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ON THE OPERATOR SOLUTIONS OF DIFFERENCE EQUATIONS

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Abstract. We analyze the character of the solutions of a class of difference equations in the field of Mikusiński operators $\mathcal{F}$. These difference equations are in fact the discrete analogues for the differential equations in the field $\mathcal{F}$ corresponding to some partial differential equations.

1. Introduction. The linear partial differential equation with constant coefficients on the set $\{(x,t)|\ x \in \mathbf{R},\ t > 0\}$:

$$\sum_{i=0}^{p} A_i \frac{\partial^{2+i} u(x,t)}{\partial x^2 \partial t^i} + \sum_{i=0}^{q} B_i \frac{\partial^{1+i} u(x,t)}{\partial x \partial t^i} + \sum_{i=0}^{r} C_i \frac{\partial^{i} u(x,t)}{\partial t^i} = f(x,t), \tag{1}$$

with some initial and/or boundary conditions is the mathematical model for different physical systems. In (1), A_i, $i = 0,1,\dots,p$, B_i, $i = 0,1,\dots,q$, and C_i, $i = 0,1,\dots,r$, are numerical constants, p, q and r are natural numbers, while the function $f(x,t)$ on the right hand side is given.

If one has the initial conditions

$$\left.\frac{\partial^{\mu+\nu} u(x,t)}{\partial x^{\mu} \partial t^{\nu}}\right|_{t=0} = 0$$

for $\mu = 0$, $\nu = 0,1,\dots,r-1$, $\mu = 1$, $\nu = 0,1,\dots,q-1$, $\mu = 2$, $\nu = 0,1,\dots,p-1$, then the Mikusiński operator calculus can be applied, in order to obtain the solution. (For reader's convenience, we give some notions from that theory in Section 2; for complete exposition, see [**2**].) In the field of Mikusiński operators $\mathcal{F}$, the following nonhomogeneous differential equation corresponds to (1):

$$\sum_{i=0}^{p} A_i s^i u''(x) + \sum_{i=0}^{q} B_i s^i u'(x) + \sum_{i=0}^{r} C_i s^i u(x) = f(x), \tag{2}$$

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where s is the differential operator, $f(x) = \{f(x,t)\}$ and $u(x) = \{u(x,t)\}$ are, respectively, the given and the unknown operator function (for the notations and notions, see Section 2). The most important case is when the operator function $f(x)$ is represented by a continuous function.

In Section 3, using the usual difference schemes, we come to a second order difference equation in the field $\mathcal{F}$, and, in particular, we analyze its characteristic equation. We find sufficient conditions on the numbers r, p and q which imply that the solutions are represented by continuous functions.

In Section 4, we use classical methods from [**1**] for finding the explicit solution of the difference equation in $\mathcal{F}$. Of course, these classical methods had to be adapted for the Mikusiński operator calculus. Similar problems were considered in [**3**], [**4**] and [**6**], though for less general difference equations.

Finally, in Section 5, analogously to [**7**] and [**8**], we estimate the approximation error, i.e., the difference between the exact solution of the differential equation (2) and the approximate one, given as the solution of the corresponding difference equation. We show that the error of approximation in the field $\mathcal{F}$ is of the same order as in the usual numeric case.

2. Notations and notions from the Mikusiński calculus. The set $\mathcal{C}_+$ of continuous functions defined on $[0,\infty)$, with the usual addition and the multiplication given by the convolution

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau, \quad t > 0,$$

is a ring. By the Tichmarsh theorem, $\mathcal{C}_+$ has no divisors of zero, hence its quotient field can be defined (see [**2**]). Elements of that field, the Mikusiński operator field $\mathcal{F}$, are called *operators*. They are quotients of the form f/g, where $f \in \mathcal{C}_+$ and $0 \not\equiv g \in \mathcal{C}_+$; the last division is observed in the sense of convolution. Clearly, a function $a = a(t)$ from $\mathcal{C}_+$ can be observed also as an operator. This operator is unique and will be denoted simply by a. We shall write then $a = \{a(t)\}$ and say that the operator a is represented by the continuous function $a(t)$.

Among the most important operators are the identical operator $I = f/f$, where f is any function from $\mathcal{C}_+$, not identically equal to zero; the integral operator $\ell = \{1\}$ together with its positive powers ℓ^α and the inverse operator of ℓ, the differential operator s. The following holds

$$\ell s = I, \quad \ell^\alpha = \{t^{\alpha-1}/\Gamma(\alpha)\}, \quad \alpha > 0,$$
$$\{x^{(n)}(t)\} = s^n x - s^{n-1}x(0) - \cdots - x^{(n-1)}(0)I.$$

Let us denote by $\mathcal{F}_c$ the subset of $\mathcal{F}$ consisting of the operators represented by elements of $\mathcal{C}_+$, and by $\mathcal{F}_I$ the subset of $\mathcal{F}$ consisting of the elements γI, for some numerical constant γ.

The only convergence in $\mathcal{F}$ that we shall use is the type I convergence, which is defined as follows: *A sequence of operators* $(p_n)_{n\in\mathbf{N}}$ *converges to* $p \in \mathcal{F}$ *if there exists an operator* $q \neq 0$ *such that the operators* qp_n, $n \in \mathbf{N}$, *as well as* qp *are in* $\mathcal{F}_c$, *and the last sequence converges uniformly on every compact set to the continuous function* qp.

The convergence of an infinite sum in the field of Mikusiński operators is defined accordingly. An example that will be used several times in this paper is the infinite series $\sum_{i=1}^{\infty} \phi^i$, where $\phi \in \mathcal{F}_c$. It is important to note that this series converges and its sum is an operator from $\mathcal{F}_c$.

The operators can be compared if they are from $\mathcal{F}_c$. So for two operators $a = \{a(t)\}$ and $b = \{b(t)\}$ from $\mathcal{F}_c$ we define $a \leq b$ if $a(t) \leq b(t)$ for each $t \geq 0$ (see [**2**, p. 237]). Analogously, for two operator functions we define

$$a(x) \leq_T b(x), \quad x \in [c, d],$$

if $a(x)$ and $b(x)$ are represented by continuous real valued functions of two variables, $a(x) = \{a(x,t)\}$, $b(x) = \{b(x,t)\}$ and $a(x,t) \leq b(x,t) \quad$ for $x \in [c,d]$, $t \in [0,T]$.

The absolute value of an operator a from $\mathcal{F}_c$, $a = \{a(t)\}$, is the operator $|a| = \{|a(t)|\}$. Also, we put $|a(x)| = \{|a(x,t)|\}$.

If the operators a and b are from $\mathcal{F}_c$, then

$$|a + b| \leq |a| + |b|,$$
$$|ab| = \Big| \int_0^t a(\tau) b(t-\tau) d\tau \Big| \leq |a||b|,$$
$$|a| \leq_T \alpha(T)\ell, \quad \alpha(T) = \max_{t\in[0,T]} |a(t)|.$$

3. Difference equations in $\mathcal{F}$. As we announced in the introduction, we consider the nonhomogeneous differential equation (2) in the field of Mikusiński operators $\mathcal{F}$.

The differential equation (2) can be written in the form

$$Pu''(x) + Qu'(x) + Ru(x) = f(x), \tag{3}$$

where $P = \sum\limits_{i=0}^{p} s^i A_i$, $Q = \sum\limits_{i=0}^{q} s^i B_i$, $R = \sum\limits_{i=0}^{r} s^i C_i$.

As is usual in numerical analysis for $h > 0$, instead of $u'(x)$ we take

$$\frac{u(x+h) - u(x-h)}{2h}$$

and also instead of $u''(x)$ we put

$$\frac{u(x+h) - 2u(x) + u(x-h)}{h^2}.$$

So we obtain the difference equation in the field $\mathcal{F}$ corresponding to (3):

$$P\frac{u(x+h)-2u(x)+u(x-h)}{h^2}+Q\frac{u(x+h)-u(x-h)}{2h}+Ru(x)=f(x). \tag{4}$$

Let us take an arbitrary real x_0. If we put $x_n = x_{n-1}+h$, $h>0$, $n = 0,\pm1,\pm2,\ldots$, and define the operator f_n by $f_n = f(x_n)$, then the equation (4) can be written as the difference equation

$$au_{n-1}+bu_n+cu_{n+1}=f_n, \tag{5}$$

where a, b and c are operators from the field $\mathcal{F}$. Putting $r_1 = \max(p,q)$ and $r_2 = \max(p,r)$, we have

(6)
$$a=\frac{I}{h^2}\left(P-\frac{Qh}{2}\right)=\frac{I}{h^2}\left(A_0-\frac{B_0h}{2}\right)I+s^{r_1}\left(a_2I+\phi_a\right)=a_1I+s^{r_1}(a_2I+\phi_a),$$
(7)
$$b=-\frac{I}{h^2}\left(2P-Rh^2\right)=\frac{I}{h^2}\left(-2A_0+h^2C_0\right)I+s^{r_2}\left(b_2I+\phi_b\right)=b_1I+s^{r_2}(b_2I+\phi_b),$$
(8)
$$c=\frac{I}{h^2}\left(P+\frac{Qh}{2}\right)=\frac{I}{h^2}\left(A_0+\frac{B_0h}{2}\right)I+s^{r_1}\left(c_2I+\phi_c\right)=c_1I+s^{r_1}(c_2I+\phi_c).$$

In the previous three relations a_1, b_1, c_1, a_2, b_2, c_2 are numerical constants and ϕ_a, ϕ_b, ϕ_c are operators from $\mathcal{F}_c$. In this paper we analyze the solution of difference equation (5) depending on p, q, r, more precisely on r_1, r_2.

The characteristic equation of the difference equation (5) has the form

$$a+b\omega+c\omega^2=0. \tag{9}$$

Let us analyze the solution of the equation (9).

It is known that the field of Mikusiński operators has very good algebraic properties, which also means that the usual addition and multiplication with operators can be treated in the same way as with real numbers.

Using relations (6), (7) and (8), the characteristic equation (9) can be written as

$$a_1I+s^{r_1}(a_2I+\phi_a)+(b_1I+s^{r_2}(b_2I+\phi_b))\omega+(c_1I+s^{r_1}(c_2I+\phi_c))\omega^2=0.$$

First, let us consider the expression b^2-4ac:

$$\begin{aligned}b^2-4ac&=(b_1I+s^{r_2}(b_2I+\phi_b))^2-4(a_1I+s^{r_1}(a_2I+\phi_a))(c_1I+s^{r_1}(c_2I+\phi_c))\\&=b_1^2I+2b_1s^{r_2}(b_2I+\phi_b)+s^{2r_2}(b_2I+\phi_b)^2-4a_1c_1I\\&\quad-4s^{r_1}(a_1c_2I+a_1\phi_c+a_2c_1I+c_1\phi_a)-4s^{2r_1}(a_2I+\phi_a)(c_2I+\phi_c).\end{aligned}$$

If $r_2 > r_1 > 0$, $b_2 \neq 0$, then it can be transformed as

$$b^2 - 4ac = s^{2r_2}(b_2 I + \phi_b)^2 \Big(I + \ell^{2r_2} \frac{b_1^2 - 4a_1c_1}{(b_2 I + \phi_b)^2} + \frac{2b_1 l^{r_2}}{b_2 I + \phi_b}$$
$$- 4s^{(r_1 - 2r_2)} \frac{a_1 c_2 I + a_1 \phi_c + a_2 c_1 I + c_1 \phi_a}{(b_2 I + \phi_b)^2} - 4s^{2(r_1 - r_2)} \frac{(a_2 I + \phi_a)(c_2 I + \phi_c)}{(b_2 I + \phi_b)^2} \Big).$$

So we can write for $r_2 > r_1 > 0$:

$$b^2 - 4ac =: s^{2r_2}(b_2 I + \phi_b)^2 (I + \psi_1). \tag{10}$$

The operator $I/(b_2 I + \phi_b)^2$ is of the form $I + \psi$, where ψ is represented by a continuous function; therefore, the operator ψ_1, given in (10), is represented by a continuous function too. From relation (10) we obtain

$$\sqrt{b^2 - 4ac} = s^{r_2}(b_2 I + \phi_b) \sum_{i=0}^{\infty} \binom{1/2}{i} (\psi_1)^i$$
$$= s^{r_2}(b_2 I + \phi_b) + s^{r_2}(b_2 I + \phi_b) \sum_{i=1}^{\infty} \binom{1/2}{i} (\psi_1)^i.$$

Hence

$$\sqrt{b^2 - 4ac} =: s^{r_2}(b_2 I + \phi_b)(I + \psi_{q,1}), \tag{11}$$

where the operator $\psi_{q,1}$ is represented by a continuous function.

If, however, $r_1 > r_2 > 0$, then we have

$$b^2 - 4ac = s^{2r_1}(-4a_2c_2) \Big(I + \ell^{2r_1} \frac{b_1^2 - 4a_1c_1}{-4a_2c_2} + \frac{s^{2(r_2 - r_1)}(b_2 I + \phi_b)^2}{-4a_2c_2} +$$
$$\frac{2b_1 s^{r_2 - 2r_1}(b_2 I + \phi_b)}{-4a_2c_2} - 4 \frac{\ell^{r_1}(a_1 c_2 I + a_1 \phi_c + a_2 c_1 I + c_1 \phi_a)}{-4a_2c_2} - 4 \frac{a_2 \phi_c + c_2 \phi_a + \phi_a \phi_c}{-4a_2c_2} \Big).$$

So we can write

$$b^2 - 4ac = s^{2r_1}(-4a_2c_2)(I + \psi_2), \tag{12}$$

and in this case ($r_1 > r_2 > 0$) we have

$$\sqrt{b^2 - 4ac} = s^{r_1} \sqrt{-4a_2c_2} \sum_{i=0}^{\infty} \binom{1/2}{i} (\psi_2)^i$$
$$= s^{r_1} \sqrt{(-4a_2c_2)} + s^{r_1} \sqrt{-4a_2c_2} \sum_{i=1}^{\infty} \binom{1/2}{i} (\psi_1)^i,$$

or

$$\sqrt{b^2-4ac} =: s^{r_1}\sqrt{-4a_2c_2}\,(I+\psi_{q,2}). \tag{13}$$

In relations (12) and (13) ψ_2 and $\psi_{q,2}$ are operators from $\mathcal{F}_c$.

If $p=r_1=r_2>0$, then we have

$$b^2-4ac = s^{2p}(b_2^2 I-4a_2c_2)(I+\psi_3)$$

So in this case we can write

$$\sqrt{b^2-4ac} =: s^p\sqrt{b_2^2 I-4a_2c_2}(I+\psi_{q,3}),$$

where ψ_3 and $\psi_{q,3}$ are operators from $\mathcal{F}_c$.

LEMMA 1. *Assume that coefficients of the equation* (5) *are of the form* (6), (7) *and* (8), *and* $r_2>r_1>0$. *Then one of the solutions of the characteristic equation* (9), *say* ω_1, *belongs to* $\mathcal{F}_c$, *the other one, say* ω_2, *does not, but* I/ω_2 *does.*

Proof. Solutions of the equation (9) have the form

$$\begin{aligned}\omega_{1,2} &= \frac{-b_1I-s^{r_2}(b_2I+\phi_b)\pm\sqrt{b^2-4ac}}{2(c_1I+s^{r_1}(c_2I+\phi_c))}\\ &= \frac{\ell^{r_1}}{2c_2}\cdot\frac{-b_1-s^{r_2}(b_2I+\phi_b)\pm s^{r_2}(b_2I+\phi_b)(I+\psi_{q,1})}{I+\dfrac{c_1}{c_2}\ell^{r_1}+\dfrac{\phi_c}{c_2}}.\end{aligned}$$

The first solution ω_1 has the form

$$\omega_1 = \frac{-b_1\ell^{r_1}+s^{r_2-r_1}(b_2I+\phi_b)\psi_{q,1}}{2c_2}\sum_{j=0}^{\infty}(-1)^j\left(\frac{c_1}{c_2}\ell^{r_1}+\frac{\phi_c}{c_2}\right)^j.$$

From (11) it follows that $s^{r_2-r_1}\psi_{q,1}$ is an operator from $\mathcal{F}_c$, and since the operators ϕ_b, $\psi_{q,1}$ and ϕ_c are from $\mathcal{F}_c$, we obtain that the operator ω_1 is from $\mathcal{F}_c$.

The second solution ω_2 of the characteristic equation has the form

$$\omega_2 = \frac{-b_1\ell^{r_1}-2s^{r_2-r_1}(b_2I+\phi_b)-s^{r_2-r_1}(b_2I+\phi_b)\psi_{q,1}}{2c_2}\sum_{j=0}^{\infty}(-1)^j\left(\frac{c_1}{c_2}\ell^{r_1}+\frac{\phi_c}{c_2}\right)^j.$$

Since $r_2-r_1>0$, the last operator does not belong to $\mathcal{F}_c$. However, the operator I/ω_2 is from $\mathcal{F}_c$, because it can be written as

$$\begin{aligned}\frac{I}{\omega_2} &= \frac{2(c_1I+s^{r_1}(c_2I+\phi_c))}{-b_1I-2s^{r_2}(b_2I+\phi_b)-s^{r_2}(b_2I+\phi_b)\psi_{q,1}}\\ &= \frac{\ell^{r_2}}{-b_2}\cdot\frac{c_1I+s^{r_1}(c_2I+\phi_c)}{I+\dfrac{b_1\ell^{r_2}}{2b_2}+\dfrac{\phi_b}{b_2}+\left(\dfrac{I}{2}+\dfrac{\phi_b}{2b_2}\right)\psi_{q,1}}\\ &= \frac{c_1\ell^{r_2}+s^{r_1-r_2}(c_2I+\phi_c)}{b_2}\sum_{j=0}^{\infty}(-1)^{j+1}\left(\frac{b_1\ell^{r_2}}{2b_2}+\frac{\phi_b}{b_2}+\left(\frac{I}{2}+\frac{\phi_b}{2b_2}\right)\psi_{q,1}\right)^j.\end{aligned}$$

LEMMA 2. *If coefficients of the equation* (5) *are of the form* (6), (7) *and* (8), *and if* $r_1 > r_2 > 0$, *then the characteristic equation* (9) *has the following two solutions:*

$$\omega_1 = \delta_1 I + \delta_{c,1}, \quad \omega_2 = \delta_2 I + \delta_{c,2}$$

where δ_1, δ_2 *are numerical constants, while* $\delta_{c,1}$ *and* $\delta_{c,2}$ *are operators from* $\mathcal{F}_c$.

Proof. When $r_1 > r_2$, the solutions of the characteristic equation are

$$\begin{aligned}
\omega_{1,2} &= \frac{-b_1 I - s^{r_2}(b_2 I + \phi_b) \pm \sqrt{b^2 - 4ac}}{2(c_1 I + s^{r_1}(c_2 I + \phi_c))} \\
&= \frac{\ell^{r_1}}{2c_2} \cdot \frac{-b_1 I - s^{r_2}(b_2 I + \phi_b) \pm s^{r_1}(\sqrt{-4a_2 c_2})(I + \psi_{q,2})}{I + \dfrac{c_1}{c_2}\ell^{r_1} + \dfrac{\phi_c}{c_2}} \\
&= \frac{-b_1 \ell^{r_1} - s^{r_2 - r_1}(b_2 I + \phi_b) \pm \sqrt{-4a_2 c_2}(I + \psi_{q,2})}{2c_2} \sum_{j=0}^{\infty} (-1)^j \left(\frac{c_1}{c_2}\ell^{r_1} + \frac{\phi_c}{c_2} \right)^j .
\end{aligned}$$

Since then $r_2 - r_1 < 0$ and $s^{r_2 - r_1} = \ell^{r_1 - r_2}$, we can introduce numerical constants δ_1, δ_2 and operators $\delta_{c,1}$, $\delta_{c,2}$ from $\mathcal{F}_c$, by

$$\omega_j = \delta_j I + \delta_{c,j}, \tag{14}$$

for $j = 1, 2$. In (14) the numerical constants δ_j are equal to $\pm\sqrt{-a_2/c_2}$.

Similarly we can prove

LEMMA 3. *If coefficients of the equation* (5) *are of the form* (6), (7) *and* (8), *and if* $r_1 = r_2 > 0$, *then the solutions* ω_j, $j = 1, 2$, *of the characteristic equation* (9) *are*

$$\omega_j = \delta_{j+2} I + \delta_{c,j+2}. \tag{15}$$

In (15), $\delta_{3,4} = \left(-b_2 \pm \sqrt{b^2 - 4a_2 c_2}\right)/(2c_2)$ are numerical constants, while $\delta_{c,3}$ and $\delta_{c,4}$ are operators from $\mathcal{F}_c$.

4. The operator particular solution of the difference equation. The particular solution of the difference equation (5) has the form

$$u_n = \sum_{k=-\infty}^{\infty} G_{n-k} f_k, \tag{16}$$

with

$$G_{n-k} = \begin{cases} \alpha_1 \omega_1^{n-k} + \beta_1 \omega_2^{n-k}, & n - k \le 0; \\ \alpha_2 \omega_1^{n-k} + \beta_2 \omega_2^{n-k}, & n - k \ge 0, \end{cases} \tag{17}$$

where α_1, β_1, α_2, β_2 are operators from $\mathcal{F}$.

LEMMA 4. *Let us suppose that the conditions of Lemma* 1 *are fulfilled* ($r_2 > r_1 > 0$). *Then, the operators* G_n *given by* (17) *can be given in the form*

$$G_n = \alpha_2 \begin{cases} \omega_2^n, & \text{for } n \leq 0, \\ \omega_1^n, & \text{for } n \geq 0, \end{cases} \tag{18}$$

where α_2 *is an operator from* $\mathcal{F}_c$. *Hence operators* G_n *are represented by continuous functions for every* $n \in \mathbf{Z}$.

Proof. By Lemma 1 the operators ω_1 and I/ω_2 are from $\mathcal{F}_c$, hence the operator G_n can be written in the form

$$G_n = \begin{cases} \beta_1 \omega_2^n, & n \leq 0; \\ \alpha_2 \omega_1^n, & n \geq 0. \end{cases} \tag{19}$$

For $n = 0$ we have $\alpha_2 = \beta_1$. The coefficient α_2 can be obtained from the equation

$$aG_{-1} + bG_0 + cG_1 = I, \tag{20}$$

where a, b, c are coefficients of the equation (5) having the forms (6), (7) and (8), respectively. From (19) it follows that

$$G_{-1} = \alpha_2/\omega_2, \quad G_0 = \alpha_2, \quad G_1 = \alpha_2\omega_1. \tag{21}$$

Using the last two relations ((20) and (21)), we obtain

$$\alpha_2 \left((a_1 I + s^{r_1}(a_2 I + \phi_a))/\omega_2 + (b_1 I + s^{r_2}(b_2 I + \phi_b)) + (c_1 I + s^{r_1}(c_2 I + \phi_c))\omega_1 \right) = I,$$

where a_1, a_2, b_1, b_2, c_1, c_2, r_1, r_2 are numerical constants and ϕ_a, ϕ_b, ϕ_c, are the operators from $\mathcal{F}_c$. So we have

$$\begin{aligned} \alpha_2 &= \frac{I}{(a_1 I + s^{r_1}(a_2 I + \phi_a))/\omega_2 + (b_1 I + s^{r_2}(b_2 I + \phi_b)) + (c_1 I + s^{r_1}(c_2 I + \phi_c))\omega_1} \\ &= \frac{\ell^{r_2}}{b_2} \cdot \frac{I}{I + \gamma}. \end{aligned}$$

Since $r_2 > r_1 > 0$, the operator γ is from $\mathcal{F}_c$; it has the form

$$\gamma = \left(\frac{a_1}{b_2}\ell^{r_2} + s^{r_1 - r_2}\left(\frac{a_2}{b_2} + \frac{\phi_a}{b_2} \right) \right) \frac{I}{\omega_2} + \frac{b_1}{b_2}\ell^{r_2} + \frac{\phi_b}{b_2} + \left(\frac{c_1}{b_2}\ell^{r_2} + s^{r_1 - r_2}\left(\frac{c_2}{b_2} + \frac{\phi_c}{b_2} \right)\omega_1 \right).$$

So we have

$$\alpha_2 = \frac{\ell^{r_2}}{b_2} \sum_{k=0}^{\infty} (-1)^k \gamma^k =: \ell^{r_2}\alpha.$$

Since $r_2 > 0$, and I/ω_2 and ω_1 are represented by continuous functions, it follows that α_2 is an operator from $\mathcal{F}_c$. Therefore the operators G_n, for each n, are from $\mathcal{F}_c$, and the relation (18) holds.

THEOREM 1. *If conditions of Lemma* 1 *and Lemma* 4 *are fulfilled* $(r_2 > r_1 > 0)$ *and operators* f_k *in the equation* (5) *are of the form* $f_k = F_k s^m$, *for* $m < r_2$, *where* F_k *are numerical constants satisfying* $|F_k| < F$ *(for some* $F > 0$*), then the particular solution of equation* (5) *exists in the field* $\mathcal{F}$ *and is represented by a continuous function.*

Proof. If $r_2 > r_1 > 0$, then we have

$$G_{n-k} = \alpha_2 \begin{cases} \omega_2^{n-k} & \text{for } n-k \leq 0, \\ \omega_1^{n-k} & \text{for } n-k \geq 0. \end{cases}$$

In that case the operators G_{n-k} are from $\mathcal{F}_c$ and therefore the solution of difference equation (5) has the form (see (16))

$$u_n = \sum_{k=-\infty}^{\infty} G_{n-k} f_k = \ell^{r_2 - m} \alpha \left(\sum_{k=-\infty}^{n} \omega_1^{n-k} F_k + \sum_{k=n+1}^{\infty} \omega_2^{n-k} F_k \right). \tag{22}$$

The last two series converge because the operators ω_1 and I/ω_2 are from $\mathcal{F}_c$.

LEMMA 5. *Suppose that* $r_1 \geq r_2 > 0$ *(compare to Lemma* 2*). If* $\delta_1 < 1$ *and* $\delta_2 > 1$*, then the operator* G_n *can be written as*

$$G_n = \alpha_3 \begin{cases} (\delta_2 I + \delta_{c,2})^n, & \text{for } n \leq 0; \\ (\delta_1 I + \delta_{c,1})^n, & \text{for } n \geq 0, \end{cases} \tag{23}$$

where α_3 *is represented by a continuous function and operators* G_n *are from* $\mathcal{F}_c$.

Proof. Since $\delta_1 < 1$ and $\delta_2 > 1$, it follows that we can take $\alpha_1 = \beta_2 = 0$, and we obtain from (24)

$$G_n = \begin{cases} \beta_1 (\delta_2 I + \delta_{c,2})^n, & n \leq 0; \\ \alpha_2 (\delta_1 I + \delta_{c,1})^n, & n \geq 0. \end{cases} \tag{24}$$

For $n = 0$ it follows that $\alpha_2 = \beta_1 =: \alpha_3$. The coefficient α_3 can be obtained from the equation

$$aG_{-1} + bG_0 + cG_1 = I,$$

where a, b, c are coefficients of the equation (5) having the forms (6), (7) and (8), respectively. From the relation (24) it follows that $G_0 = \alpha_3$ and we have

$$(a_1 I + s^{r_1}(a_2 + \phi_a))\alpha_3(\delta_2 I + \delta_{c,2})^{-1} + \alpha_3 + (c_1 I + s^{r_1}(c_2 + \phi_c))\alpha_3(\delta_1 I + \delta_{c,1}) = I.$$

Since $r_1 \geq r_2 > 0$, we have

$$\alpha_3 = \frac{I}{(a_1 I + s^{r_1}(a_2 + \phi_a))(\delta_2 I + \delta_{c,2})^{-1} + I + (c_1 I + s^{r_1}(c_2 + \phi_c))(\delta_1 I + \delta_{c,1})}.$$

If we denote by $\gamma_2 = \dfrac{a_2}{\delta_2} + c_2\delta_1$, then we have $\alpha_3 = \dfrac{\ell^{r_1}}{\gamma_2} \cdot \dfrac{I}{I+\lambda}$, where

$$\lambda = \left(\frac{a_1\ell^{r_1}}{\gamma_2} + \frac{\phi_a}{\gamma_2}\right) \cdot \frac{I}{\delta_2 I + \delta_{c,2}} + \frac{a_2}{\gamma_2}\sum_{k=1}^{\infty}(-1)^k\left(\frac{\delta_{c,2}}{\delta_2}\right)^k + \frac{\ell^{r_1}}{\gamma_2} + \left(\frac{c_1\ell^{r_1}}{\gamma_2} + \frac{\phi_c}{\gamma_2}\right)(\delta_1 I + \delta_{c,1}) + c_2\delta_{c,1}.$$

From the last expression it follows that the operator λ is represented by a continuous function and therefore the operator α_3 has the form

$$\alpha_3 = \frac{\ell^{r_1}}{\gamma_2} \cdot \sum_{j=0}^{\infty}(-1)^j\lambda^j =: \ell^{r_1}\alpha^1$$

and is represented by a continuous function too. This implies the relation (23). Therefore, the operators G_n given by the relation (23) are from $\mathcal{F}_c$.

Similarly we can prove

Lemma 6. *Suppose that $r_1 \geq r_2 > 0$. Then we have*

$$G_n = \alpha_4\begin{cases} (\delta_1 I + \delta_{c,1})^n, \; for \; n \leq 0; \\ (\delta_2 I + \delta_{c,2})^n, \; for \; n \geq 0, \end{cases} \qquad for \; \delta_1 > 1, \delta_2 < 1;$$

$$G_n = \alpha_5\begin{cases} (\delta_1 I + \delta_{c,1})^n - (\delta_2 I + \delta_{c,2})^n, & for \; n \leq 0; \\ 0, & for \; n \geq 0; \end{cases} \qquad for \; \delta_1 > 1, \delta_2 > 1;$$

$$G_n = \alpha_6\begin{cases} 0, & for \; n \leq 0; \\ (\delta_1 I + \delta_{c,1})^n - (\delta_2 I + \delta_{c,2})^n, & for \; n \geq 0, \end{cases} \qquad for \; \delta_1 < 1, \delta_2 < 1,$$

where α_4, α_5 and α_6 are represented by continuous functions. In all three cases the operators G_n are from $\mathcal{F}_c$.

Finally, we can give the following

Theorem 2. *If the conditions of Lemma 5 (and Lemma 6) are fulfilled in the equation (5), i.e. $r_1 \geq r_2 > 0$, and operators f_k satisfy the condition $f_k = F_k l^m$ for $m < r_1$, where F_k are numerical constants satisfying $|f_k| < F$, then the particular solution of the equation (5) is represented by a continuous function for $r_1 - m > 0$.*

Proof. We shall prove only the case when $\delta_2 > 1$ and $\delta_1 < 1$; the other cases are handled similarly. Then we have

$$G_{n-k} = \alpha_3\begin{cases} (\delta_2 I + \delta_{c,2})^{n-k} & \text{for } n-k \leq 0, \\ (\delta_1 I + \delta_{c,1})^{n-k} & \text{for } n-k \geq 0. \end{cases}$$

In this case we can write $(\delta_1 I + \delta_{c,1})^{n-k} = \delta_1^{n-k} + \rho_{c,n-k,1}$ for $n-k > 0$, and $I(\delta_2 I + \delta_{c,2})^{-(k-n)} = I\delta_2^{-(k-n)} + \rho_{c,k-n,2}$, for $k-n > 0$, where $\rho_{c,k-n,2}$ are some operators from $\mathcal{F}_c$.

So the solution of the difference equation (5) has the form

$$u_n = \sum_{k=-\infty}^{\infty} G_{n-k} f_k = \ell^{r_1-m}\alpha^1$$

$$\times \left(\sum_{k=-\infty}^{n} F_k \delta_1^{k-n} + \sum_{k=-\infty}^{n} F_k \rho_{c,n-k,1} + \sum_{k=n+1}^{\infty} F_k \delta_2^{n-k} + \sum_{k=n+1}^{\infty} F_k \rho_{c,n-k,2} \right).$$

The first and the third series converge as numerical series and the second and the forth series converge in the field $\mathcal{F}$. Therefore in this case ($r_1 > m$) the solution u_n is again represented by a continuous function.

5. The error of approximation. In this section we shall keep x in some fixed interval $[A, B]$. Also, we shall consider in this section only the case when $r > p$.

Let us suppose that the solution of the equation (2) is from $\mathcal{F}_c$ and has a continuous fourth derivative in the field of Mikusiński operators. Let us denote by $u(x_j)$ the exact solution (the solution of the equation(3)) and by u_j the approximate solution of the same equation (which by Theorems 1 and 2 also belongs to $\mathcal{F}_c$). In fact u_j is the solution of the difference equation (5).

In order to give the error of approximation, we have to estimate the difference between the equations (2) (or (3)) and (5). So we obtain

$$\sum_{i=0}^{p} A_i s^i \left(u''(x_j) - \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} \right) + \sum_{i=0}^{q} B_i s^i \left(u'(x_j) - \frac{u_{j+1} - u_{j-1}}{2h} \right)$$

$$+ \sum_{i=0}^{r} C_i s^i \left(u(x_j) - u_j \right) = 0.$$

From the previous relation we have

$$|u(x_j) - u_j| = \left| \Big(\sum_{i=0}^{r} C_i s^i \Big)^{-1} \left(\sum_{i=0}^{p} A_i s^i \Big(u''(x_j) - \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} \Big) \right.\right.$$

$$\left.\left. + \sum_{i=0}^{q} B_i s^i \Big(u'(x_j) - \frac{u_{j+1} - u_{j-1}}{2h} \Big) \right) \right|.$$

If $r > p$, then

$$\left| \left(\sum_{i=0}^{p} A_i s^i \right) \Big/ \left(\sum_{i=0}^{r} C_i s^i \right) \right| = \left| \left(\sum_{i=0}^{p} A_i s^i \right) \Big/ \left(s^r \left(C_r + \sum_{i=0}^{r-1} C_i \ell^{r-i} \right) \right) \right|$$

is represented by a continuous function and it can be estimated by

$$\left| \left(\sum_{i=0}^{p} A_i s^i \right) \Big/ \left(\sum_{i=0}^{r} C_i s^i \right) \right| \le_T R_1(T)\ell.$$

Also, assuming that $r > q$, from the estimation

$$\left| \left(\sum_{i=0}^{q} B_i s^i \right) \Bigg/ \left(\sum_{i=0}^{r} C_i s^i \right) \right| \leq_T R_2(T)\ell,$$

we have

$$|u(x_j) - u_j|_T \leq \frac{h^2}{6} \left(R_1(T) \frac{M_4(T)}{2} + R_2 M_3(T) \right) \ell^2,$$

where

$$M_i(T) = \max_{x \in [A,B], \quad t \in [0,T]} |u^{(i)}(x,t)|, \quad i = 3, 4.$$

Therefore the solution of the equation (5) given by (22) can be treated as the approximate solution of the differential equation (2).

Let us remark that the error of approximation is of order h^2 in the field of Mikusiński operators, analogously to the case when we are working with numerical constants.

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References

1. S.K. Godunov, V.S. Rabenkii, *Raznostnye shemy,* Nauka, Moskva, 1977 .
2. J. Mikusiński, *Operational Calculus, Vol. 1*, Pergamon Press, Warszawa, 1983.
3. Dj. Takači, *The approximate solution of the operator differential equation*, in: *Proceedings of the International Conference on Generalized Functions,* Katowice, Poland, 1988, 241–251.
4. Dj. Takači, *The approximate solution of a PDE II,* Z. Angew. Math. Mech. **69**(4)(1989), 150–152.
5. Dj. Takači, A. Takači, *The numerical solution of the operator differential equation*, in: *Proceedings of Complex Analysis and Generalized Functions,* Bulgarian Academy of Sciences, Sofia, Bulgaria, 1993, 315–327.
6. Dj. Takači, A. Takači, *The character of the solution of difference equation in the field of Mikusiński operators,* Publ. Math. Debrecen **45** (1995), 379–394.
7. Dj. Takači, A. Takači, *The numerical solution of differential equations in the field of Mikusiński operators,* Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. **24**(1) (1994), 91–102.

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