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ON A THEOREM OF FROBENIUS

Radoš Bakić

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Abstract. Applying the Frobenius criterion for p-nilpotency we find a relatively large class of finite solvable groups as semidirect products of two of their Hall subgroups.

If P is a finite group, then we shall denote by $d(P)$ the minimal number of generators of P. Let n be a positive integer and p a prime. We say that n has the (p,m)-property if for each prime divisor q of n, q does not divide p^k-1 for $1 \leq k \leq m$.

The following theorem is well known:

THEOREM (Frobenius). *Let G be a finite group and P its Sylow p-subgroup. If any subgroup P is generated by at most d elements and $|G|$ has the (p,d)-property, then G is p-nilpotent* [2, p. 437].

Remark: The theorem is used with d equal to the maximal exponent m of p in $|G|$, since $d(H) \leq m$ for any subgroups $H < P$.

THEOREM 1. *Let G be a finite solvable group and $|G| = ab$, where $(a,b) = 1$. Let A have the (p,d)-property, for any prime p dividing b and d maximal exponent of p in b. Then G is a semidirect product of two normal Hall's subgroups: one of order a and one of order b.*

Proof. Note that if G is a solvable finite group and A its subgroup of order a, than the order of $N_G(A)$ depends only on a since conjugate subgroups have conjugate normalizers. Therefore, we shall put $|N_{G^A}| = n(a)$. Let $b = p_1^{a_1} \dots p_m^{a_m}$ be the prime factorization of b. Choose any subgroup K of order $ap_i^{a_i}$. Such a group exists by Hall's theorem. By the theorem of Frobenius, K has a normal subgroup A of order a. Since $K < N_G(A)$, by Lagrange's theorem we have that $|K|$ divides $n(a)$ and so p_i^{ai} divides $n(a)$ for all i. Hence, b divides $n(a)$ and so $n(a) = ab$, i.e.,

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A is normal in G. Therefore, G is a semidirect product of two Hall subgroups of orders a and b.

Remark 1. The above theorem can be proved by induction, in the usual manner for solvable groups, without using the theorem of Frobenius, but we find the given proof more elegant.

Remark 2. If G is not solvable, then the Theorem 1 need not be true. For example we can take $G = A_5$. Then $a = 5$ and $b = 12$, and one can see that the 5-Sylow subgroup is not normal in A_5.

As a special case of Frobenius theorem we have:

THEOREM (Burnside). *If G is a finite group with a cyclic p-Sylow subgroup and if for all primes q dividing $|G|$, q does not divide $p-1$, then G is p nilpotent.*

Now we prove the following:

THEOREM 2. *Let G be a finite group and H its subgroup of index p, where p is a prime and $(p-1, |G|) = 1$. Then H is normal in G.*

Proof. By induction on $|G|$. If p does not divide $|H|$, then Theorem 2 is a special case of the theorem of Frobenius. Let p divide $|G|$. Then there exists a homomorphism $h: G \to S_p$, where S_p is the symmetric group on p elements. Since p^2 divides $|G|$ and p^2 does not divide $p!$, we have $H_1 = \ker(h) \neq \{1\}$, and $H_1 < H$. Then H/H_1 is a subgroup of G/H_1 of index p, so by the induction hypothesis, H/H_1 is normal in G/H_1 and so is H in G.

We continue with two (known) characterizations of nilpotency of finite groups.

LEMMA 1. *A finite group G is nilpotent iff any two of its elements having relatively prime orders commute.*

LEMMA 2. *A finite group G is nilpotent iff it is p-nilpotent for any prime p dividing $|G|$.*

We only sketch the proofs of ($\Leftarrow$) parts.

For Lemma 1: if P is Sylow subgroup of G, then P is centralized by all Sylow subgroups of G with orders not equal to $|P|$. Therefore, $N(P)$ has Sylow subgroups of all possible prime divisors of $|G|$, and they are of the same size as those in G. By Lagrange's theorem, $|N_P| = G$, and P is normal in G.

For Lemma 2: each Sylow subgroup of G has a normal complement. Let G_P be a normal complement of a p-Sylow subgroup. If P is the intersection of all G_q's where $q \neq p$, then P is a Sylow subgroup of G and P is normal since G_q's are normal.

In [1] Pazderski proved the following two theorems; here we give different proofs.

THEOREM 3. *All the groups of order n are p-nilpotent iff n has the (p, m)-property where $n = p^m b$, $(p, b) = 1$.*

Proof. Let $n = p^m b$ where $(p, b) = 1$. Suppose $q \mid p^k - 1$, where qp^k divides n and p_1 is prime. We shall construct a group of order n which is not p-nilpotent. By assumption, q divides $|\mathrm{Aut}\,((C_P)^m)| = p^{m(m-1)/2}(p^m-1)\cdots(p^k-1)\cdots(p-1)$, and so in $\mathrm{Aut}\,((C_P)^m)$ we have a subgroup H isomorphic to C_q. Now we can construct a semidirect product $K = (C_p)^m \times_I H$ where I is the inclusion homomorphism $I\colon H \to \mathrm{Aut}\,((C_P)^m)$. If we take any $h \in H$ such that $h \neq 1$, then we can find some $g \in (C_P)^m$ with $h(g) \neq g$. Two elements $(g, 1)$ and $(1, h)$ of k, have relatively prime orders, but do not commute, and so by Lemma 1, K is not nilpotent. Therefore, H is not normal in K since H is a q-Sylow subgroup. Then K must contain at least two subgroups of order q, and $G = K \times C_t$ (where $t = n/|K|$) contains at least two subgroups of order b. Hence, G is not p-nilpotent.

THEOREM 4. *Each group of order n is nilpotent iff n has the (p, m)-property for any prime p, such that $n = p^m t$ and $(p, t) = 1$.*

Proof. Direction $(\Rightarrow)$ follows from Theorem 3 and Lemma 2, and direction $(\Leftarrow)$ follows from the Frobenius theorem and Lemma 2.

REFERENCES

1. G. Pazderski, *Die Ordnungen, zu denen nur Gruppen mit gegebenen Eigenshaften gehoren,* Arch. Math. **10** (1959), 331–343.
2. B. Huppert, *Endliche Gruppen I,* Springer-Verlag, Berlin, Heidelberg, New York, 1967.

Matematički institut
Kneza Mihaila 35
11001 Beograd, p.p. 367
Yugoslavia

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