PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 57 (71), 1995, 189–195
Đuro Kurepa memorial volume

THERE ARE INFINITELY MANY COUNTABLE MODELS OF STRICTLY STABLE THEORIES WITH NO DENSE FORKING CHAINS

Predrag Tanović

Abstract. We prove that a countable, complete, strictly stable theory with no dense forking chains has infinitely many pairwise nonisomorphic countable models.

Let T denote a complete countable stable theory and $I(T,\aleph_0)$ the number of its countable pairwise nonisomorphic models. In [**4**] Lachlan proved that if T is superstable then it is either $\aleph_0$-categorical or $I(T,\aleph_0) \geq \aleph_0$. In [**5**] he proved that an $\aleph_0$-categorical superstable theory is $\aleph_0$-stable, and he conjectured that the same is true for stable theories, namely that there is no strictly stable $\aleph_0$-categorical theory. By the time being, it has become clear that the strictly stable theories are much more complicated than superstable ones; Hrushovski has constructed a counterexample to Lachlan's Conjecture.

Some work was done to extract those strictly stable theories that share some of the nice properties of superstables. In [**7**] Pillay proved that if T is strictly stable and 1-based, then $I(T,\aleph_0) \geq \aleph_0$. In [**3**] Hrushovski has introduced theories which admit finite coding and proved that such a T is either $\aleph_0$-categorical or $I(T,\aleph_0) \geq \aleph_0$ holds.

In [**2**] Pillay introduced the class of theories with no dense forking chains, which seems to be a reasonable approximation of superstability; for example, every type in such a theory has a regular decomposition. We prove that the class satisfies Lachlan's conjecture, namely that there are no $\aleph_0$-categorical theories in it; because of the existence of regular decompositions, or just finiteness of the weight, the argument from the superstable case goes through and we get $I(T,\aleph_0) \geq \aleph_0$.

We assume some basic knowledge of stability theory as can be found in [**1**] or [**6**]. Below, we define Pillay's notions of dimension of U_α-rank in terms of partial

---

Supported by Ministry of Sciences and Technology RS
AMS Subject Classification (1991): Primary 03C45

orders in order to prove Proposition 7, which is the main technical result used in the proof of Theorem.

Let $(P, \leq)$ be a partial order. For $p, q \in P$ we denote by $[p, q]_P$ the interval $\{x \in P | p \leq x \leq q\}$ ordered by (the restriction of) $\leq$; $(\leq, p]_P$ denotes $\{x \in P | x \leq p\}$ ordered by $\leq$, and similarly we define $(<, p)_P$, $(p, <)_P$ and $[p, \leq)_P$.

Definition. Let $(P, \leq)$ be a nonempty partial order. Inductively, define the dimension $\dim(P)$ which is an ordinal or ∞:

(1) $\dim(P) \geq 0$

(2) $\dim(P) \geq \alpha + 1$ if there is an infinite decreasing chain $p_0 > p_1 > p_2 > \dots$ such that for every $i \in \omega$ $\dim([p_{i+1}, p_i]_P) \geq \alpha$.

(3) $\dim(P) \geq \lambda$, where λ is a limit ordinal, if $\dim(P) \geq \alpha$ for every $\alpha < \lambda$.

(4) $\dim(P) = \alpha$ iff α is the greatest ordinal for which $\dim(P) \geq \alpha$ holds; $\dim(P) = \infty$ iff $\dim(P) \geq \alpha$ holds for all ordinals α.

Definition. Let α be an ordinal and let $(P, \leq)$ be a partial order. Inductively we define U_α-rank of $(P, \leq)$:

(1) $U_\alpha(P) \geq 0$ if $P \neq \emptyset$.

(2) $U_\alpha(P) \geq \beta + 1$ iff there exists a $p \in P$ such that $U_\alpha((\leq, p]_P) \geq \beta$ and $\dim([p, \leq)_P) \geq \alpha$.

(3) $U_\alpha(P) \geq \lambda$, where λ is a limit ordinal, iff $U_\alpha(P) \geq \beta$ for all ordinals $\beta < \lambda$.

(4) $U_\alpha(P) = \xi$, where ξ is an ordinal, if ξ is the greatest ordinal for which $U_\alpha(P) \geq \xi$. If no such ordinal exists, let $U_\alpha(P) = \infty$

Lemma 1. *Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be partial orders.*

(a) *If $f: P \to Q$ is strictly increasing then* $\dim(P) \leq \dim(Q)$ *and* $U_\alpha(P) \leq U_\alpha(Q)$.

(b) *If $\alpha \geq 1$ then $U_\alpha(P) + U_\alpha(Q) \leq U_\alpha(P \oplus Q)$ where $P \oplus Q$ is the set $P \times \{0\} \cup Q \times \{1\}$ ordered by $\{((p,0),(p',0)) | p \leq p'\} \cup \{((q,1),(q',1)) | q \leq q'\} \cup \{((p,0),(q,1)) | p \in P, q \in Q\}$.*

Proof. The part (a) is an easy induction on $\dim(P)$ and $U_\alpha(P)$; we prove only (b). Q is embedded in $P \oplus Q$, so if $U_\alpha(Q) = \infty$ the conclusion follows by the part (a).

Let $\xi = U_\alpha(Q)$. We use induction on ξ. For $\xi = 0$ it is obvious and for $\xi = 1$ it follows from the definition of U_α. Suppose that $\xi = \eta + 1$ and let $q \in Q$ be such that $U_\alpha((\leq, q]_Q) = \eta$ and $U_\alpha([q, \leq)_Q) = 1$. By the induction hypothesis

$$U_\alpha(P) + \xi = U_\alpha(P) + U_\alpha((\leq, q]_Q) \leq U_\alpha(P \oplus (\leq, q]_Q) = U_\alpha((\leq, q]_{P \oplus Q}).$$

On the other hand $1 = U_\alpha([q, \leq)_Q) = U_\alpha([q, \leq)_{P \oplus Q})$, and from the definition of U_α we get $U_\alpha(P) + U_\alpha(Q) = U_\alpha(P) + \xi + 1 \leq U_\alpha(P \oplus Q)$.

The case when ξ is a limit ordinal is similar.

Lemma 2. *Let $(P, \leq)$ be a partial order. Then* $\dim(P) = \infty$ *if and only if there exists an embedding of rationals into $(P, \leq)$.*

Proof. $\leftarrow$ is clear, so we prove only $\rightarrow$. Assume that $\dim(P) = \infty$. Let α be an ordinal such that for all $p, q \in P$ $\dim([p,q]_P) \geq \alpha$ implies $\dim([p,q]_P) = \infty$. Since $\dim(P) \geq \alpha + 1$ there is an infinite decreasing chain $p_0 > p_1 > p_2 > \dots$ such that for all $i \in \omega$ $\dim([p_{i+1}, p_i]_P) \geq \alpha$, thus $\dim([p_{i+1}, p_i]_P) = \infty$. Applying the same reasoning to each $[p_{i+1}, p_i]_P$ for $i \in \omega$ in place of P we get infinite descending chains $p_0^i > p_1^i > p_2^i > \dots$ in $[p_{i+1}, p_i]_P$ so that $\dim([p_{j+1}^i, p_j^i]_P) = \infty$. Continuing in this way we get a chain in P isomorphic to the rationals.

LEMMA 3. *If $(P_i, \leq_i)$ are nonempty partial orders for $i \leq n$, then:*

$$\dim(P_1 \times P_2 \times \dots \times P_n) = \max\{\dim(P_1), \dim(P_2), \dots, \dim(P_n)\}.$$

(Here $P_1 \times P_2 \times \dots \times P_n$ is ordered by the product order, i.e.

$$(p_1, p_2, \dots, p_n) \leq (p_1', p_2', \dots p_n') \text{ iff } p_1 \leq_1 p_1' \;\; p_2 \leq_2 p_2' \dots p_n \leq_n p_n').$$

Proof. Assume $n = 2$. Then $\dim(P_1 \times P_2) \geq \max\{\dim(P_1), \dim(P_2)\}$ follows immediately from Lemma 1, so we prove the reverse inequality. Actually, we show by induction on ordinals α that $\dim(P_1 \times P_2) \geq \alpha$ implies $\max\{\dim(P_1), \dim(P_2)\} \geq \alpha$. For $\alpha = -1$ or 0 the claim is obvious, so we distinguish the following two cases:

Case 1: $\alpha = \beta + 1$. Assume that $\dim(P_1 \times P_2) \geq \beta + 1$. Then there is an infinite decreasing sequence $(p_0, p_0') > (p_1, p_1') > (p_2, p_2') > \dots$ such that for all $i \in \omega$ $\dim([(p_{i+1}, p_i), (p_{i+1}', p_i')]_{P_1 \times P_2}) \geq \beta$. By the induction hypothesis for each $i \in \omega$ either $\dim([p_{i+1}, p_i]_{P_1}) \geq \beta$ or $\dim([p_{i+1}', p_i')]_{P_2}) \geq \beta$ holds. Therefore either for infinitely many $i \in \omega$ $\dim([p_{i+1}, p_i]_{P_1}) \geq \beta$ or for infinitely many $i \in \omega$ $\dim([p_{i+1}', p_i'))]_{P_2}) \geq \beta$. Thus either $\dim(P_1) \geq \beta + 1$ or $\dim(P_2) \geq \beta + 1$ holds.

Case 2: α is a limit ordinal. Let $\alpha = \cup\{\alpha_\xi | \xi < \varkappa\}$ where $\varkappa = \mathrm{cf}(\alpha)$. By the induction hypothesis for each $\xi < \varkappa$ at least one of $\dim(P_1) \geq \alpha_\xi$ and $\dim(P_2) \geq \alpha_\xi$ holds. Thus at least one of the sets $\{\xi < \varkappa | \dim(P_1) \geq \alpha_\xi\}$ and $\{\xi < \varkappa | \dim(P_2) \geq \alpha_\xi\}$ is cofinal in $\varkappa$ and that means that either $\dim(P_1) \geq \alpha$ or $\dim(P_2) \geq \alpha$.

Thus we proved the Lemma for $n = 2$. The general case follows rather easily from this one.

From now on we assume that T is stable, $\mathcal{M}$ is a monster model of T and we operate in $\mathcal{M}^{eq}$. All the sets and tuples mentioned below are 'small' subsets of $\mathcal{M}^{eq}$; models are elementary submodels of $\mathcal{M}$.

Definition. Let $A \subseteq B$, $p \in S(A)$ and $p \subseteq q \in S(B)$.

(a) $\dim(p|q) = \dim([\mathrm{bnd}(q), \mathrm{bnd}(p)]_{o(T)})$.

(b) $U_\alpha(p|q) = U_\alpha([\mathrm{bnd}(q), \mathrm{bnd}(p)]_{o(T)})$.

(c) $\dim(p) = \dim(p|r)$ where r is any algebraic extension of r, and $U_\alpha(p) = U_\alpha(p|r)$.

Further in the text, we will write $\dim(\bar{a}/B)$ instead of $\dim(\mathrm{tp}(\bar{a}/B))$ and $\dim(\bar{a})$ instead of $\dim(\bar{a}/\emptyset)$. Similarly we do for U_α-rank.

If we allow infinitary types, not just types, in the previous definitions, then we get the notions of dim and U_α-rank of infinitary types as well.

Note that U_0 is the usual U-rank and $\dim(p) = 0$ means exactly that p has ordinal U-rank. Also, $U_\alpha(p|q) = 0$ implies $\dim(p|q) < \alpha$.

LEMMA 4. *If $A \subseteq B$ and $C \subseteq \mathrm{acl}(DA)$, then*

$$\dim(C/A|C/B) \leq \dim(D/A|D/B) \quad \textit{and} \quad U_\alpha(C/A|C/B) \leq U_\alpha(D/A|D/B).$$

Proof. By induction on $\dim(C/A|C/B)$. Suppose that $\beta_1 > \beta_2 > \dots$ is an infinite descending chain between $\mathrm{bnd}(C/B)$ and $\mathrm{bnd}(C/A)$ such that $\dim(C/A|C/B) > \dim([\beta_{i+1}, \beta_i]) = \xi_i$. Pick a sequence $A \subseteq M_1 \subseteq M_2 \subseteq \dots$ such that for all $i < j$ $\mathrm{bnd}(C/M_i) = \beta_i$, $C \downarrow_{M_1} A$ and $C \downarrow_{M_i} M_j$. Moreover, assume that $\cup M_i \downarrow_{CA} DB$. By the induction hypothesis for all $j < i$:

$$\dim(C/M_i|C/M_j) \leq \dim(D/M_i|D/M_j)$$

Hence $\dim(D/M_i|D/M_j) \geq \xi_i$. From the independence assumptions we derive $D \downarrow_{M_o} A$, $D \downarrow_{M_i} M_j$ and $D \downarrow_B M_i$, for all $i < j$. Therefore

$$\mathrm{bnd}(D/A) \geq \mathrm{bnd}(D/M_0) > \mathrm{bnd}(D/M_1) > \dots \geq \mathrm{bnd}(D/B).$$

If $\dim(C/A|C/B) = \xi + 1$ then we could choose $\beta_i' s$ so that $\xi_i = \xi$, and if $\dim(C/A|C/B)$ is a limit ordinal then it can be chosen so that ξ_i' s form a cofinal sequence. In both cases the conclusion follows.

A similar argument works for U_α.

LEMMA 5. *If $p \subseteq q \subseteq r$, then $U_\alpha(q|r) + U_\alpha(p|q) \leq U_\alpha(p|r)$. If r is algebraic, then $U_\alpha(q) + U_\alpha(p|q) \leq U_\alpha(p)$.*

Proof. Follows from Lemma 1 (b).

Definition. T has no dense forking chains if the order type of the rationals can not be embedded into $O(T)$.

As an immediate consequence of Lemma 2 we have that if T has no dense forking chains and $p \subseteq q$, then $\dim(p|q) < \infty$.

THEOREM 6. *Suppose that T has no dense forking chains.*

(a) *For any a, b and $A \subseteq B$ and $\alpha \geq 0$,*

$$\dim(ab/A|ab/B) = \sup\{\dim(b/aA|b/aB), \dim(a/A|a/B)\}.$$

(b) *(U_α-rank inequalities)*

$$U_\alpha(b/aA) + U_\alpha(a/A) \leq U_\alpha(ab/A) \leq U_\alpha(b/aA) \oplus U_\alpha(a/A).$$

(c) *Every type decomposes into a product of regular types.*

Proof. The part (a) is Lemma 10, (b) is Proposition 11 and (c) is Theorem 14 from [**2**].

We note the following instance of Theorem 6 (a) and Lemma 4 that we will use often: if $B \subseteq \operatorname{acl}(C_1C_2\dots C_nA)$ and $\dim(C_i/A) \le \alpha$ for $1 \le i \le n$, then $\dim(B/A) \le \alpha$.

PROPOSITION 7. *If $A \subseteq B$, $p \in S(A)$ and $p \subseteq q \in S(B)$, then*

$$\dim(p|q) \ge \sup\{\dim(\bar{c}/A) | \bar{c} \in Cb(q)\}.$$

Proof. Without loss of generality, assume that $A = \emptyset$ and we operate in $\mathcal{M}^{eq}$. Let $\bar{c} \in Cb(q)$ and let $I = \bar{a}_1\bar{a}_2\dots\bar{a}_n$ be a Morley sequence in (a stationarization of) q long enough so that $\bar{c} \in \operatorname{dcl}(I)$. Let $C = \operatorname{acl}(\bar{c})$ and we show that $\dim(p|q) \ge \dim(C)$; since $\dim(\bar{c}) \le \dim(C)$ (by Lemma 4) this will imply the conclusion of the Proposition.

Let $I_k = \bar{a}_1\bar{a}_2\dots\bar{a}_{k-1}$ for $k \le n$, let $P = \{\beta \in O(T) | \beta < \operatorname{bnd}(C)\}$ and for $\beta \in P$ let D_β be such that $\operatorname{bnd}(C/D_\beta) = \beta$. For $k \le n$, $\beta \in P$, define $p^k_\beta = \operatorname{bnd}(\bar{a}_k/I_kE_\beta)$ where E_β satisfies $\operatorname{tp}(E_\beta/C) = \operatorname{tp}(D_\beta/C)$ and $E_\beta \downarrow_C I$. We note that p^k_β does not depend on the particular choice of E_β. Actually, since C is algebraically closed, $\operatorname{tp}(D_\beta/C)$ is stationary so it has a unique nonforking extension over CI, thus $\operatorname{tp}(E_\beta/CI)$ is uniquely determined and hence $\operatorname{tp}(I/CE_\beta)$ is uniquely determined, too.

For natural $k \le n$ let $P_k = \{p^k_\beta | \beta \in P\}$ with the inherited order from $O(T)$. Now we show that $P_k \subseteq [\operatorname{bnd}(q), \operatorname{bnd}(p)]_{O(T)}$. From the definition of p^k_β we have $p^k_\beta \le \operatorname{bnd}(p)$, and $\operatorname{bnd}(q) \le p^k_\beta$ follows from:

$$\operatorname{bnd}(q) \le \operatorname{bnd}(\bar{a}_k/I_kC) = \operatorname{bnd}(\bar{a}_k/I_kCE_\beta) \le \operatorname{bnd}(\bar{a}_k/I_kE_\beta) = p^k_\beta.$$

The first inequality above is true since I is a Morley sequence in q. From $E_\beta \downarrow_C I$ we have $E_\beta \downarrow_{I_kC} \bar{a}_k$ and the first equality follows. The second inequality is clear and hence $P_k \subseteq [\operatorname{bnd}(q), \operatorname{bnd}(p)]_{O(T)}$.

Further, order $P_1 \times P_2 \times \dots \times P_n$ with the product order and define a mapping $f: P \to P_1 \times P_2 \times \dots \times P_n$ by $f(\beta) = (p^1_\beta, p^2_\beta, \dots, p^n_\beta)$. We show that f is strictly increasing. Assume that $\beta, \gamma \in P$ and $\gamma \le \beta$. Choose E_β and E_γ such that:

$$\operatorname{tp}(E_\beta/C) = \operatorname{tp}(D_\beta/C),\ \ \operatorname{tp}(E_\gamma/C) = \operatorname{tp}(D_\gamma/C),\ \ E_\beta \downarrow_{E_\gamma} C \ \text{ and } \ E_\beta E_\gamma \downarrow_C I.$$

Then $p^k_\beta = \operatorname{bnd}(\bar{a}_k/I_kE_\beta)$ and $p^k_\gamma = \operatorname{bnd}(\bar{a}_k/I_kE_\gamma)$. From the independence assumptions we derive $I \downarrow_{E_\gamma} E_\beta$, and thus $\bar{a}_k \downarrow_{I_kE_\gamma} E_\beta$. We have:

$$(!)_k \qquad p^k_\beta = \operatorname{bnd}(\bar{a}_k/I_kE_\beta) \ge \operatorname{bnd}(\bar{a}_k/I_kE_\beta E_\gamma) = \operatorname{bnd}(\bar{a}_k/I_kE_\gamma) = p^k_\gamma.$$

Thus $p^k_\beta \ge p^k_\gamma$ and f is increasing. Now, if $\gamma < \beta$ then $C \not\downarrow_{E_\beta} E_\gamma$ and since $C \subseteq \operatorname{acl}(I)$ we have $I \not\downarrow_{E_\beta} E_\gamma$ so for some $j \le n$ we have $\bar{a}_j \not\downarrow_{I_jE_\beta} E_\gamma$ and $\operatorname{bnd}(\bar{a}_j/I_jE_\beta E_\gamma) < \operatorname{bnd}(\bar{a}_j/I_jE_\beta)$. We conclude that in $(!)_j$ the strict inequality holds and $p^k_\gamma < p^k_\beta$. This proves that f is strictly increasing.

By Lemma 1 we have $\dim(P) \leq \dim(P_1 \times P_2 \times \ldots \times P_n)$ and by Lemma 3 we have $\dim(P_1 \times P_2 \times \ldots \times P_n) = \dim(P_k)$, for some $k \leq n$. Therefore $\dim(P) \leq \dim(P_k)$. But $P_k \subseteq [\mathrm{bnd}(q), \mathrm{bnd}(p)]_{O(T)}$ thus $\dim(P_k) \leq \dim(p|q)$ and we have:

$$\dim(C) = \dim(P) \leq \dim(P_k) \leq \dim(p|q)$$

completing the proof of the Proposition.

From now on we assume that T is strictly stable and has no dense forking chains. Consider all complete types whose domain is finite. Let $\alpha \geq 1$ be the smallest ordinal such that at least one of the types considered has dimension α and let ξ be the smallest possible U_α-rank of such a type. We say that a type is an (α, ξ)-type if its domain is finite, its dimension is α and its U_α-rank is ξ.

LEMMA 8. *If $p = \mathrm{tp}(\bar{a}/B)$ is an (α, ξ)-type, then there is a $\bar{c}$ such that $\bar{c} \in \mathrm{dcl}(\bar{a}B) \backslash \mathrm{acl}(B)$ and $\dim(\bar{c}/B) = 0$. In particular, every (α, ξ)-type is nonorthogonal to a type of dimension 0.*

Proof. Without loss of generality assume that $B = \emptyset$. Since $\alpha > 0$, there exists an infinite sequence $\beta_1 > \beta_2 > \ldots$ below $\mathrm{bnd}(p)$ in $O(T)$. Let $r = \mathrm{tp}(\bar{a}/C)$ be such that $\mathrm{bnd}(r) = \beta_2$. Note that $\beta_3 > \beta_4 > \ldots$ is an infinite descending sequence below $\mathrm{bnd}(r)$ so that $\dim(r) \geq 1$. If we replace C by a large enough finite subset of $Cb(r)$ we can assume that C is finite, r is a forking extension of p and $\dim(r) \geq 1$.

By the minimality assumptions on α and ξ we have $\dim(r) = \alpha$ and $U_\alpha(r) = U_\alpha(p) = \xi$. By Lemma 5 $U_\alpha(r) + U_\alpha(p|r) \leq U_\alpha(p)$ and it follows that $U_\alpha(p|r) = 0$. Thus, $\dim(p|r) < \alpha$. By Proposition 7 we have

$$\sup\{\dim(\bar{d}) | \bar{d} \in Cb(r)\} \leq \dim(p|r).$$

Therefore $\sup\{\dim(\bar{d}) | \bar{d} \in Cb(r)\} < \alpha$ and by the minimality assumption on α we have $\dim(\bar{d}) = 0$ for all $\bar{d} \in Cb(r)$. Let $\bar{d} \in Cb(r)$ be such that $\bar{a} \not\perp \bar{d}$ and let $\bar{c}' \in Cb(\bar{d}/\bar{a}) \backslash \mathrm{acl}(\emptyset)$. $\bar{c}'$ is definable in a finite Morley sequence $\bar{d}_1 \bar{d}_2 \ldots \bar{d}_k$ in $\mathrm{stp}(\bar{d}/\bar{a})$. Also $\dim(\bar{d}_i) = 0$, so $\dim(\bar{d}_1 \bar{d}_2 \ldots \bar{d}_k) = 0$ and $\dim(\bar{c}') = 0$. Let c be the name for the set of all $\{\bar{a}\}$-conjugates of $\bar{c}'$. Since $\bar{c}' \in \mathrm{acl}(\bar{a})$ this set is finite, so $c \in \mathrm{dcl}(\bar{a})$; also, every $\{\bar{a}\}$-conjugate of $\bar{c}'$ has the dimension 0 so that $\dim(c) = 0$. Finally, from $\bar{c}' \in \mathrm{acl}(c) \backslash \mathrm{acl}(\emptyset)$ we have $c \notin \mathrm{acl}(\emptyset)$ completing the proof of the Lemma.

THEOREM. *If T is a countable, complete, strictly stable theory with no dense forking chains then $I(T, \aleph_0) \geq \aleph_0$.*

Proof. Let B be finite, let $p = \mathrm{tp}(\bar{a}/B)$ be an (α, ξ)-type, let $A = \{\bar{d} \in \mathrm{dcl}(\bar{a}B) | \dim(\bar{d}/B) = 0\}$ and let $q = \mathrm{tp}(\bar{a}/AB)$. We *claim* that q is nonisolated.

Suppose, on the contrary, that $\varphi(\bar{x}, \bar{b})$ is a formula over AB which isolates q; here $\varphi(\bar{x}, \bar{y})$ is an L-formula $\bar{b} \subseteq AB$ and without any loss of generality we assume that $B \subseteq \bar{b}$. Clearly $\dim(\bar{b}/B) = 0$ holds, so that $\dim(\bar{a}/\bar{b}) > 0$ by Theorem 6 (a). By the minimality assumptions on α and ξ we must have $\dim(\bar{a}/\bar{b}) = \alpha$ and

$U_\alpha(\bar{a}/\bar{b}) = \xi$. By Lemma 8 there exists $\bar{c} \in A\backslash\mathrm{acl}(\bar{b})$. Choose $\bar{a}_1 \vdash \mathrm{stp}(\bar{a}/\bar{b})$ such that $\bar{a}_1 \downarrow_b \bar{c}$. From $\vdash \varphi(\bar{a}_1, \bar{b})$ we get $\mathrm{tp}(\bar{a}_1/AB) = q$, and from the independence assumption on $\bar{a}_1$ we get $\bar{c} \notin \mathrm{acl}(\bar{a}_1\bar{b})$; on the other hand $\bar{c} \in \mathrm{dcl}(\bar{a}\bar{b})$, so that $\mathrm{tp}(\bar{a}_1/\bar{c}B) \neq \mathrm{tp}(\bar{a}/\bar{c}B)$ and $\mathrm{tp}(\bar{a}_1/AB) \neq \mathrm{tp}(\bar{a}/AB) = q$. This is a contradiction and the *claim* is proved.

Continuing the proof of the Theorem, let $r \in S(\mathrm{dcl}(\bar{a}B))$ be a nonforking extension of q. Then, by the Open Mapping Theorem r must be nonisolated too, and $r|B\bar{a}$ is nonisolated as well. We have found a nonisolated type over a finite domain, hence there exists a nonisolated type over $\emptyset$. To complete the proof of the Theorem, we repeat the proof from the superstable case:

Suppose that T is small, otherwise $I(T, \aleph_0) = 2^{\aleph_0}$. Let $\mathrm{tp}(\bar{d})$ be nonisolated and let $\bar{d}_1\bar{d}_2\dots$ be an infinite Morley sequence in $\mathrm{tp}(\bar{d})$. For each n let M_n be prime over $\bar{d}_1\bar{d}_2\dots\bar{d}_n$. By Theorem 6 (c) $m = wt(\bar{d}) < \omega$. We show that in M_n there is no Morley sequence in $\mathrm{tp}(\bar{d})$ of lenght $m \cdot n + 1$, which clearly implies the conclusion of the Theorem.

If $\bar{e} \vdash \mathrm{tp}(\bar{d})$ and $\bar{e} \in M_n$ then $\mathrm{tp}(\bar{e}/\bar{d}_1\bar{d}_2\dots\bar{d}_n)$ is isolated, hence by the Open Mapping Theorem $\bar{e}$ forks with $\bar{d}_1\bar{d}_2\dots\bar{d}_n$. On the other hand $wt(\bar{d}_1\bar{d}_2\dots\bar{d}_n) = m \cdot n$, hence there is no independent set of realizations of $\mathrm{tp}(\bar{d})$ of size $m \cdot n + 1$ in M_n.

REFERENCES

[1] J. Baldwin, *Fundamentals of Stability Theory*, Springer-Verlag, Berlin 1988.

[2] B. Herwig, J.G. Loveys, A. Pillay, P. Tanović, F.O. Wagner, *Stable theories without dense forking chains*, Arch. for Math. Logic **31** (1992), 297–304.

[3] E. Hrushovski, *Finitely based theories*, J. Symbolic Logic **54** (1989), 221–225.

[4] A.H. Lachlan, *The number of countable models of a countable superstable theory,* in: *Proc. of the International Congress on Logic, Methodology and Philosophy of Science*, Roumania 1971, North Holland, Amsterdam 1973, pp. 45–56.

[5] A.H. Lachlan, *Two conjectures on stability of* $\aleph_0$*-categorical theories*, Fund. Math. **81** (1974), 133–145.

[6] M. Makkai, *A survey of basic stability theory with particular emphasis on orthogonality and regular types*, Israel J. Math. **49** (1984),

[7] A. Pillay, *Stable theories pseudoplanes and the number of countable models*, Ann. Pure Appl. Logic **43** (1989), 147–160.

Matematički institut
Kneza Mihaila 35
11001 Beograd, p.p. 367
Yugoslavia

(Received 27 04 1995)