined on V. We define *integral functionals* on the set

$$D = \{x(\cdot) \in C^1([t_0, t_1], R^n) | \Gamma^1(x(\cdot)) \subseteq V\},$$

where

$$\Gamma^1(x(\cdot)) = \{(t, x(t), \dot{x}(t)) | t \in [t_0, t_1]\}$$

is the *extended graph of the function* $x(\cdot)$, in the following way:

$$I_i(x(\cdot)) = \int_{t_0}^{t_1} L_i(t, x(t), \dot{x}(t))dt, \quad i = 0, 1, \dots, m.$$

AMS Subject Classification (1991): Primary 49B10.

Key Words: isoperimetric problem, weak local minimum, strong local minimum, Lagrange multipliers.

The *General isoperimetric problem* is the following one:

$$(P) \qquad I_0(x(\cdot)) \to \inf; \quad \begin{aligned} &I_i(x(\cdot)) \le 0, \quad i = 1, \dots, k, \\ &I_i(x(\cdot)) = 0, \quad i = k+1, \dots, m, \\ &x(t_0) = x_0, \;\; x(t_1) = x_1. \end{aligned}$$

An admissible function $\hat{x}(\cdot)$ is a *weak local solution* of the problem (P), if there exists an $\epsilon > 0$, such that $I_0(x(\cdot)) \ge I_0(\hat{x}(\cdot))$ for each admissible function $x(\cdot)$, satisfying $\|x(\cdot) - \hat{x}(\cdot)\|_{C^1} < \epsilon$.

THEOREM 1. *Suppose integrands L_i, $i = 0, 1, \dots, m$, are continuous and have continuous partial derivatives in t and $\dot{x}$. If $\hat{x}(\cdot)$ is weak local solution of the problem (P), then there exist Lagrange multipliers $\hat{\lambda}_i \in R$, $i = 0, 1, \dots, m$, not all of them equal to zero, such that*

$$\hat{\lambda}_i \ge 0, \quad i = 0, 1, \dots, k,$$
$$\hat{\lambda}_i I_i(\hat{x}(\cdot)) = 0, \quad i = 1, \dots, k,$$

the function $t \to [\hat{L}_{\dot{x}} x(t), \dot{\hat{x}}(t) - \hat{L}(t)]$ is differentiable on $[t_0, t_1]$, and

$$\frac{d}{dt}[\hat{L}_{\dot{x}}(t)\dot{\hat{x}}(t) - \hat{L}(t)] = -\hat{L}_t(t), \quad t \in [t_0, t_1],$$

where $L(t, x, \dot{x}) = \sum\limits_{i=0}^{m} \hat{\lambda}_i L_i(t, x, \dot{x})$.

In the sequel we shall use the following abbreviations:

$$\hat{L}(t) = L(t, \hat{x}(t), \dot{\hat{x}}(t)), \;\; \hat{L}_t(t) = L_t(t, \hat{x}(t), \dot{\hat{x}}(t)), \;\; \hat{L}_{\dot{x}}(t) = L_{\dot{x}}(t, \hat{x}(t), \dot{\hat{x}}(t)), \dots$$

1.2. General isoperimetric problem with strong local extreme. We shall assume that $[t_0, t_1]$ is an interval of the real line, x_0 and x_1 are points from R^n, G is an open set in $R \times R^n$, K is a cone in R^n, $U \subseteq K$, and $L_i(t, x, \dot{x})$, $i = 0, 1, \dots, m$, are continuous integrands defined on $G \times K$. We define *integral functionals* on the set

$$\bar{D} = \{x(\cdot) \in \bar{C}^1([t_0, t_1], R^n) | \Gamma(x(\cdot)) \subseteq G, \;\; (\forall t \in [t_0, t_1]) \dot{x}_-(t), \dot{x}_+(t) \in K\}$$

where $\bar{C}^1([t_0, t_1], R^n)$ is the set of piecewise smooth functions from $[t_0, t_1]$ to R^n and

$$\Gamma(x(\cdot)) = \{(t, x(t)) | t \in [t_0, t_1]\}$$

is the *graph of the function* $x(\cdot)$, in the following way:

$$I_i(x(\cdot)) = \int_{t_0}^{t_1} L_i(t, x(t), \dot{x}(t)) dt, \;\; i = 0, 1, \dots, m.$$

We shall study the following variant of isoperimetric problem

$$(\bar{P}) \qquad \begin{aligned} I_0(x(\cdot)) \to \inf; \quad & I_i(x(\cdot)) \le 0, \ \ i = 1, \dots, k, \\ & I_i(x(\cdot)) = 0, \ \ i = k+1, \dots, m, \\ & x(t_0) = x_0, \ \ x(t_1) = x_1, \\ & \dot{x}(t) \in U, \ \ t \in T, \end{aligned}$$

where T is the set of points in which $x(\cdot)$ is differentiable.

The admissible function $\hat{x}(\cdot)$ is the *strong local solution* of the problem $(\bar{P})$, if $\epsilon > 0$ exists, such that for every admissible function $x(\cdot)$, for which $\|x(\cdot) - \hat{x}(\cdot)\|_C < \epsilon$, we have $I_0(x(\cdot)) \ge I_0(\hat{x}(\cdot))$.

THEOREM 2. *Suppose integrands L_i, $i = 0, 1, \dots, m$, are continuous and have continuous partial derivatives in t. If $\hat{x}(\cdot)$ is strong local solution of the problem $(\bar{P})$, then there exist Lagrange multipliers $\hat{\lambda}_i \in R$, $i = 0, 1, \dots, m$, not all of them equal to zero, and piecewise continuous function $\hat{r}(\cdot)\colon [t_0, t_1] \to R$, such that*

$$\begin{aligned} & \hat{\lambda}_i \ge 0, \ \ i = 0, 1, \dots, k, \\ & \hat{\lambda}_i I_i(\hat{x}(\cdot)) = 0, \ \ i = 1, \dots, k, \\ & \dot{\hat{r}}(t) = \hat{L}_t(t), \ \ t \in T, \\ & L(t, \hat{x}(t), \dot{\hat{x}}(t)/w)w - L(t, \hat{x}(t), \dot{\hat{x}}(t)) - \hat{r}(t)(w-1) \ge 0, \ \ t \in T, \ \ w \in W(t), \end{aligned}$$

where $T \subseteq [t_0, t_1]$ is the set of points in which $\hat{x}(\cdot)$ is differentiable,

$$L(t, x, \dot{x}) = \sum_{i=0}^{m} \hat{\lambda}_i L_i(t, x, \dot{x}),$$

$$W(t) = cl \bigcup_{\delta > 0} \bigcap_{t < s < t+\delta} \{w > 0 \mid \dot{\hat{x}}(s)/w \in U\}.$$

2. Comments

2.1. The equation

$$\text{(E2)} \qquad \frac{d}{dt}[L_{\dot{x}}(t, x(t), \dot{x}(t))\dot{x}(t) - L(t, x(t), \dot{x}(t))] = -L_t(t, x(t), \dot{x}(t))$$

is sometimes called the *second Euler equation*, and we shall use that name further on.

2.2. In many books on calculus of variations the validity of the equation

$$\frac{d}{dt}[\hat{L}_{\dot{x}}(t)\dot{\hat{x}}(t) - \hat{L}(t)] = -\hat{L}_t(t), \ \ t \in [t_0, t_1],$$

is proved to be a necessary condition for weak local extreme in the simplest problem of the calculus of variations, case $n = 1$, providing that integrand L is twice continuously differentiable and regular ($L_{\dot{x}\dot{x}}(t,x,\dot{x}) > 0$). Under these assumptions all extremals are twice continuously differentiable. It is not difficult to prove that for every function $x(\cdot) \in C^2[t_0,t_1]$ the following equality holds

$$\frac{d}{dt}[L_{\dot{x}}(t,x(t),\dot{x}(t))\dot{x}(t) - L(t,x(t),\dot{x}(t))] + L_t(t,x(t),\dot{x}(t))$$
$$= \dot{x}(t)\left[\frac{d}{dt}L_{\dot{x}}(t,x(t),\dot{x}(t)) - L_x(t,x(t),\dot{x}(t))\right], \ \ t \in [t_0,t_1],$$

and mentioned necessary condition immediately follows from the above equality and Euler condition.

From the equality above it is possible to obtain a sharper result: If the integrand L is twice continuously differentiable then the second Euler equation (E2) is equivalent to the (first) Euler equation

$$\frac{d}{dt}L_{\dot{x}}(t,x(t),\dot{x}(t)) = L_x(t,x(t),\dot{x}(t)),$$

on the class of functions $x(\cdot) \in C^2[t_0,t_1]$ satisfying $\dot{x}(t) \neq 0$, $t \in [t_0,t_1]$.

2.3. The *Pontryagin function* $H\colon V \times R^{n*} \times R^{m+1*} \to R$ in (P) is given by

$$H(t,x,\dot{x},p,\lambda) = p\dot{x} - \sum_{i=0}^{m}\lambda_i L_i(t,x,\dot{x}).$$

Since $\hat{p}(t) = \hat{L}_{\dot{x}}(t)$, then

$$\hat{H}(t) = \hat{L}_{\dot{x}}(t)\dot{\hat{x}}(t) - \hat{L}(t), \ \ \hat{H}_t(t) = -\hat{L}_t(t).$$

The second Euler equation can be written in the form:

$$\frac{d}{dt}\hat{H}(t) = \hat{H}_t(t), \ \ t \in [t_0,t_1].$$

It was proved in [**2**] that, assuming smoothness of all relevant functions, the preceding equation could be added to the Lagrange principle of the Lagrange problem [**1**, §4.1]. When we apply that result from [**2**] to general isoperimetric problem introduced in 1.1, we get the following proposition:

PROPOSITION. *Suppose integrands L_i, $i = 0,1,\dots,m$, are continuously differentiable. If $\hat{x}(\cdot)$ is weak local solution of the problem (P), then there exist Lagrange multipliers $\hat{\lambda}_i \in R$, $i = 0,1,\dots,m$, not all of them equal to zero, such that*

$$\hat{\lambda}_i \geq 0, \ \ i = 0,1,\dots,k,$$
$$\hat{\lambda}_i I_i(\hat{x}(\cdot)) = 0, \ \ i = 1,\dots,k,$$

the functions $t \to \hat{L}_{\dot{x}}(t)$ and $t \to [\hat{L}_{\dot{x}}(t)\dot{\hat{x}}(t) - \hat{L}(t)]$ are differentiable on $[t_0,t_1]$, and

$$\frac{d}{dt}\hat{L}_{\dot{x}}(t) = \hat{L}_x(t), \ \ t \in [t_0, t_1],$$
$$\frac{d}{dt}[\hat{L}_{\dot{x}}(t)\dot{\hat{x}}(t) - \hat{L}(t)] = -\hat{L}_t(t), \ \ t \in [t_0, t_1].$$

One of our aims is to prove that the second Euler equation is the necessary condition for weak local extremum, without the assumption that integrands L_i have partial derivatives in x – since those partial derivatives do not appear in the second Euler equation. General isoperimetric problem is the most general problem for which we succeeded to prove such a thing.

2.4. Suppose in $(\bar{P})$ integrands L_i, $i = 0, 1, \ldots, m$ are continuous and have continuous partial derivatives in t and $\dot{x}$. If $1 \in \operatorname{int} W(t)$, then in Theorem 2 we have

$$-\hat{r}(t) = \hat{L}_t(t)\dot{\hat{x}}(t) - \hat{L}(t), \ \ t \in T.$$

Consequently, from $\dot{\hat{r}}(t) = \hat{L}_t(t), \ \ t \in T$, we get

$$\frac{d}{dt}\left[\hat{L}_{\dot{x}}(t)\dot{\hat{x}}(t) - \hat{L}(t)\right] = -\hat{L}_t(t), \ \ t \in T.$$

If, additionally, we assume that $n = 1$ and $K = U = (0, +\infty)$, then the inequality

$$L(t, \hat{x}(t), \dot{\hat{x}}(t)/w)w - L(t, \hat{x}(t), \dot{\hat{x}}(t)) - \hat{r}(t)(w-1) \geq 0, \ \ t \in T, \ \ w \in W(t)$$

is equivalent to the Weierstrass inequality

$$L(t, \hat{x}(t), v) - L(t, \hat{x}(t), \dot{\hat{x}}(t)) - L_{\dot{x}}(t, \hat{x}(t), \dot{\hat{x}}(t))(v - \dot{\hat{x}}(t)) \geq 0, \ \ t \in T, \ \ v > 0.$$

Note that the expression defining Weierstrass function contains partial derivative in $\dot{x}$ of the integrand L, and does not contain its partial derivatives in t and x. Here we get that Weierstrass inequality is the necessary condition for strong local extremum under the assumption that integrand L has continuous partial derivatives in t and $\dot{x}$ Usually it is proved that Weierstrass inequality is the necessary condition for strong local extremum under the assumption that integrand L has continuous partial derivatives in x and $\dot{x}$ (see [**1**, 1.4.4].

It would be interesting to investigate whether Weierstrass inequality is the necessary condition for strong local solution only under the assumption that integrand L has continuous partial derivative in $\dot{x}$.

2.5. Lack of smoothness on x is often the case in mathematical analysis, approximation theory etc. For example, consider the classical Didona problem (which is a variant of isoperimetric problem), written in the following way

$$\int_0^1 |x(t)|dt \to \sup; \ \ \int_0^1 \sqrt{1 + \dot{x}(t)^2}dt \leq L, \ \ x(0) = x(1) = 0.$$

2.6. When we solve problems of the isoperimetric type with integrands which have partial derivatives in the phase variable, in some cases we obtain more by applying theorems 1 and 2 than by classical theorems. Here is an example:

$$\int_0^1 [\dot{x}(t)^2 - x(t)\dot{x}(t)^3]dt \to \inf; \quad x(0) = 0, x(1) = 0.$$

Using conditions of second order it is easy to prove that the admissible extremal $\hat{x}(\cdot) = 0$ is weak local solution of the above problem. Using Weierstrass condition it is easy to prove that $\hat{x}(\cdot)$ is not strong local solution. Using Theorem 1 we shall prove that $\hat{x}(\cdot)$ is the only weak local solution of that problem.

Necessary condition from Theorem 1 applied to our problem gives the following differential equation

$$\frac{d}{dt}[\dot{x}(t)^2 - 2x(t)\dot{x}(t)^3] = 0,$$

which is equivalent to

$$\dot{x}(t)^2[1 - 2x(t)\dot{x}(t)] = \text{const}.$$

By Rolle's theorem the derivative of $x(\cdot)$ vanishes in at least one point of the interval $(0,1)$. It follows that

$$\dot{x}(t)^2[1 - 2x(t)\dot{x}(t)] = 0.$$

Therefore we have $\dot{x}(0) = \dot{x}(1) = 0$. Suppose the derivative $\dot{x}(\cdot)$ is not identically zero. Then there exists interval $[a,b] \subseteq [0,1]$, such that $\dot{x}(a) = \dot{x}(b) = 0$ and $\dot{x}(t) \neq 0$ for $t \in (a,b)$. Then $1 - 2x(t)\dot{x}(t) = 0,\ \ t \in (a,b)$. It follows that $\lim_{t\to a+} x(t)\dot{x}(t) = 1/2$. On the other hand, we have $\lim_{t\to a+} x(t)\dot{x}(t) = x(a)\dot{x}(a) = 0$. Contradiction! Therefore $\dot{x}(\cdot) = 0$, i.e. the function $x(\cdot)$ is constant. Having in mind that $x(0) = 0$, we obtain that $x(\cdot) = 0$.

2.7. To prove Theorem 1 we shall first transform our problem by changing the time variable. Then we shall apply Lagrange principle [**1**, §4.1] to the transformed problem.

The first step in the proof of Theorem 2 will be the same transformation as in the proof of Theorem 1. The second step will be an application of the maximum principle to the transformed problem. In order to do that, we shall formulate maximum principle for the optimal control problem with variable control set.

3. Proof of Theorem 1

Suppose the function $\hat{x}(\cdot)$ is extended beyond $[t_0, t_1]$ so that its smoothness is preserved. Together with (P) we shall consider the problem

$$(P^*) \qquad \begin{aligned} I_0^*(z(\cdot)) \to \inf; \quad & I_i^*(z(\cdot)) \le 0, \quad i = 1, \dots, k, \\ & I_i^*(z(\cdot)) = 0, \quad i = k+1, \dots, m, \\ & z(t_0) = t_0, \ \ z(t_1) = t_1, \end{aligned}$$

where integral functionals are defined by

$$I_i^*(z(\cdot)) = \int_{t_0}^{t_1} L_i(z(t), \hat{x}(t), \dot{\hat{x}}(t)/\dot{z}(t))\dot{z}(t)dt, \quad i = 0, 1, \ldots, m.$$

In that problem the integrands

$$L_i^*(t, z, \dot{z}) = L_i(z, \hat{x}(t), \dot{\hat{x}}(t)/\dot{z})\dot{z}, \quad i = 0, 1, \ldots, m,$$

are defined and continuous on the open set

$$V^* = \{(t, z, \dot{z}) \in R \times R \times R | (z, \hat{x}(t), \dot{\hat{x}}(t)/\dot{z}) \in V, \quad \dot{z} > 0\}.$$

LEMMA 1. *If the function $\hat{x}(\cdot)$ is a weak local solution of the problem (P), then $\hat{z}(\cdot)$, $\hat{z}(t) = t$, is a weak local solution of the problem (P^*).*

Proof of the lemma. There exists an $\epsilon > 0$ such that weak ϵ-neighborhood of $\hat{x}(\cdot)$ belongs to D and on that neighborhood $\hat{x}(\cdot)$ is global solution of the problem (P). There exists δ such that the following inequalities are valid

$$0 < \delta < 1, \quad \omega(\hat{x}(\cdot), \delta) < \epsilon, \quad \|\hat{x}(\cdot)\| \frac{\delta}{1-\delta} + \omega(\dot{\hat{x}}(\cdot), \delta) < \epsilon.$$

Let $z(\cdot) \in C^1[t_0, t_1]$, $\|z(\cdot) - \hat{z}(\cdot)\|_{C^1} < \delta$, be an admissible function for the problem (P^*). Since $\dot{z}(t) > 0$, $t \in [t_0, t_1]$, there exists inverse function $z^{-1}(\cdot)$. Put $x(\cdot) = \hat{x} \circ z^{-1}(\cdot)$. Obviously $x(\cdot)$ belongs to $C^1([t_0, t_1], R^n)$. For $t \in [t_0, t_1]$, we have

$$|z^{-1}(t) - t| = |\hat{z}(z^{-1}(t)) - z(z^{-1}(t))| < \delta,$$
$$|\dot{z}(t) - 1| = |\dot{z}(t) - \dot{\hat{z}}(t)| < \delta.$$

Using these inequalities, we get

$$\begin{aligned}
\|x(t) - \hat{x}(t)\| &= \|\hat{x}(z^{-1}(t)) - \hat{x}(t)\| \leq \omega(\hat{x}(\cdot), \delta) < \epsilon, \\
\|\dot{x}(t) - \dot{\hat{x}}(t)\| &= \|\dot{\hat{x}}(z^{-1}(t))/\dot{z}(z^{-1}(t)) - \dot{\hat{x}}(t)\| \\
&\leq \|\dot{\hat{x}}(z^{-1}(t))/\dot{z}(z^{-1}(t)) - \dot{\hat{x}}(z^{-1}(t))\| + \|\dot{\hat{x}}(z^{-1}(t)) - \dot{\hat{x}}(t)\| \\
&= \|\dot{\hat{x}}(z^{-1}(t))\| \frac{|1 - \dot{z}(z^{-1}(t))|}{|\dot{z}(z^{-1}(t))|} + \|\dot{\hat{x}}(z^{-1}(t)) - \dot{\hat{x}}(t)\| \\
&\leq \|\hat{x}(\cdot)\| \frac{\delta}{1-\delta} + \omega(\dot{\hat{x}}(\cdot), \delta) < \epsilon.
\end{aligned}$$

It follows that $\|x(\cdot) - \hat{x}(\cdot)\|_{C_1} < \epsilon$, and therefore $x(\cdot) \in D$. Since

$$\begin{aligned}
I_i(x(\cdot)) &= \int_{t_0}^{t_1} L_i(t, x(t), \dot{x}(t))dt \\
&= \int_{t_0}^{t_1} L_i(t, \hat{x}(z^{-1}(t)), \dot{\hat{x}}(z^{-1}(t))/\dot{z}(z^{-1}(t)))dt \\
&= \int_{t_0}^{t_1} L_i(z(s), \hat{x}(s), \dot{\hat{x}}(s)/\dot{z}(s))\dot{z}(s)ds \\
&= I_i^*(z(\cdot)),
\end{aligned}$$

for $i = 0, 1, \dots, m$, then

$$I_i(x(\cdot)) = I_i^*(z(\cdot)) \le 0,\ \ i = 1, \dots, k,$$
$$I_i(x(\cdot)) = I_i^*(z(\cdot)) = 0,\ \ i = k+1, \dots, m.$$

We also have

$$x(t_0) = \hat{x}(z^{-1}(t_0)) = \hat{x}(t_0) = x_0,$$
$$x(t_1) = \hat{x}(z^{-1}(t_1)) = \hat{x}(t_1) = x_1.$$

Therefore $x(\cdot)$ is admissible for the problem (P). It follows that

$$I_0^*(z(\cdot)) = I_0(x(\cdot)) \ge I_0(\hat{x}(\cdot)) = I_0^*(\hat{z}(\cdot)). \quad \square$$

Integrands L_i^*, $i = 0, 1, \dots, m$, have continuous partial derivatives in z and $\dot{z}$. We can apply the Lagrange principle for the Lagrange problem [**1**, §4.1] on (P^*). Consequently, there exist Lagrange multipliers $\hat{\lambda}_i \in R$, $i = 0, 1, \dots, m$, not all of them equal to zero, such that

$$\hat{\lambda}_i \ge 0,\ \ i = 0, 1, \dots, k,$$
$$\hat{\lambda}_i I_i^*(\hat{z}(\cdot)) = \hat{\lambda}_i I_i(\hat{x}(\cdot)) = 0,\ \ i = 1, \dots, k,$$
$$\frac{d}{dt}\hat{L}_{\dot{z}}^*(t) = \hat{L}_z^*(t),\ \ t \in [t_0, t_1].$$

From

$$L_z^*(t, z, \dot{z}) = L_t(z, \hat{x}(t), \dot{\hat{x}}(t)/\dot{z})\dot{z},$$
$$L_{\dot{z}}^*(t, z, \dot{z}) = -L_{\dot{x}}(z, \hat{x}(t), \dot{\hat{x}}(t)/\dot{z})\dot{\hat{x}}(t)/\dot{z} + L(z, \hat{x}(t), \dot{\hat{x}}(t)/\hat{z})$$

it follows that

$$\hat{L}_z^*(t) = \hat{L}_t(t), \quad \hat{L}_{\dot{z}}^*(t) = -\hat{L}_{\dot{x}}(t)\dot{\hat{x}}(t) + \hat{L}(t).$$

Consequently,

$$\frac{d}{dt}[\hat{L}_{\dot{x}}(t)\dot{\hat{x}}(t) - \hat{L}(t)] = -\hat{L}_t(t),\ \ t \in [t_0, t_1]. \quad \square$$

4. Optimal control problem with variable control set

Let G be an open set in $R \times R^n$, W an open set in $R^n \times R^n$, Y a topological space, and $[t_0, t_1]$ an interval of the real line. We say that function $\phi(t, x, u)$, mapping the set $G \times Y$ into a topological space, is *piecewise continuous* if

a) there exists a finite set $S \subseteq R$ such that the function ϕ is continuous in every point $(\bar{t}, \bar{x}, \bar{u}) \in G \times Y$, where $\bar{t} \notin S$;

b) there exist limits of the function ϕ, as $t \to \bar{t}_-$, $x \to \bar{x}$, $u \to \bar{u}$ and as $t \to \bar{t}_+$, $x \to \bar{x}$, $u \to \bar{u}$ for every point $(\bar{t}, \bar{x}, \bar{u}) \in G \times Y$.

We say that $\bar{t} \in R$ is a *continuity point* of the function ϕ if ϕ is continuous in every point $(t, x, u) \in G \times Y$, such that $t = \bar{t}$.

Suppose $f: G \times Y \to R^n$ is a piecewise continuous function, integrands $L_i: G \times Y \to R$, $i = 0, 1, \ldots, m$, are piecewise continuous and terminants $l_i: W \to R$, $i = 0, 1, \ldots, m$, are continuous. Let $U(\cdot)$ be a multivalued mapping of the set of real numbers into a topological space Y.

The pair $(x(\cdot),\, u(\cdot))$ is called a *process* if $x(\cdot)$ is a piecewise smooth function mapping $[t_0, t_1]$ into R^n, such that $\Gamma(x(\cdot)) \subseteq G$, $(x(t_0), x(t_1)) \in W$, and $u(\cdot)$ is piecewise continuous function mapping $[t_0, t_1]$ into Y. *Bolza functionals* are defined on the set of processes in the following way:

$$B_i(x(\cdot), u(\cdot)) = \int_{t_0}^{t_1} L_i(t, x(t), u(t))dt + l_i(x(t_0), x(t_1)), \;\; i = 0, 1, \ldots, m.$$

We consider the following optimal control problem

$$\begin{aligned} B_0(x(\cdot), u(\cdot)) \to \inf; \quad & B_i(x(\cdot), u(\cdot)) \leq 0, \;\; i = 1, \ldots, k, \\ & B_i(x(\cdot), u(\cdot)) = 0, \;\; i = k+1, \ldots, m, \\ & \dot{x}(t) = f(t, x(t), u(t)), \;\; t \in T, \\ & u(t) \in U(t), \;\; t \in [t_0, t_1], \end{aligned}$$

where $T \subseteq [t_0, t_1]$ is the set of continuity points of control $u(\cdot)$, of functions f and of integrands L_i, $i = 0, 1, \ldots, m$.

An admissible process $(\hat{x}(\cdot), \hat{u}(\cdot))$ is called *optimal in the strong sense* if there exists $\epsilon > 0$, such that for each admissible process $(x(\cdot), u(\cdot))$, with $\|x(\cdot) - \hat{x}(\cdot)\|_C < \epsilon$, the inequality $B_0(x(\cdot), u(\cdot)) \geq B_0(\hat{x}(\cdot), \hat{u}(\cdot))$ is valid.

Functions $H : G \times Y \times R^{n*} \times R^{m+1*} \to R$, $l : W \times R^{m+1*} \to R$ and $\bar{U}(\cdot) : R \to \mathcal{P}Y$ are defined by

$$\begin{aligned} H(t, x, u, p, \lambda) &= pf(t, x, u) - \sum_{i=0}^{m} \lambda_i L_i(t, x, u), \\ l(x_0, x_1, \lambda) &= \sum_{i=0}^{m} \lambda_i l_i(x_0, x_1), \\ \bar{U}(t) &= \mathrm{cl} \bigcup_{\delta > 0} \bigcap_{t < s < t+\delta} U(s). \end{aligned}$$

Theorem 3. *Suppose that the function f, integrands L_i, $i = 0, 1, \ldots, m$, and their partial derivatives in x are piecewise continuous, and that terminants l_i, $i = 0, 1, \ldots, m$, are continuously differentiable. If the process $(\hat{x}(\cdot), \hat{u}(\cdot))$ is optimal in the strong sense, then there exist Lagrange multipliers $\hat{\lambda} \in R$, $i = 0, 1, \ldots, m$,* *not all of them equal to zero and piecewise smooth function* $\hat{p}(\cdot):[\hat{t}_0,\hat{t}_1]\to R^*$, *such that*

$$\begin{aligned}
&\hat{\lambda}_i \geq 0,\ \ i=0,1,\dots,k,\\
&\hat{\lambda}_i\hat{B}_i = 0,\ \ i=1,\dots,k,\\
&\dot{\hat{p}}(t) = -\hat{H}_x(t),\ \ t\in T,\\
&H(t,\hat{x}(t),u,\hat{p}(t),\hat{\lambda}) \leq \hat{H}(t),\ \ t\in T,\ \ u\in\bar{U}(t),\\
&\hat{p}(\hat{t}_0) = \hat{l}_{x_0},\\
&\hat{p}(\hat{t}_1) = -\hat{l}_{x_1},
\end{aligned}$$

where $T\subseteq[t_0,t_1]$ *is the set of continuity points of the function* f, *integrands* L_i, $i=0,1,\dots,m$, *their partial derivatives in* x *and of the control* $\hat{u}(\cdot)$.

Theorem 3 can be proved in the same way as the maximum principle from [**1**, §4.2].

5. Proof of Theorem 2

Suppose $\hat{x}(\cdot)$ is extended beyond $[t_0,t_1]$ in such a way that piecewise smoothness is preserved. Together with $(\bar{P})$ we shall study the problem

$$(\bar{P}^*)\qquad I_0^*(z(\cdot))\to\inf;\qquad \begin{aligned}&I_i^*(z(\cdot))\leq 0,\ \ i=1,\dots,k,\\ &I_i^*(z(\cdot))=0,\ \ i=k+1,\dots,m,\\ &z(t_0)=t_0,\ \ z(t_1)=t_1,\\ &\dot{z}(t)\in U^*(t),\end{aligned}$$

where integral functionals are defined by

$$I_i^*(z(\cdot)) = \int_{t_0}^{t_1} L_i(z(t),\hat{x}(t),\dot{\hat{x}}(t)/\dot{z}(t))\dot{z}(t)dt,\ \ i=0,1,\dots,m.$$

The integrands in $(\bar{P}^*)$

$$L_i^*(t,z,\dot{z}) = L_i(z,\hat{x}(t),\dot{\hat{x}}(t)/\dot{z})\dot{z},\ \ i=0,1,\dots,m,$$

are defined and piecewise continuous on the set $G^*\times K^*$, where

$$G^* = \{(t,z)\in R\times R \mid (z,\hat{x}(t))\in G\},\ \ K^* = (0,+\infty),$$
$$U^*(t) = \{\dot{z}>0 \mid \dot{\hat{x}}(t)/\dot{z}\in U\}.$$

LEMMA 2. *If* $\hat{x}(\cdot)$ *is a strong local solution of* $(\bar{P})$, *then* $\hat{z}(\cdot)$, $\hat{z}(t)=t$, *is a strong local solution of* $(\bar{P}^*)$.

Proof of the lemma. There exists $\epsilon > 0$ such that ϵ-neighborhood of the set $\Gamma(\hat{x}(\cdot))$ belongs to G and $\hat{x}(\cdot)$ is a global solution of the problem $(\bar{P})$ on the strong ϵ-neighborhood of $\hat{x}(\cdot)$. There exists $\delta > 0$ such that $\omega(\hat{x}(\cdot), \delta) < \epsilon$.

Suppose $z(\cdot) \in \bar{C}^1[t_0, t_1]$, $\|z(\cdot) - \hat{z}(\cdot)\|_C < \delta$, is an admissible function for the problem $(\bar{P}^*)$. Since $\dot{z}(t) > 0$, there exists inverse $z^{-1}(\cdot)$. Set $x(\cdot) = \hat{x} \circ z^{-1}(\cdot)$. The function $x(\cdot)$ obviously belongs to the space $\bar{C}^1([t_0, t_1], R^n)$. For every $t \in [t_0, t_1]$ the following inequality is valid

$$|z^{-1}(t) - t| = |\hat{z}(z^{-1}(t)) - z(z^{-1}(t))| < \delta.$$

Using the inequality above we get

$$\|x(t) - \hat{x}(t)\| = \|\hat{x}(z^{-1}(t)) - \hat{x}(t)\| \leq \omega(\hat{x}(\cdot), \delta) < \epsilon.$$

Therefore $\|x(\cdot) - \hat{x}(\cdot)\|_C < \epsilon$. Consequently, $\Gamma(x(\cdot)) \subseteq G$. Besides, for every $t \in [t_0, t_1]$

$$\dot{x}_{\pm}(t) = \dot{\hat{x}}_{\pm}(z^{-1}(t))/\dot{z}_{\pm}(z^{-1}(t)) \in K.$$

It follows that $x(\cdot) \in \bar{D}$. Since

$$\begin{aligned} I_i(x(\cdot)) &= \int_{t_0}^{t_1} L_i(t, x(t), \dot{x}(t))dt \\ &= \int_{t_0}^{t_1} L_i(t, \hat{x}(z^{-1}(t)), \dot{\hat{x}}(z^{-1}(t))/\dot{z}(z^{-1}(t)))dt \\ &= \int_{t_0}^{t_1} L_i(z(s), \hat{x}(s), \dot{\hat{x}}(s)/\dot{z}(s))\dot{z}(s)ds \\ &= I_i^*(z(\cdot)), \end{aligned}$$

for $i = 0, 1, \dots, m$, then

$$\begin{aligned} I_i(x(\cdot)) &= I_i^*(z(\cdot)) \leq 0, \ \ i = 1, \dots, k, \\ I_i(x(\cdot)) &= I_i^*(z(\cdot)) = 0, \ \ i = k+1, \dots, m. \end{aligned}$$

Moreover,

$$\begin{aligned} x(t_0) &= \hat{x}(z^{-1}(t_0)) = \hat{x}(t_0) = x_0, \\ x(t_1) &= \hat{x}(z^{-1}(t_1)) = \hat{x}(t_1) = x_1. \\ \dot{x}(t) &= \dot{\hat{x}}(z^{-1}(t))/\dot{z}(z^{-1}(t)) \in U, \end{aligned}$$

provided $\dot{\hat{x}}(z^{-1}(t))$ and $\dot{z}(z^{-1}(t))$ exist. Therefore $x(\cdot)$ is admissible for $(\bar{P})$. It follows that

$$I_0^*(z(\cdot)) = I_0(x(\cdot)) \geq I_0(\hat{x}(\cdot)) = I_0^*(\hat{z}(\cdot)). \quad \square$$

We finish the proof of Theorem 2 by applying to the problem $(\bar{P}^*)$ the maximum principle from Section 4. We get that there exist Lagrange multipliers $\hat{\lambda}_i \in R$,

$i = 0, 1, \dots, m$, not all of them equal to zero, and piecewise smooth function $\hat{r}(\cdot) : [t_0, t_1] \to R$, such that

$$\begin{aligned}
&\hat{\lambda}_i \geq 0, \;\; i = 0, 1, \dots, k, \\
&\hat{\lambda}_i I_i^*(\hat{z}(\cdot)) = \hat{\lambda}_i I_i(\hat{x}(\cdot)) = 0, \;\; i = 1, \dots, k, \\
&\dot{\hat{r}}(t) = \hat{L}_z^*(t), \;\; t \in T, \\
&L^*(t, \hat{z}(t), w) - L^*(t, \hat{z}(t), \dot{\hat{z}}(t)) - \hat{r}(t)(w - \dot{\hat{z}}(t)) \geq 0, \;\; t \in T, \;\; w \in \bar{U}^*(t).
\end{aligned}$$

From

$$L_z^*(t, z, \dot{z}) = L_t(z, \hat{x}(t), \dot{\hat{x}}(t)/\dot{z})\dot{z},$$

it follows that $\hat{L}_z^*(t) = \hat{L}_t(t)$. Consequently $\dot{\hat{r}}(t) = \hat{L}_t(t), \;\; t \in T$. From

$$\begin{aligned}
L^*(t, \hat{z}(t), w) &= L(t, \hat{x}(t), \dot{\hat{x}}(t)/w)w, \\
L^*(t, \hat{z}(t), \dot{\hat{z}}(t)) &= L(t, \hat{x}(t), \dot{\hat{x}}(t)), \\
W(t) &= \bar{U}^*(t),
\end{aligned}$$

we have

$$L(t, \hat{x}(t), \dot{\hat{x}}(t)/w)w - L(t, \hat{x}(t), \dot{\hat{x}}(t)) - \hat{r}(t)(w - 1) \geq 0, \;\; t \in T, \;\; w \in W(t). \quad \square$$

References

1. В.М. Алексеев, В.М. Тихомиров, В.С. Фомин, *Оптимальное управление*, Наука, Москва, 1979.
2. V. Janković, *Necessary conditions in a problem of calculus of variations*, Publ. Inst. Math. (Beograd) (N.S.) **45** (**59**) (1989), 143–151.

Matematički fakultet
Studentski trg 16
11001 Beograd, p.p. 550
Yugoslavia

(Received 04 12 1995)