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ON CESARO MEANS IN HARDY SPACES

Miroslav Pavlović

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Abstract. In the case $1/2 < p < 1$ we generalize the Hardy-Littlewood theorem on (C, α) means in H^p, $\alpha > 1/p - 1$, by proving that $M_p(\sigma_n^\alpha u, r) \leq C_{p,\alpha} M_p(u, r)$, $0 < r < 1$, where u is a harmonic function such that $\hat{u}(k) = 0$ for $k < -2n$. In the case $p \leq 1/2$ such a generalization is not possible, but the above estimate is valid if $\sigma_n^\alpha u$ are replaced by Riesz type means.

1. Introduction and results

Let H^p denote the usual Hardy space of analytic functions on the unit disc (cf. [**1**]). A result of Hardy and Littlewood [**3**] states that

$$\|\sigma_n^\alpha \phi\|_{H^p} \leq C_{p,\alpha} \|\phi\|_{H^p} \qquad (0 < p < 1, \ \alpha > 1/p - 1), \tag{1}$$

where $\sigma_n^\alpha \phi$ are the Cesáro means of order α of the Taylor series of ϕ. Gwilliam [**2**] extended this result to the case of harmonic functions by proving that

$$\sup_{r<1} M_p(\sigma_n^\alpha u, r) \leq C_{p,\alpha} \sup_{r<1} M_p(u, r), \tag{2}$$

where p and α are as in (1), and

$$M_p(u, r) = \left\{ \int_0^{2\pi} |u(re^{it})|^p \frac{dt}{2\pi} \right\}^{1/p}, \qquad r \geq 0.$$

If u is the Poisson kernel, then $M_p(u, r) \to 0$ $(r \to 1^-)$, when $p < 1$, which shows that (2) cannot be improved to obtain, for an arbitrary harmonic function u,

$$M_p(\sigma_n^\alpha u, r) \leq C_{p,\alpha} M_p(u, r) \qquad (0 < r < 1). \tag{3}$$

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In the case $p \leq 1/2$ the situation is even worse because then there exist harmonic polynomials u_n, $\deg(u_n) < n$, such that $M_p(\sigma_n^\alpha, l)/M_p(u_n, 1) \to \infty$ $(n \to \infty)$ (see Theorem 3 below). However, if $1/2 < p < 1$, then (3) holds provided $\hat{u}(k) = 0$ for $k < -2n$. We state this fact in the following form.

Theorem 1. *Let $f \in L_1(0, 2\pi)$ be such that $\hat{f}(k) = 0$ for $k < -2n$, where n is a positive integer. If $1/2 < p < 1$ and $\alpha > 1/p - 1$, then*

$$\|\sigma_n^\alpha f\|_p \leq C_{p,\alpha}\|f\|_p, \tag{4}$$

where $C_{p,\alpha}$ is a constant depending only on p and α.

Here

$$\|f\|_p = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(t)|^p dt \right\}^{1/p},$$

and $\hat{f}$ is the Fourier transformation of $f \in L_1 = L_1(0, 2\pi)$.

We will deduce Theorem 1 from the following result by using the known estimates for Fejer's kernels [**5**].

Theorem 2. *Let $n \geq 1$, $0 < p < 1$ and $f, g \in L_1$ be such that $\hat{f}(k) = 0$ for $k < -n$ and $\hat{g}(k) = 0$ for $k > n$. Then*

$$\|f * g\|_p \leq C_p n^{1/p-1}\|f\|_p\|g\|_p. \tag{5}$$

Here $f * g$ denotes the convolution of f and g. Under the conditions of Theorem 2 this is a trigonometric polynomial of degree $\leq n$; we have $(f * g)\hat{\ }(k) = \hat{f}(k)\hat{g}(k)$.

In Section 4 we will show how (1) can easily be deduced from another important result of Hardy and Littlewood (cf. [**1**]):

$$|\hat{\phi}(n)| \leq C_p n^{1/p-1}\|\phi\|_{H_p} \qquad (\phi \in H^p,\ 0 < p < 1). \tag{6}$$

(Here $\hat{\phi}(n)$ are Taylor's coefficients of ϕ.) Although (6) is contained in (5) we will deduce (5) from (6) very simply.

Theorem 2 will be used to prove part of the following.

Theorem 3. *Let n and f be as in Theorem 1 and $\alpha \geq 1$. Then*

$$\|\sigma_n^\alpha f\|_{1/2} \leq C_\alpha(\log(2n))^2\|f\|_{1/2} \tag{7}$$

$$\|\sigma_n^\alpha f\|_p \leq C_{p,\alpha} n^{1/p-2}\|f\|_p \qquad (0 < p < 1/2). \tag{8}$$

These inequalities are the best possible in the sense that there are (nontrivial) trigonometric polynomials f_n, independent of p, α, such that $\deg(f_n) \leq n/2$ and

$$\|\sigma_n^\alpha f_n\|_{1/2} \geq c_\alpha\|f_n\|_{1/2}(\log n)^2 \tag{9}$$

$$\|\sigma_n^\alpha f_n\|_p \geq c_{p,\alpha}\|f_n\|_p n^{1/p-2} \qquad (0 < p < 1/2). \tag{10}$$

As a further application of Theorem 2 we shall prove that Theorem 1 is true for all $p<1$ if the Cesáro means are replaced by the Riesz (spherical) means. The latter are defined as

$$(R_n^\alpha f)(t)=\sum_{|k|<n}\left(1-\left(\frac{k}{n}\right)^2\right)^\alpha \hat{f}(k)e^{ikt}, \qquad n\geq 1. \tag{11}$$

THEOREM 4. *If f is as in Theorem 1, $0<p<1$ and $\alpha>1/p-1$, then*

$$\|R_n^\alpha f\|_p \leq C_{p,\alpha}\|f\|_p. \tag{12}$$

COROLLARY. *If $f\in H^p$, $0<p<1$ and $\alpha>1/p-1$, then $\|R_n^\alpha f\|_{H^p}\leq C_{p,\alpha}\|f\|_{H^p}$.*

Proof of Theorems 2, 1 and 4

Proof of Theorem 2. Assuming, as we may, that f and g are trigonometric polynomials we define the analytic polynomials F and G by $\hat{F}(k)=\hat{f}(k-n)$ and $\hat{G}(k)=\hat{g}(n-k)$, $k\geq 0$. It follows from the hypotheses that $g(-t)=e^{-\int}G(e^{it})$ and $f(t)=e^{-\int}F(e^{it})$. Now write $f*g$ as

$$\begin{aligned}(f*g)(t)&=\sum_{k=0}^{2n}\hat{f}(n-k)\hat{g}(n-k)e^{i(k-n)t}\\&=\sum_{k=0}^{2n}\hat{G}(k)\hat{F}(2n-k)e^{i(k-n)t}.\end{aligned}$$

Hence, for a fixed t, we have that $(f*g)(t)=\hat{\phi}(2n)e^{-\int}$, where ϕ is the analytic function defined by

$$\phi(z)=G(e^{it}z)F(z).$$

Now we apply (6) to obtain

$$\begin{aligned}|(f*g)(t)|^p&\leq C_p(2n)^{1-p}\int_0^{2\pi}\left|G\left(e^{i(t+\theta)}\right)F(e^{i\theta})\right|^p d\theta\\&=C_p n^{1-p}\int_0^{2\pi}|g(-t-\theta)f(\theta)|^p d\theta.\end{aligned}$$

Integrating this over the interval $0\leq t\leq 2\pi$ and using Fubini's theorem we get (5). $\square$

Remark. Somewhere we use C to denote constants which may vary from line to line.

Proof of Theorem 1. Recall that $\sigma_n^\alpha f = K_n^\alpha * f$, where K_n^α are Fejer's kernels,

$$K_n^\alpha(t) = \sum_{-n}^{n} \frac{B(\alpha, n+1)}{B(\alpha, n+1-|k|)} e^{ikt}, \tag{13}$$

where B is the Euler Beta function. By Theorem 2, inequality (4) follows from the inequality $\|K_n^\alpha\|_p \le C n^{1-1/p}$ $(1/2 < p < 1, \alpha > 1/p - 1)$. If $0 < \alpha \le 1$, this is easily obtained by integration from

$$|K_n^\alpha(t)| \le C_\beta \min\left(n, n^{-\beta}|t|^{-\beta-1}\right) \qquad (0 < \beta \le 1, |t| < \pi)$$

(see [**5**, p. **48**]). If $\alpha > 1$, we use the formula

$$K_n^\alpha(t) = \left(\sum_{j=0}^{n} A_{n-j}^{\alpha-2} A_j^l K_j^l(t)\right) / A_n^\alpha,$$

where, for $\beta > -1$,

$$A_n^\beta = \binom{n+\beta}{n} \sim \frac{n^\beta}{\Gamma^{(\beta+1)}} \qquad (n \to \infty) \tag{14}$$

(see [**5**, p. 42 and Ch. 3.13]). Combining these relations we find that $|K_n^\alpha(t)| \le C_\alpha n^{-1}|t|^{-2}$ $(|t| < \pi)$. Since also $|K_n^\alpha(t)| \le 2n + 1$ (by (13)) we obtain

$$|K_n^\alpha(t)| \le C_\alpha \min(n, n^{-1}|t|^{-2}) \qquad (\alpha \ge 1,\ |t| < \pi). \tag{15}$$

Now integration yields the desired estimate for $\|K_n\|_p$. This completes the proof of Theorem 1. □

Proof of Theorem 4. In this case we have $R_n^\alpha f = T_n^\alpha * f$, where

$$T_n^\alpha(t) = \sum_{-n}^{n} \left(1 - \frac{k}{n}\right)^\alpha \left(1 + \frac{k}{n}\right)^\alpha e^{ikt}.$$

Hence T_n^α is the convolution of the functions

$$\sum_{k=-n}^{\infty} \left(1 + \frac{k}{n}\right)^\alpha r_n^k e^{ikt} =: h(t)$$
$$\sum_{k=-\infty}^{n} \left(1 - \frac{k}{n}\right)^\alpha r_n^{-k} e^{ikt} = h(-t)$$

where $r_n = 1 - 1/(n+1)$. Hence, by Theorem 2,

$$\|T_n^\alpha * f\|_p \le C_p n^{1/p-1} \|T_n^\alpha\|_p \|f\|_p \le C_p n^{2/p-2} \|h\|_p^2 \|f\|_p.$$

So it suffices to prove that $\|h\|_p \le C_{p,\alpha} n^{1-1/p}$, for $0 < p < 1$, $\alpha > 1/p - 1$. We have

$$h(t) = r_n^{-n} n^{-\alpha} e^{-int} \phi(r_n e^{it}),$$

where

$$\phi(z) = \sum_{k=1}^{\infty} k^\alpha z^k,$$

and hence

$$\|h\|_p \le C n^{-\alpha} M_p(\phi, r_n).$$

Now we use two familiar estimates,

$$|\phi(z)| \le C_\alpha |1-z|^{-\alpha-1} \qquad (|z| < 1)$$

and

$$\int_0^{2\pi} |1 - re^{it}|^{-\beta} dt \le C_\beta (1-r)^{1-\beta} \qquad (\beta > 1),$$

to obtain

$$M_p^p(\phi, r_n) \le C_{p,\alpha} (1 - r_n)^{1-(\alpha+1)p}.$$

Combining these inequalities we conclude the proof. □

3. Proof of Theorem 3

Inequalities (7) and (8) follow from Theorem 2 and (15). To prove the rest define trigonometric polynomials f_n by $\hat{f}_n(k) = \eta(k/n)$, where η is an even C^∞-function on the real line such that $\eta(x) = 1$ for $|x| < 1/4$ and $\eta(x) = 0$ for $|x| > 1/2$. It is easily shown (see, for example, [**4**, p. 177]) that $\|f_n\|_p \le C_p n^{1-1/p}$. So it remains to prove that

$$\|\sigma_n^\alpha f_n\|_p \ge c_\alpha n^{-1} (\log n)^2 \qquad (p = 1/2) \tag{16}$$

$$\|\sigma_n^\alpha f_n\|_p \ge c_{p,\alpha} n^{-1} \qquad (p < 1/2), \tag{17}$$

where c_α and $c_{p,\alpha}$ are positive constants.

Let

$$F_n(x) = \frac{\eta(x/n)}{B(\alpha, n+1-|x|)}, \quad -\infty < x < \infty, \tag{18}$$

$$G_n(t) = (1 - e^{it})^3 \sum_{-\infty}^{\infty} F_n(k) e^{ikt}.$$

We will show that there exists a constant $A > 0$ such that

$$|G_n(t)| \geq A^{-1} n^{\alpha-1} t \qquad (n > A, \;\; A/n < t < \pi), \tag{19}$$

which will imply

$$|\sigma_n^\alpha f_n(t)| \geq c_\alpha n^{-1} t^{-2} \qquad (A/n < t < \pi) \tag{20}$$

because

$$\sigma_n^\alpha f_n(t) = (1 - e^{it})^{-3} B(\alpha, n+1) G_n(t).$$

Inequalities (16) and (17) are immediate consequences of (20).

To prove (19) observe first that G_n is a trigonometric polynomial of degree $< n/2 + 3$. The coefficients of G_n are given by

$$\hat{G}_n(k) = F_n(k) - 3F_n(k-1) + 3F_n(k-2) - F_n(k-3).$$

Using this we get

$$\hat{G}_n(1)e^{it} + \hat{G}_n(2)e^{2it} = (4F_n(1) - 3F_n(0) - F_n(2))e^{it}(1 - e^{it}).$$

Since, by (18), for $n > 4$,

$$4F_n(1) - 3F_n(0) - F_n(2) = \left(4 - \frac{3(\alpha+n)}{n} - \frac{n}{\alpha+n-1}\right) \frac{1}{B(\alpha, n)} > c_\alpha n^{\alpha-1},$$

we see that

$$|\hat{G}_n(1)e^{it} + \hat{G}_n(2)e^{2it}| \geq c_\alpha n^{\alpha-1}|t| \qquad (|t| < \pi). \tag{21}$$

On the other hand, if $k \neq 1, 2$ and $|k| < n/2 + 3$, then we can apply Lagrange's theorem for symmetric differences to obtain

$$|\hat{G}_n(k)| \leq \sup\{|F_n'''(x)| \colon 0 < |x| < n/2 + 3\}.$$

(It follows from (18) that $F_n'''(x)$ exists for $x \neq 0$.) The formula

$$\left(\frac{d}{dx}\right)^m B(\alpha, n+1-x) = \int_0^1 t^{\alpha-1}(1-t)^{n-x} \left(\log \frac{1}{1-t}\right)^m dt$$

$(0 < x < n+1)$ together with the inequality $\log(1/(1-t)) \leq t/(1-t)$, $0 < t < 1$, shows that

$$(d/dx)^m B(\alpha, n+1-x) \leq B(\alpha+m, n+1+m-x) \leq C_\alpha n^{-\alpha-m}$$

for $m = 1, 2, 3$ and $0 < x < n/2 + 3$, $n > 10$. Using this we can show, after an elementary but rather long computation which we omit, that $|F_n'''(x)| \le C_\alpha n^{\alpha-3}$. Hence $|\hat{G}_n(k)| \le C_\alpha n^{\alpha-3}$ and hence

$$\Big|\sum_{k\neq 1,2} \hat{G}_n(k)e^{ikt}\Big| \le C_\alpha n^{\alpha-2}.$$

Combining this with (21) we obtain

$$|G_n(t)| \ge c_\alpha n^{\alpha-1}|t| - C_\alpha n^{\alpha-2} \qquad (|t| < \pi),$$

which implies (19). This completes the proof of Theorem 3. $\square$

4. Remarks

(A) A simple proof of (1) can be given by using the identity

$$\phi(z\zeta)(1-z)^{-\alpha-1} = \sum_{k=0}^{\infty} A_k^\alpha(\sigma_k^\alpha\phi)(\zeta)z^n, \quad |z| < 1, \ |\zeta| = 1,$$

where A_k^α is defined by (14). From this and an obvious modification of (6) it follows that

$$r^{np}|A_n^\alpha(\sigma_n^\alpha\phi)(\zeta)|^p \le C_p n^{1-p} \int_0^{2\pi} |\phi(re^{it}\zeta)|^p|1-re^{it}|^{-(\alpha+1)p}dt$$

Now let $r = 1 - 1/n$ and integrate over the circle $|\zeta| = 1$ to obtain (1).

(B) Inequality (1) can also be deduced from (5) by considering $\sigma_n^\alpha\phi$ as the convolution of the functions $\phi(r_n e^{it})$, $r_n = 1 - 1/(n+1)$, and

$$\sum_{k=-\infty}^{n} \frac{B(\alpha, n+1)}{B(\alpha, n+1-k)} r_n^{-k}e^{ikt} = r_n^{-n}B(\alpha, n+1)\alpha(1-r_ne^{-it})^{-(\alpha+1)}e^{int}.$$

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Matematički fakultet
Studentski trg 16
11001 Beograd, p.p. 550
Yugoslavia

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