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FINITE EQUATIONS IN N UNKNOWNS

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Abstract. S. Prešić determined in [**6**] all reproductive general solutions of a finite equation, supposing that particular solutions are known. In [**2**] all general solutions (including all reproductive general solutions) of a finite equation were determined, without the above supposition. In this paper we consider finite equations in n unknowns and we solve them by the method of successive eliminations.

The study of general and reproductive solutions of Boolean equations, initiated by Löwenheim, was continued and generalized to the several subjects. The first result within a set-theoretical framework was obtained by Prešić [**5**]. We firstly state the definition of the general solution and reproductive general solution from [**5**].

Definition 1. Let E be a given nonempty set and R be a given unary relation of E. A formula $x = \phi(t)$, where $\phi : E \to E$ is a given function, represents a general solution of the x-equation $R(x)$ if and only if

$$(\forall t)R(\phi(t)) \wedge (\forall x)(R(x) \Rightarrow (\exists t)(x = \phi(t))).$$

A formula $x = \psi(t)$, where $\psi : E \to E$ is a given function, represents a reproductive general solution of the x-equation $R(x)$ if and only if

$$(\forall t)R(\psi(t)) \wedge (\forall t)(R(t) \Rightarrow t = \psi(t)).$$

Let $Q = \{q_0, q_1, \dots, q_m\}$ be a given set of $m+1$ elements and $S = \{0, 1\}$. Define the operations $+$ and $\circ$ in the following way:

$$\begin{array}{c|cc} + & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 1 \end{array} \qquad \begin{array}{c|cc} \circ & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array} \qquad x = \begin{cases} 1, \text{if} \quad R(X) \\ 0, \text{otherwise} \end{cases}$$

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Assuming that

$$(\forall x \in S \cup Q)(x + 0 = x \ \wedge\ 0 + x = x \ \wedge\ x \circ 0 = 0 \ \wedge\ x \circ 1 = x \ \wedge\ 1 \circ x = x),$$

S. Prešić considered the following x-equation

$$s_0 \circ x^{q_0} + s_1 \circ x^{q_1} + \cdots + s_m \circ x^{q_m} = 0 \qquad (s_i \in \{0,1\},\, x \in Q).$$

The latter equation is consistent (has a solution) if and only if $\prod_{k \in M} s_k = 0$, where $M = \{0, 1, \ldots, m\}$. In the sequel $\circ$ will be omitted.

Theorem 1. [**1**] *Let the equation*

$$\sum_{k \in M} s_k x^{q_k} \qquad (s \in \{0,1\}) \tag{1}$$

be consistent (i.e., $\prod_{k \in M} s_k = 0$*). Then the formula*

$$x = \sum_{k \in M} \big(s_k^0 q_k + s_k s_{k \oplus 1}^0 q_{k \oplus 1} + \cdots + s_k s_{k \oplus 1} \cdots s_{k \oplus (m-1)}^0 q_{k \oplus (m-1)} + \cdots + s_k s_{k \oplus 1} \cdots s_{k \oplus (m-1)} q_{k \oplus m} \big) t^{q_k},$$

where $\oplus$ *is the addition mod*$(m+1)$*, defines the reproductive general solution of the equation (1).*

Comment. Theorem 1 is a special case of Theorem 2 in [**2**].

Theorem 2. *Every function* $f : Q \to \{0,1\}$ *satisfies the identity*

$$f(x) = \sum_{B \in Q^n} f(B) X^B . \tag{2}$$

Proof. Let the right-hand side of (2) be denoted by h(X). For every $X \in Q^n$ there is $B_0 \in Q^n$ such that $X = B_0$. Hence

$$h(X) = \sum_{B \in Q^n} f(B) B_0^B = f(B) B_0^{B_0} = f(B_0) = f(X). \qquad \square$$

Remark 1. Any given n-ary relation R on Q is equivalent to an equation i.e.,

$$R(X) \Leftrightarrow \sum_{B \in Q^n} f(B) X^B = 0, \qquad \text{where } f(B) = \begin{cases} 1, & \text{if} \quad R(X) \\ 0, & \text{otherwise} \end{cases} .$$

Lemma 1. *Let* $f : Q \to \{0,1\}$*. Then, for every* $i \in \{1, \ldots, n\}$*, we have*

$$f(x_1, \ldots, x_n) = \sum_{k \in M} x_i^{q_k} f(x_1, \ldots, x_{i-1}, q_k, x_{i+1} \ldots, x_n). \tag{3}$$

Proof. Let the right-hand side of (3) be denoted by $g(x_1,\dots,x_n)$. If $(x_1,\dots,x_n)\in Q^n$, then

$$\begin{aligned} g(x_1,\dots,x_{i-1},x_i,x_{i+1}\dots,x_n) &= \sum_{k\in M} x_i^{q_k} f(x_1,\dots,x_{i-1},q_k,x_{i+1}\dots,x_n)\\ &= x_i^{x_i} f(x_1,\dots,x_{i-1},x_i,x_{i+1}\dots,x_n)\\ &= f(x_1,\dots,x_{i-1},x_i,x_{i+1}\dots,x_n). \quad \square \end{aligned}$$

We consider the following equation:

$$\sum_{B\in Q^n} f(B)X^B = 0. \tag{4}$$

Note that (4) is consistent if and only if

$$\prod_{B\in Q^n} f(B) = 0. \tag{5}$$

THEOREM 3. *Let $f(x_1,\dots,x_n)=0$ $(f:Q^n\to\{0,1\}$ be a consistent equation. Set*

$$f_p(x_1,\dots,x_n) = \prod_{(b_{p+1},\dots,b_n)\in Q^{n-p}} f(x_1,\dots,x_p,b_{p+1},\dots,b_n).$$

for $p=1,\dots,n$. Formulas obtained from

$$\begin{aligned} x_p = \sum_{k\in M} \big(& f_p^0(x_1,\dots,x_{p-1},q_k)q_k +\\ & + f_p(x_1,\dots,x_{p-1},q_k) f_p^0(x_1,\dots,x_{p-1},q_{k\oplus 1})q_{k\oplus 1} + \cdots\\ & + f_p(x_1,\dots,x_{p-1},q_k) f_p(x_1,\dots,x_p,q_{k\oplus 1})\cdots\\ & \quad f_p^0(x_1,\dots,x_{p-1},q_{k\oplus(m-1)})q_{k\oplus(m-1)}\\ & + f_p(x_1,\dots,x_{p-1},q_k) f_p(x_1,\dots,x_{p-1},q_{k\oplus 1})\cdots\\ & \quad f_p^0(x_1,\dots,x_{p-1},q_{k\oplus(m-1)})q_{k\oplus m}\big) t_p^{q_k} \end{aligned}$$

by performing the successive substitutions, define the reproductive general solutions of $f(x_1,\dots,x_n)=0$.

Proof. We start from the equation $f_n(x_1,\dots,x_n)=0$ $(f_n=f)$. At step $n-p+1$ $(p=n,n-1,\dots,1)$ we consider the equation $f_p(x_1,\dots,x_p)=0$ which can be written as

$$\sum_{k\in M} x_p^{q_k} f_p(x_1,\dots,x_{p-1},q_k) = 0,$$

by Lemma 1. Solving this equation we get, by Theorem 1, the reproductive general solution

$$
\begin{aligned}
x_p = \sum_{k\in M} \big(& f_p^0(x_1,\dots,x_{p-1},q_k)q_k + \\
& + f_p(x_1,\dots,x_{p-1},q_k)f_p^0(x_1,\dots,x_{p-1},q_{k\oplus 1})q_{k\oplus 1} + \cdots \\
& + f_p(x_1,\dots,x_{p-1},q_k)f_p(x_1,\dots,x_p,q_{k\oplus 1})\cdots \\
& \quad f_p^0(x_1,\dots,x_{p-1},q_{k\oplus(m-1)})q_{k\oplus(m-1)} \\
& + f_p(x_1,\dots,x_{p-1},q_k)f_p(x_1,\dots,x_{p-1},q_{k\oplus 1})\cdots \\
& \quad f_p^0(x_1,\dots,x_{p-1},q_{k\oplus(m-1)})q_{k\oplus m}\big)t_p^{q_k}
\end{aligned}
$$

provided the consistency condition

$$\prod_{b_p\in Q} f_p(x_1,\dots,x_{p-1},b_p) = 0,$$

i.e.,

$$\prod_{(b_p,\dots,b_n)\in Q^{n-p}} f_n(x_1,\dots,x_{p-1},b_p,\dots,b_n) = 0,$$

i.e.,

$$f_{p-1}(x_1,\dots,x_{p-1}) = 0.$$

At step n we solve the equation $f_1(x_1) = 0$ which can be written in the form

$$\sum_{k\in M} x_1^{q_k} f_1(q_k) = 0.$$

Solving the latter equation we get

$$
\begin{aligned}
x_1 = \sum_{k\in M} \big(& f_1(q_k)q_k + f_1(q_k)f_1^0(q_{k\oplus 1})q_{k\oplus 1} + \cdots \\
& + f_1(q_k)f_1(q_{k\oplus 1})\cdots f_1^0(q_{k\oplus(m-1)})q_{k\oplus(m-1)} \\
& + f_1(q_k)f_1(q_{k\oplus 1})\cdots f_1^0(q_{k\oplus(m-1)})q_{k\oplus m}\big)t_p^{q_k}
\end{aligned}
$$

provided the consistency condition

$$\prod_{b_1\in Q} f_1(b_1) = 0$$

i.e.,

$$\prod_{(b_1,\dots,b_n)\in Q^n} f(b_1,\dots,b_n) = 0) = 0 \tag{6}$$

i.e.,

$$f_0 = 0.$$

Note that (6) is the consistency condition (5), which is fulfilled, by assumption of Theorem. $\square$

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