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ON THE EXPONENTIAL DIVISOR FUNCTION

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Abstract. We investigate the exponential divisor function and establish several asymptotic formulas involving this function.

1. Introduction

The notions of exponential divisor and exponential divisor function was introduced by Subbarao [**10**]. Let p, with or without subscript, denote a prime number. For $n = p_1^{\nu_1} \cdots p_k^{\nu_k}$ (canonical decomposition of the integer $n > 1$), we call d an *exponential divisor* of n if $d = p_1^{\mu_1} \cdots p_k^{\mu_k}$ with $\mu_j | \nu_j$ $(1 \le j \le k)$. Let $\tau^{(e)}(n)$ be the number of such divisors of n with convention $\tau^{(e)}(1) = 1$ and we call it the *exponential divisor function*. This function is multiplicative and satisfies

$$\tau^{(e)}(n) = \prod_{p^\nu \| n} \tau(\nu), \tag{1.1}$$

where $\tau(n)$ is the usual divisor function and $p^\nu \| n$ means that $p^\nu \mid n$, but $p^{\nu+1} \nmid n$. In particular, $\tau^{(e)}(p^\nu) = \tau(\nu)$ so that $\tau^{(e)}(n)$ is *prime independent*. Moreover $\tau^{(e)}(n)$ depends only on the squarefull kernel of n. More precisely, each integer $n > 1$ has the unique representation $n = qs$ with $(q, s) = 1$, where $q = q(n)$ is squarefree and $s = s(n)$ is squarefull, and we have

$$\tau^{(e)}(n) = \tau^{(e)}(s). \tag{1.2}$$

Such a function is called *an arithmetical function with squarefull kernel*, or simply an *s-function*.

It seems interesting to make a systematic investigation of $\tau^{(e)}(n)$. For maximal order of $\tau^{(e)}(n)$, Erdős (cf. Theorem 6.2 of [**10**]) showed

$$\limsup_{n\to\infty} \frac{\log \tau^{(e)}(n) \, \log_2 n}{\log n} = \frac{\log 2}{2}, \tag{1.3}$$

AMS Subject Classification (1991): Primary 11N37 where $\log_k$ is the k-fold iterated logarithm. Recently Wu [**12**, Théorème 1] proved, by a simple convolution argument, the asymptotic formula

$$\sum_{n\le x}\tau^{(e)}(n)=A_1\,x+A_2\,x^{1/2}+O(x^{2/9}\log x), \tag{1.4}$$

where $A_1:=\prod_p\big(1+\sum_{\nu=2}^{\infty}\{\tau(\nu)-\tau(\nu-1)\}p^{-\nu}\big), A_2$ are two effective constants. This answers an open question in [**10**].

The aim of this paper is to consider further other analogues of some known results on $\tau(n)$ in the case of $\tau^{(e)}(n)$: Titchmarsh's exponential divisor problem, mean value of $\tau^{(e)}(n-1)$ over integers free of large prime factors, $\cdots$ etc. Most of our results can be generalized to other prime-independent multiplicative s-functions f, only if $f^*(\nu):=f(p^\nu)$ does not increase too rapidly. To avoid unnecessary length, we restrict ourselves to the case of $\tau^{(e)}(n)$.

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2. Exponential divisor problem of Titchmarsh

The Titchmarsh divisor problem consists of the evaluation of $T(x):=\sum_{p\le x}\tau(p-1)$. The best result known to date is due to Fouvry [**4**, Corollaire 1], Bombieri, Friedlander and Iwaniec [**1**, Corollary 2], who independently proved, by an application of a theorem of Bombieri–Vinogradov type, that for any fixed $A>0$, we have

$$T(x)=B_1\,x+B_2\ \mathrm{li}\,x+O_A\big(x/(\log x)^A\big),$$

where

$$\mathrm{li}\,x:=\int_2^x\frac{\delta t}{\log t},\qquad B_1:=\prod_p\Big(1+\frac{1}{p(p-1)}\Big),\qquad B_2:=\gamma-\sum_p\frac{\log p}{1+p(p-1)},$$

and γ is the Euler constant.

We propose here to consider *the exponential divisor problem of Titchmarsh*, i.e. to evaluate the summatory function $T^{(e)}(x):=\sum_{p\le x}\tau^{(e)}(p-1)$. Our result is as follows.

Theorem 1. *For any fixed $A>0$, we have*

$$T^{(e)}(x)=C\ \mathrm{li}\,x+O_A\big(x/(\log x)^A\big)$$

with $C:=\prod_p\big(1+\sum_{\nu=2}^{\infty}\{\tau(\nu)-1\}p^{-\nu}\big)$.

Proof. Here and in the sequel, the letters s and q denote respectively generic squarefull and squarefree integers. As usual, let $\mu(d)$ be the Möbius function and

$\varphi(d)$ the Euler function. Writing $p-1=qs$ with $(q,s)=1$ and in view of (1.2), we find that

$$T^{(e)}(x)=\sum_{s\le x}\tau^{(e)}(s)\sum_{\substack{q\le(x-1)/s\\ qs+1=p,\,(q,s)=1}}1=\sum_{s\le x}\tau^{(e)}(s)\sum_{\substack{n\le(x-1)/s\\ ns+1=p,\,(n,s)=1}}\mu(n)^2.$$

With the aid of the relation

$$\mu(n)^2=\sum_{d^2|n}\mu(d), \tag{2.1}$$

we can show, by interchanging the summations and the Möbius inversion formula, that

$$\begin{aligned}T^{(e)}(x)&=\sum_{s\le x}\tau^{(e)}(s)\sum_{\substack{d\le\sqrt{(x-1)/s}\\ (d,s)=1}}\mu(d)\sum_{\substack{\ell\le(x-1)/d^2s\\ d^2\ell s+1=p,\,(\ell,s)=1}}1\\ &=\sum_{s\le x}\tau^{(e)}(s)\sum_{\substack{d\le\sqrt{(x-1)/s}\\ (d,s)=1}}\mu(d)\sum_{m|s}\mu(m)\sum_{\substack{n\le(x-1)/d^2ms\\ d^2msn+1=p}}1.\end{aligned}$$

Obviously, the last sum over n is equal to the number of primes not to exceed x and congruent to 1 modulo d^2ms. Defining $\pi(x;a,\ell):=|\{p\le x: p\equiv a \pmod{\ell}\}|$ for $(a,\ell)=1$, it follows

$$T^{(e)}(x)=\sum_{s\le x}\tau^{(e)}(s)\sum_{\substack{d\le\sqrt{(x-1)/s}\\ (d,s)=1}}\mu(d)\sum_{m|s}\mu(m)\,\pi(x;1,d^2ms). \tag{2.2}$$

Let Y, $Z\in[1,(\log x)^{10A}]$ be two parameters to be chosen later. We divide the triple sums on the right-hand side of (2.2) into three parts:

$$\begin{aligned}T_1^{(e)}(x)&:=\sum_{s\le Y}\tau^{(e)}(s)\sum_{\substack{d\le Z\\ (d,s)=1}}\mu(d)\sum_{m|s}\mu(m)\,\pi(x;1,d^2ms),\\ T_2^{(e)}(x)&:=\sum_{s\le Y}\tau^{(e)}(s)\sum_{\substack{Z<d\le\sqrt{(x-1)/s}\\ (d,s)=1}}\mu(d)\sum_{m|s}\mu(m)\,\pi(x;1,d^2ms),\\ T_3^{(e)}(x)&:=\sum_{Y<s\le x}\tau^{(e)}(s)\sum_{\substack{d\le\sqrt{(x-1)/s}\\ (d,s)=1}}\mu(d)\sum_{m|s}\mu(m)\,\pi(x;1,d^2ms).\end{aligned}$$

Using the trivial estimate

$$\pi(x;1,d^2ms)\le x/d^2ms \tag{2.3}$$

and noticing that

$$\sum_{s\le Y}\frac{\tau^{(e)}(s)}{\sqrt{s}}\sum_{m|s}\frac{|\mu(m)|}{m}\ll\prod_{p\le Y}\Big(1+\frac{2}{p}+O\Big(\frac{1}{p^{3/2}}\Big)\Big)\ll(\log Y)^2,$$

we deduce, by Abel summation, that

$$T_3^{(e)}(x)\ll x\sum_{s>Y}\frac{\tau^{(e)}(s)}{s}\sum_{m|s}\frac{|\mu(m)|}{m}\ll x\frac{(\log Y)^2}{\sqrt{Y}}. \tag{2.4}$$

Similarly, we have

$$T_2^{(e)}(x)\le x\sum_{s\le Y}\frac{\tau^{(e)}(s)}{s}\sum_{m|s}\frac{|\mu(m)|}{m}\sum_{d>Z}\frac{|\mu(d)|}{d^2}\ll\frac{x}{Z}. \tag{2.5}$$

It remains to evaluate $T_1^{(e)}(x)$. For this, we write $T_1^{(e)}(x)=P_1(x,y)+R_1(x,y)$, where

$$\begin{aligned}P_1(x,y)&:=\sum_{s\le Y}\tau^{(e)}(s)\sum_{\substack{d\le Z\\(d,s)=1}}\mu(d)\sum_{m|s}\mu(m)\,\frac{\mathrm{li}\,x}{\varphi(d^2ms)},\\ R_1(x,y)&:=\sum_{s\le Y}\tau^{(e)}(s)\sum_{\substack{d\le Z\\(d,s)=1}}\mu(d)\sum_{m|s}\mu(m)\Big\{\pi(x;1,d^2ms)-\frac{\mathrm{li}\,x}{\varphi(d^2ms)}\Big\}.\end{aligned}$$

Since $d^2ms\le(ds)^2\le(\log x)^{20A}$, Siegel–Walfisz' theorem [**11**, Theorem II.8.5] gives us

$$\begin{aligned}R_1(x,y)&\ll x\,e^{-c_1\sqrt{\log x}}\,Z\sum_{s\le Y}\tau^{(e)}(s)2^{\omega(s)}\ll x\,e^{-c_1\sqrt{\log x}}\,Z\sqrt{Y}\sum_{s\le Y}\tau^{(e)}(s)2^{\omega(s)}/\sqrt{s}\\ &\ll x\,e^{-c_1\sqrt{\log x}}\,Z\sqrt{Y}\prod_{p\le Y}\Big(1+4p^{-1}+O\big(p^{-3/2}\big)\Big)\ll x\,e^{-\frac{c_1}{2}\sqrt{\log x}},\end{aligned}$$

where c_1 is an absolute positive constant.

For $(d,s)=1$ and $m|s$, we easily show that $\varphi(d^2ms)=d\varphi(d)m\varphi(s)$. Recalling the relation $\sum_{m|s}\mu(m)/m=\varphi(s)/s$, we have

$$P_1(x,y)=\mathrm{li}\,x\sum_{s\le Y}\frac{\tau^{(e)}(s)}{s}\sum_{\substack{d\le Z\\(d,s)=1}}\frac{\mu(d)}{d\varphi(d)}=C\;\mathrm{li}\,x+O\Big(\frac{(\log Y)^2}{\sqrt{Y}}\,\mathrm{li}\,x+\frac{1}{Z}\,\mathrm{li}\,x\Big),$$

where we have used the following identities

$$\sum_s \frac{\tau^{(e)}(s)}{s} \sum_{(d,s)=1} \frac{\mu(d)}{d\varphi(d)} = \prod_p \Big(1 - \frac{1}{p(p-1)}\Big) \sum_s \frac{\tau^{(e)}(s)}{s} \prod_{p|s} \Big(1 - \frac{1}{p(p-1)}\Big)^{-1}$$
$$= \prod_p \Big(1 - \frac{1}{p(p-1)}\Big)\Big(1 + \Big(1 - \frac{1}{p(p-1)}\Big)^{-1} \sum_{\nu=2}^{\infty} \frac{\tau(\nu)}{p^\nu}\Big) = C.$$

These estimates imply that

$$T_1^{(e)}(x) = C \ \mathrm{li}\, x + O\Big(\frac{(\log Y)^2}{\sqrt{Y}} \,\mathrm{li}\, x + \frac{1}{Z} \,\mathrm{li}\, x + x\, e^{-\frac{c_1}{2}\sqrt{\log x}}\Big). \tag{2.6}$$

Combining (2.4)–(2.6) with (2.2), we obtain that

$$T^{(e)}(x) = C \ \mathrm{li}\, x + O\Big(\frac{x}{Z} + \frac{x(\log Y)^2}{\sqrt{Y}} + x\, e^{-\frac{c_1}{2}\sqrt{\log x}}\Big).$$

Now the required result follows from on taking $Y = (\log x)^{3A}$ and $Z = (\log x)^A$. The proof of Theorem 1 is finished.

Remark 1. (i) The relation (1.4) and Theorem 1 show that the integers n and $p-1$ possess respectively A_1 and C exponential divisors, in the average sense. The fact of $C > A_1$ attests to the bad distribution of prime numbers p in the corresponding congruence class.

(ii) It is worth indicating that we use only Siegel–Walfisz' theorem and the trivial estimate (2.3) instead of the Bombieri–Vinogradov theorem and Brun–Titchmarsh's inequality, as in the classical divisor problem of Titchmarsh.

3. Mean value of $\tau^{(e)}(n-1)$ over integers free of large prime factors

Let $P(n)$ be the largest prime factor of the integer $n > 1$ with the convention $P(1) = 1$. For $x \ge y \ge 2$, we define $u := (\log x)/\log y$ and

$$S(x,y) := \{n \le x : P(n) \le y\}, \quad \Psi(x,y) := |S(x,y)|, \quad T(x,y) := \sum_{n \in S(x,y)} \tau(n-1).$$

Fouvry and Tenenbaum [**6**] proved that there exists a positive constant η such that the asymptotic formula (see (1.16) of [**6**])

$$T(x,y) = \Psi(x,y) \log x \Big\{1 + O\Big(\frac{\log(u+1)}{\log y}\Big)\Big\}$$

holds uniformly in the region: $x \ge 3$, $x^{\eta \log_3 x/\log_2 x} \le y \le x$.

In this section, we shall consider an analogue: $T^{(e)}(x,y) := \sum_{n \in S(x,y)} \tau^{(e)}(n-1)$. We have the following result.

Theorem 2. *Let A_1 be defined as in (1.4). For any $\varepsilon > 0$, the asymptotic formula*

$$T^{(e)}(x,y) = A_1\,\Psi(x,y)\big\{1 + O_\varepsilon\big((\log_2 y)^2/\log y\big)\big\}$$

holds uniformly for

$$(\mathcal{C}_\varepsilon) \qquad x \ge 3, \qquad \exp\big\{(\log x)^{2/3+\varepsilon}\big\} \le y \le x.$$

Proof. Let $\Psi(x,y;a,\ell) := |\{n \in S(x,y) : n \equiv a \pmod{\ell}\}|$. As before, we can prove that

$$T^{(e)}(x,y) = \sum_{s\le x} \tau^{(e)}(s) \sum_{\substack{d\le\sqrt{(x-1)/s}\\ (d,s)=1}} \mu(d) \sum_{m|s} \mu(m)\,\Psi(x,y;1,d^2ms). \tag{3.1}$$

Let Y, $Z \in [1, x^{1/10}]$ be two parameters to be chosen later. We divide the triple sums on the right-hand side of (3.1) into three parts:

$$\begin{aligned}
T_1^{(e)}(x,y) &:= \sum_{s\le Y} \tau^{(e)}(s) \sum_{\substack{d\le Z\\ (d,s)=1}} \mu(d) \sum_{m|s} \mu(m)\,\Psi(x,y;1,d^2ms),\\
T_2^{(e)}(x,y) &:= \sum_{s\le Y} \tau^{(e)}(s) \sum_{\substack{Z<d\le\sqrt{(x-1)/s}\\ (d,s)=1}} \mu(d) \sum_{m|s} \mu(m)\,\Psi(x,y;1,d^2ms),\\
T_3^{(e)}(x,y) &:= \sum_{Y<s\le x} \tau^{(e)}(s) \sum_{\substack{d\le\sqrt{(x-1)/s}\\ (d,s)=1}} \mu(d) \sum_{m|s} \mu(m)\,\Psi(x,y;1,d^2ms).
\end{aligned}$$

Using the inequality $\Psi(x,y;1,\ell) \le x/\ell + 1$, we can prove, as in the proof of Theorem 1, that

$$T_2^{(e)}(x,y) \ll x/Z, \qquad T_3^{(e)}(x,y) \ll x(\log Y)^2/\sqrt{Y}. \tag{3.2}$$

It remains to evaluate $T_1^{(e)}(x,y)$. For this, we write

$$T_1^{(e)}(x,y) = P_1(x,y) + R_1(x,y), \tag{3.3}$$

where

$$\begin{aligned}
P_1(x,y) &:= \sum_{s\le Y} \tau^{(e)}(s) \sum_{d\le Z,\,(d,s)=1} \mu(d) \sum_{m|s} \mu(m)\,\frac{\Psi_{d^2ms}(x,y)}{\varphi(d^2ms)},\\
R_1(x,y) &:= \sum_{s\le Y} \tau^{(e)}(s) \sum_{d\le Z,\,(d,s)=1} \mu(d) \sum_{m|s} \mu(m) E(x,y;1,d^2ms),\\
\Psi_\ell(x,y) &:= |\{n \in S(x,y) : (n,\ell)=1\}|, \qquad E(x,y;a,\ell) := \Psi(x,y;a,\ell) - \frac{\Psi_\ell(x,y)}{\varphi(\ell)}.
\end{aligned}$$

In order to control the error term $R_1(x,y)$, we need an estimate of Bombieri–Vinogradov type for $S(x,y)$: *For any fixed $A>0$ and $\varepsilon>0$, the inequality*

$$\sum_{\ell\le\sqrt{x}/\exp\{(\log x)^{1/3}\}} \tau(\ell)^3 \max_{(a,\ell)=1} |E(x,y;a,\ell)| \ll_{A,\varepsilon} \frac{\Psi(x,y)}{(\log x)^A} \tag{3.4}$$

holds uniformly in the region $(\mathcal{C}_\varepsilon)$. This is (7.1) of Fouvry and Tenenbaum [**7**]. Introducing

$$w(\ell) := \sum_{s\le Y}\sum_{d\le Z}\sum_{\substack{m\le Y\\ d^2ms=\ell}} \tau^{(e)}(s)\mu(d)^2\mu(m)^2$$

we can write $|R_1(x,y)| \le \sum_{\ell\le(YZ)^2} w(\ell)|E(x,y;1,\ell)|$. Obviously we have $w(\ell)\le\tau(\ell)^3$. Since $(YZ)^2\le x^{2/5}$, the estimate (3.4) implies that

$$|R_1(x,y)| \le \sum_{\ell\le x^{2/5}} \tau(\ell)^3|E(x,y;1,\ell)| \ll_{A,\varepsilon} \frac{\Psi(x,y)}{(\log x)^A}. \tag{3.5}$$

In order to approximate to the quantity $\Psi_{d^2ms}(x,y)$ in the principal term $P_1(x,y)$, we shall need Theorem 1 of Fouvry and Tenenbaum [**5**]: *Under the following conditions*

$$x\ge 3, \qquad \exp\left\{(\log_2 x)^{5/3+\varepsilon}\right\}\le y\le x, \tag{H_ε}$$

$$\log_2(\ell+2)\le\Big(\frac{\log y}{\log(u+1)}\Big)^{1-\varepsilon}, \tag{Q_ε}$$

we have uniformly

$$\Psi_\ell(x,y)=\frac{\varphi(\ell)}{\ell}\Psi(x,y)\Big\{1+O\Big(\frac{\log_2(\ell y)\,\log_2 x}{\log y}\Big)\Big\}.$$

Since $d^2ms\le(YZ)^2\le x^{2/5}$ and (x,y) is in the region $(\mathcal{C}_\varepsilon)$, it is clear that the conditions (H_ε) and (Q_ε) are satisfied. Hence we have

$$\begin{aligned} P_1(x,y) &= \Psi(x,y)\Big\{\sum_{s\le Y}\frac{\tau^{(e)}(s)\varphi(s)}{s^2}\sum_{d\le Z,\,(d,s)=1}\frac{\mu(d)}{d^2}+O_\varepsilon\Big(\frac{\log_2(YZy)\,\log_2 x}{\log y}\Big)\Big\} \\ &= A_1\,\Psi(x,y)\Big\{1+O_\varepsilon\Big(\frac{\log_2(YZy)\,\log_2 x}{\log y}+\frac{(\log Y)^2}{\sqrt{Y}}+\frac{1}{Z}\Big)\Big\}. \end{aligned} \tag{3.6}$$

The estimates (3.3)–(3.6) imply that $T^{(e)}(x,y)=A_1\,\Psi(x,y)\{1+O_\varepsilon(R)\}$, with

$$R:=\frac{\log_2(YZy)\,\log_2 x}{\log y}+\frac{x(\log Y)^2}{\Psi(x,y)\sqrt{Y}}+\frac{x}{\Psi(x,y)Z}.$$

Taking $Y = Z = x^{1/10}$ and using $\Psi(x,y) \gg x\,u^{-3u}$ valid for $x \ge y \ge 2$ [**11**, Theorem III.5.13], we can show that

$$x(\log Y)^2/\Psi(x,y)\sqrt{Y} + x/\Psi(x,y)Z \ll 1/\log x.$$

This concludes the proof of Theorem 2.

4. Maximal orders for $\Omega\big(\tau^{(e)}(n)\big)$, $\omega\big(\tau^{(e)}(n)\big)$ and $\tau^{(e)}\big(\tau^{(e)}(n)\big)$

As usual, let $\Omega(n)$ and $\omega(n)$ be the number of prime factors of n and the number of distinct prime factors of n, i.e. $\Omega(n) := \sum_{p^\nu\|n}\nu$ and $\omega(n) := \sum_{p|n}1$. Erdős and Ivić [**3**] investigated the maximal orders for $\omega\big(\tau(n)\big)$ and $\log\tau\big(\tau(n)\big)$. Recently Ivić [**9**] further developed the method of [**3**] to study the maximal orders for $\omega\big(f(n)\big)$ and $\log f\big(f(n)\big)$ for a fairly wide class of prime independent, integer valued multiplicative functions f. Their results (cf. (3.3) and (3.4) of [**3**], (11) and (12) of [**9**]) are approximate. As they indicated, it seems difficult to determine precisely these maximal orders, even in the case of $f(n) = \tau(n)$, $a(n)$ (the number of nonisomorphic abelian groups of order n).

In this section, we consider another interesting example: $f(n) = \tau^{(e)}(n)$.

Theorem 3. *(i) A maximal order for $\Omega\big(\tau^{(e)}(n)\big)$ is $(\log n)/2\log_2 n$.*

(ii) A maximal order for $\omega\big(\tau^{(e)}(n)\big)$ is $(\log_2 n)/(\log 2)\log_3 n$.

(iii) We have

$$\log\tau^{(e)}\big(\tau^{(e)}(n)\big) \le \big\{1+o(1)\big\}\Big(\frac{\log_2 n}{\log_3 n}\Big)^2. \tag{4.1}$$

In addition, the inequality

$$\log\tau^{(e)}\big(\tau^{(e)}(n)\big) \ge \big\{\log 2+o(1)\big\}\frac{\log_2 n}{\log_3 n} \tag{4.2}$$

holds for infinitely many integers n.

Proof. On the one hand, using the relation (1.3), we immediately see that

$$\Omega\big(\tau^{(e)}(n)\big) \le \frac{\log\tau^{(e)}(n)}{\log 2} \le \big\{\tfrac{1}{2}+o(1)\big\}\frac{\log n}{\log_2 n}.$$

On the other hand, putting $n_k := (p_1p_2\cdots p_k)^2$ $(k = 1,2,\cdots)$, where p_j denotes the jth prime number, we have $\Omega\big(\tau^{(e)}(n_k)\big) = \Omega\big(2^k\big) = k$. It is clear that $\log n_k \le 2k\log p_k$ and Chebyshev's estimate implies that $\log n_k \gg p_k$. Thus it follows that $k \ge (\log n_k)/(2\log_2 n_k)\big\{1+O(1/\log_2 n_k)\big\}$. This proves the first assertion.

In view of (1.1), we can write that

$$\omega\big(\tau^{(e)}(n)\big) = \omega\Big(\prod_{p^\nu\|n}\tau(\nu)\Big) = \omega\Big(\prod_{p^\nu\|n}\prod_{p'^\mu\|\nu}(\mu+1)\Big). \tag{4.3}$$

Noticing that $\mu \le (\log \nu)/\log 2$ and $\nu \le (\log n)/\log 2$, we have $\mu \le (\log_2 n)/\log 2 + O(1)$. It is clear that the right-hand side of (4.3) does not exceed the number of prime numbers $\le \log_2 n/\log 2 + O(1)$, i.e.

$$\omega\big(\tau^{(e)}(n)\big) \le \pi\big(\log_2 n/\log 2 + O(1)\big) \le \Big\{\frac{1}{\log 2} + O\Big(\frac{1}{\log_3 n}\Big)\Big\}\frac{\log_2 n}{\log_3 n}.$$

In order to establish the lower bound, we consider $n_k := \prod_{j=1}^{k} p_j^{2^{p_j-1}}$ $(k \in \mathbf{N})$. We have $\omega\big(\tau^{(e)}(n_k)\big) = \omega\big(\prod_{j=1}^{k} \tau(2^{p_j-1})\big) = \omega\big(\prod_{j=1}^{k} p_j\big) = k$ and $2^{p_k-1}\log p_k \le \log n_k \le 2^{p_k-1} k \log p_k$. Using the relation $p_k \sim k\log k$, we find $k \sim \dfrac{(\log_2 n_k)}{(\log 2)}\log_3 n_k$ $(k \to \infty)$.

Finally we consider (iii). We write $\log \tau^{(e)}\big(\tau^{(e)}(n)\big) = \sum_{p^\nu \| \tau^{(e)}(n)} \log \tau(\nu)$. For $p^\nu \,\|\, \tau^{(e)}(n)$, the relation (1.3) implies that $\nu \le (\log \tau^{(e)}(n))/\log 2 \le \{\frac{1}{2} + o(1)\}(\log n)/\log_2 n$. Thus by a well-known result, we get

$$\log \tau(\nu) \le \big\{\log 2 + o(1)\big\}\frac{\log \nu}{\log_2 \nu} \le \big\{\log 2 + o(1)\big\}\frac{\log_2 n}{\log_3 n}.$$

This and (ii) yield that

$$\log \tau^{(e)}\big(\tau^{(e)}(n)\big) \le \big\{\log 2 + o(1)\big\}\frac{\log_2 n}{\log_3 n}\omega\big(\tau^{(e)}(n)\big) \le \big\{1 + o(1)\big\}\Big(\frac{\log_2 n}{\log_3 n}\Big)^2.$$

This proves the inequality (4.1).

Next let $n_k := (p_1 p_2 \cdots p_{p_1 p_2 \cdots p_k})^2$ $(k = 1, 2, \cdots)$, we have

$$\log \tau^{(e)}\big(\tau^{(e)}(n_k)\big) = \log \tau^{(e)}\big(2^{p_1 p_2 \cdots p_k}\big) = \log \tau(p_1 p_2 \cdots p_k) = k \log 2. \tag{4.4}$$

We easily see that

$$\log n_k = 2 \sum_{j \le p_1 p_2 \cdots p_k} \log p_j \asymp p_{p_1 p_2 \cdots p_k} \asymp p_1 p_2 \cdots p_k \log(p_1 p_2 \cdots p_k),$$

thus $\log_2 n_k = \log(p_1 p_2 \cdots p_k) + O(\log_3 n_k) \le k \log p_k + O(\log_3 n_k)$. It is clear that $\log_2 n_k \gg \log(p_1 p_2 \cdots p_k) \gg p_k$ and $\log p_k \le \log_3 n_k + O(1)$. Therefore

$$k \ge \Big\{1 + O\Big(\frac{1}{\log_3 n_k}\Big)\Big\}\frac{\log_2 n_k}{\log_3 n_k}. \tag{4.5}$$

Now the required estimate (4.2) follows from (4.4) and (4.5), completing the proof.

As application of Theorem 3(ii), we state the asymptotic formula, which contains a better error term than that in [**8**] for this special function $\tau^{(e)}(n)$ (see (4.4) and Theorem 5 of [**8**]).

Corollary 1. *Let $g(k) := \prod_{p\mid k} p/(p+1)$, we have*

$$\sum_{n\le x}\omega\big(\tau^{(e)}(n)\big)=\frac{6x}{\pi^2}\sum_{s=1}^{\infty}\frac{\omega\big(\tau^{(e)}(s)\big)g(s)}{s}+O\Big(\sqrt{x}\,\frac{(\log x)^2\log_2 x}{\log_3 x}\Big).$$

Proof. This can be verified by the same argument as in [**8**].

5. Values of $\tau^{(e)}(n)$ compared to $\omega(n)$

In this section, we shall make a comparison between exponential divisors and prime factors of integers. Although the maximal order of $\tau^{(e)}(n)$ is much larger than that of $\omega(n)$, but the average order of $\tau^{(e)}(n)$ is A_1 and the average order of $\omega(n)$ is $\log_2 n$, so that almost all n satisfy $\omega(n)>\tau^{(e)}(n)$. Precisely, we have the following result.

Theorem 4. *For any fixed $A>0$, we have*

$$\sum_{n\le x,\,\omega(n)>\tau^{(e)}(n)} 1 = x+O_A\big(x/(\log_2 x)^A\big). \tag{5.1}$$

Proof. Putting $S := \sum_{n\le x,\,\omega(n)\le\tau^{(e)}(n)} 1$, it follows, by Cauchy–Schwarz' inequality, that

$$S\le 1+\sum_{1<n\le x}\left(\frac{\tau^{(e)}(n)}{\omega(n)}\right)^A\le 1+\Big\{\sum_{1<n\le x}\tau^{(e)}(n)^{2A}\Big\}^{1/2}\Big\{\sum_{1<n\le x}\omega(n)^{-2A}\Big\}^{1/2}. \tag{5.2}$$

Let $h(n)$ be the multiplicative function defined by $\mathbf{1}*h(n)=\tau^{(e)}(n)^{2A}$. It is easy to see that $h(p)=0$ and $h(p^\nu)=\tau(\nu)^{2A}-\tau(\nu-1)^{2A}$ for $\nu\ge 2$. Thus the series $\sum_{n=1}^{\infty}h(n)n^{-s}$ converges absolutely for $\operatorname{Re}e\, s>\frac12$, and this implies

$$\sum_{n\le x}\tau^{(e)}(n)^{2A}\le x\sum_{m\le x}|h(m)|/m\ll x. \tag{5.3}$$

Without loss of generality, we can suppose that A is an integer. By Theorem 12 of [**2**], we have $\sum_{1<n\le x}\omega(n)^{-2A}\ll x/(\log_2 x)^{2A}$. Now the relation (5.1) follows from (5.2) and (5.3). This completes the proof of Theorem 4.

The following result, due to Ivić, exhibits integers n for which $\omega(n)=\tau^{(e)}(n)$.

Theorem 5. *For each $A>0$, there exists two positive constants $C_1(A)$, $C_2(A)$ such that*

$$C_1(A)x(\log_2 x)^A/\log x\le\sum_{n\le x,\,\omega(n)=\tau^{(e)}(n)}1\le C_2(A)x/(\log_2 x)^A. \tag{5.4}$$

Proof. Obviously the second inequality of (5.4) immediately follows from Theorem 4. In order to prove the first one, we suppose, as before, that A is an integer.

Considering the integer of type $n = 2^{2^A} p_1 \cdots p_A$, where $p_1, \cdots, p_A$ are distinct odd prime numbers, then we have $\tau^{(e)}(2^{2^A} p_1 \cdots p_A) = \tau^{(e)}(2^{2^A}) = \tau(2^A) = A + 1 = \omega(2^{2^A} p_1 \cdots p_A)$. Thus we deduce that

$$\sum_{n \le x,\, \omega(n) = \tau^{(e)}(n)} 1 \ge \sum_{n \le x/2^{2^A},\, \omega(n) = A,\, 2 \nmid n} \mu(n)^2. \tag{5.5}$$

Introducing the function $H_r(x) := \sum_{n \le x,\, \omega(n)=r,\, 2 \nmid n} \mu(n)^2$ we easily see that (cf. (5.14) of [**8**])

$$H_r(x) + H_{r-1}(x/2) = \sum_{n \le x,\, \omega(n) = r} \mu(n)^2 = \{1 + o(1)\} \frac{x}{\log x} \frac{(\log_2 x)^{r-1}}{(r-1)!}.$$

Since (5.5) holds for any positive integer A, we must have that

$$\begin{aligned} \sum_{n \le x,\, \omega(n) = \tau^{(e)}(n)} 1 &\ge \tfrac{1}{2} \left\{ H_{A+1}\big(x/2^{2^{A+1}}\big) + H_A\big(x/2^{2^A}\big) \right\} \\ &\ge \tfrac{1}{2} \left\{ H_{A+1}\big(x/2^{2^{A+1}}\big) + H_A\big(x/2^{2^{A+1}+1}\big) \right\} \gg_A x (\log_2 x)^A / \log x. \end{aligned}$$

This proves Theorem 5.

Opposite to the usual divisor function $\tau(n)$, we shall show that $\sum_{n \le x} \tau^{(e)}(n)$ is dominated by a large number (actually almost all inyegers $\le x$) of normal integers.

Theorem 6. *For any $\varepsilon \in (0,1)$, we have*

$$\sum_{\substack{n \le x \\ |\omega(n) - \log_2 x| > \varepsilon \log_2 x}} \tau^{(e)}(n) \ll x/(\log x)^{\eta} \tag{5.6}$$

with $\eta := \min\big\{(1+\varepsilon)\log(1+\varepsilon) - \varepsilon,\ (1-\varepsilon)\log(1-\varepsilon) + \varepsilon\big\} > 0$. In particular, the mean value $(1/x)\sum_{n \le x} \tau^{(e)}(n)$ is given by the integers such that $\omega(n) = \log_2 x + O\big(\xi(x)\sqrt{\log_2 x}\big)$, where $\xi(x) \to \infty$, $\xi(x) = o\big(\sqrt{\log_2 x}\big)$.

Proof. For each $z > 0$, the function $\tau^{(e)}(n) z^{\omega(n)}$ is multiplicative and $\tau^{(e)}(p) z^{\omega(p)} = z$ for all prime numbers p. Thus we have that $\sum_{n=1}^{\infty} \tau^{(e)}(n) z^{\omega(n)} n^{-s} = \zeta(s)^z G(s)$ for $\mathrm{Re}\, s > 1$, where $\zeta(s)$ is the Riemann zeta-function and $G(s)$ is a Dirichlet series absolutely convergent for $\mathrm{Re}\, s > \frac{1}{2}$. By a standard analytic argument (see the proof of Theorem II.5.3 of [**11**]), we can show that

$$\sum_{n \le x} \tau^{(e)}(n) z^{\omega(n)} \ll x (\log x)^{z-1}.$$

Hence we deduce that

$$\sum_{\substack{n\le x\\ |\omega(n)-\log_2 x|>\varepsilon\log_2 x}} \tau^{(e)}(n) \le \sum_{n\le x} \tau^{(e)}(n)\big\{(1+\varepsilon)^{\omega(n)-(1+\varepsilon)\log_2 x} + (1-\varepsilon)^{\omega(n)-(1-\varepsilon)\log_2 x}\big\} \ll x(\log x)^{-\eta},$$

where η is defined as in Theorem 6. Noticing that $\eta = \min\Big\{\int_1^{1+\varepsilon} \log t\,\delta t,\ \int_{1-\varepsilon}^{1} \log\frac{1}{t}\,\delta t\Big\}$, we immediately see $\eta > 0$ for any $\varepsilon \in (0,1)$. This proves the inequality (5.6). Combining this with (1.4) yields the second assertion. The proof of Theorem 6 is finished.

Remark 2. In view of (1.4), it is easy to show that the function $\tau^{(e)}(n)$ does not have a monotone normal order.

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