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COMPLEMENTARY PAIRS OF GRAPHS WITH THE SECOND LARGEST EIGENVALUE NOT EXCEEDING $(\sqrt{5}-1)/2$

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Dedicated to Professor Đuro Kurepa

Abstract. We characterize (in terms of minimal forbidden subgraphs) graphs having the following property: both the graph and its complement have the second largest eigenvalue not exceeding $(\sqrt{5}-1)/2$, i.e. the golden section. This characterization also enables us to find explicitely all graphs in question.

0. Introduction

Let G be an undirected graph (without loops and/or multiple edges) having n vertices, and let $\lambda_1(G), \dots, \lambda_n(G)$ $(\lambda_1(G) \geq \cdots \geq \lambda_n(G))$ be its eigenvalues, i.e. the eigenvalues of its adjacency matrix. For everything related to graph spectra see [**3**] (other terminology follows [**8**]).

We shall introduce (for golden section) the notation $\sigma = (\sqrt{5}-1)/2$ (≈ 0.618033989). Recently, the graphs with the second largest eigenvalue not exceeding σ have attracted considerable interest (see [**2**, **6**, **7**, **15**]). In particular, it was noted in [**6**] (see also [**15**]) that the graphs whose second largest eigenvalue does not exceed σ (to be called *σ-graphs*) can be characterized by a finite collection of forbidden induced subgraphs. Although this collection is finite, it is very large (for more details, see the forthcoming paper [**7**]). Here our aim is more moderate: namely, to characterize in terms of minimal forbidden subgraphs those pairs of graphs $(G, \overline{G})$ such that both of them are σ-graphs (or, for short, σ^*-graphs). In addition, we determine all σ^*-graphs explicitly.

For similar results on a graph and its complement which share, in particular, some spectral property, see [**11**, **13**]. Generaly, this sort of problems was systematically treated by J. Akiyama and F. Harary (see, for example, [**1**]).

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[1] *AMS Subject Classification* (1991): Primary 05C50

1. Preliminaries

In this section, besides necessary notation, we reproduce some results from [**6**, **15**] in order to keep the paper more selfcontained.

A *rooted tree* T is the tree with one vertex, say r (also called the *root*), distinguished. In describing some relations between the vertices of a rooted tree, we shall (besides the usual terminology) use the terminology of family trees (see, for example, [**14**]). Thus all vertices of T are the *descendants* of the root r, while r is their *ancestor*. We can also imagine that the edges of a tree are oriented from the root to its descendants. If f is joined with s by an (oriented) edge, then s is regarded as a *son* of f (while f is the father of s). The vertices without sons are called *leaves*; other vertices, except the root, are called *internal vertices*. Two vertices of a tree are called *incomparable* if they are not connected by an oriented path. The *height* of a tree T, also denoted by $h\,(= h(T))$, is the maximal distance between the root and the leaves.

Weighted rooted trees (with weights assigned to vertices) were used in [**15**] in representing graphs from the class $\mathcal{C}$, which is recursively defined as follows:

(i) $\emptyset \in \mathcal{C}$ ($\emptyset$ being an empty graph (or null-graph) [1]);

(ii) if $G \in \mathcal{C}$, then $G \cup nK_1 \in \mathcal{C}$ $(\backslash \in \mathcal{N})$;

(iii) if $G_1, G_2 \in \mathcal{C}$, then $G_1 \bigtriangledown G_2 \in \mathcal{C}$;

(iv) any graph from $\mathcal{C}$ can be obtained only by using the rules $(i) - (iii)$ (finitely many times).

Here $\bigtriangledown$ denotes the join of two graphs, while $\cup$ refers to union of two disjoint graphs. Notice that $G_1 \bigtriangledown G_2 = \overline{\overline{G_1} \cup \overline{G_2}}$

Remark: An alternative way to describe graphs from the class $\mathcal{C}$ is in terms of minimal forbidden induced subgraphs. Actually, $\mathcal{C}$ is a class of graphs having no induced subgraphs equal to E $(= 2K_2)$ or P $(= P_4)$.

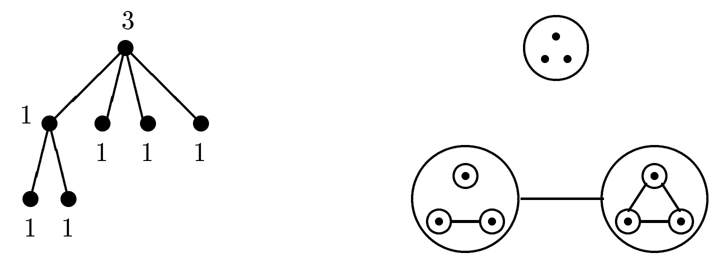
To any graph G from $\mathcal{C}$ we associate a weighted rooted tree T_G (also called an *expression tree* of G) in the following way:

> if $H = (H_1 \bigtriangledown \ldots \bigtriangledown H_m) \cup nK_1$ is any subexpression of a graph G (i.e. a graph obtained by using the above rules), then a subtree T_H with a root v corresponds to H; $n(= w(v))$ is a weight of v, whereas for each i $(i = 1, \ldots, m)$ there is a vertex v_i (a son of v) representing a root of H_i.

[1]This is the only place were we use the empty graph; later we shall ignore the fact that it belongs to $\mathcal{C}$. For more accounts on this question see [**9**]

Remark: It is also worth mentioning (see Lemma 3.4 of [**15**]) that this representation may be turned to a canonical one. Then all vertices except possibly the root have non-zero weights, and each father has at least two sons. If so, then any canonical representation determines the graph up to isomorphism. The corresponding tree is called the *canonical expression tree*.

Example: If $G = ((K_2 \cup K_1) \bigtriangledown K_3) \cup 3K_1$ (= $(((((K_1 \bigtriangledown K_1) \cup K_1) \bigtriangledown K_1) \bigtriangledown K_1) \bigtriangledown K_1) \cup 3K_1$), then the corresponding expression tree is depicted in Fig. 1(a). In Fig. 1(b) the same graph is represented alternatively, as a set diagram (a line between two circumscribed sets of vertices denotes that each vertex inside one set is adjacent to each vertex inside the other set).



(a) (b)

Fig. 1.

It was proved in [**15**] that the collection of minimal forbidden induced subgraphs for the σ^--graphs (graphs with the second largest eigenvalue strictly less than σ) is finite. The following minimal forbidden graphs (up to 7 vertices) were found in [**6**].

E	$= 2K_2$	(1.0000)	R_1	$= (K_2 \cup 4K_1) \bigtriangledown K_1$	(0.6421)
P	$= P_4$	(0.6180)	R_2	$= (K_{1,2} \cup 3K_1) \bigtriangledown K_1$	(0.6648)
Q	$= (K_3 \cup 2K_1) \bigtriangledown K_1$	(0.6784)	R_3	$= (K_4 \cup K_1) \bigtriangledown K_2$	(0.6532)

Graphs $S_1 - S_{16}$ from the list below are also minimal forbidden subgraphs for σ^--graphs.

S_1	$= (((K_2 \cup K_1) \bigtriangledown K_1) \cup K_1) \bigtriangledown K_3$	(0.6320)
S_2	$= (K_2 \cup 3K_1) \bigtriangledown K_3$	(0.6216)
S_3	$= (K_3 \cup K_1) \bigtriangledown (K_3 \cup K_1)$	(0.6277)
S_4	$= (K_3 \cup K_1) \bigtriangledown K_4$	(0.7321)
S_5	$= (K_{1,2} \cup 2K_1) \bigtriangledown K_3$	(0.6268)
S_6	$= (K_{2,4} \cup 2K_1) \bigtriangledown K_1$	(0.6222)
S_7	$= (K_{3,3} \cup 2K_1) \bigtriangledown K_1$	(0.6318)
S_8	$= (K_7 \cup K_1) \bigtriangledown 2K_1$	(0.6205)
S_9	$= (K_{2,3} \cup K_1) \bigtriangledown (K_{3,3} \cup K_1)$	(0.6230)
S_{10}	$= (K_{2,4} \cup K_1) \bigtriangledown (K_{2,4} \cup K_1)$	(0.6262)

$$
\begin{array}{lll}
S_{11} & = (K_{2,4} \cup K_1) \bigtriangledown K_{53} & (0.6182) \\
S_{12} & = (K_{3,3} \cup K_1) \bigtriangledown K_{17} & (0.6191) \\
S_{13} & = (K_{1,3} \cup 2K_1) \bigtriangledown K_2 & (0.6401) \\
S_{14} & = ((2K_1 \bigtriangledown K_2) \cup K_1) \bigtriangledown K_3 & (0.6380) \\
S_{15} & = (((K_2 \cup 2K_1) \bigtriangledown K_1) \cup K_1) \bigtriangledown K_2 & (0.6376) \\
S_{16} & = (((K_{1,2} \cup K_1) \bigtriangledown K_1) \cup K_1) \bigtriangledown K_2 & (0.6514)
\end{array}
$$

Graphs $S_1 - S_{12}$ from above were found in [**15**]; all other graphs appended to the list were found in this paper.

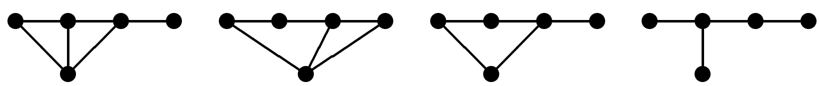
The numbers in brackets were computed by the programming package GRAPH (see [**4**], and also [**5**)]; they are approximately equal to the second largest eigenvalue of the corresponding graphs.

The next two propositions are taken from [**6**] and [**15**], respectively.

Proposition 1.1. *If H is a minimal forbidden (induced) subgraph for the σ-property, then:*

1^o *H is one of the graphs $E\,(=2K_2), F_1, F_2, F_3, F_4$ (see Fig. 2), or*

2^o *H belongs to the class $\mathcal{C}$.*



F_1 F_2 F_3 F_4

Fig. 2.

Proposition 1.2. *$G = ((G_1 \bigtriangledown \ldots \bigtriangledown G_k) \cup K_1) \bigtriangledown K_1$ is a σ^--graph if and only if*

$$
G_i = \begin{cases} K_2 \cup 2K_1 \\ K_{1,n} \cup K_1 \\ K_{2,2} \cup K_1 & (i = 1, \ldots, k). \\ K_{2,3} \cup K_1 \\ nK_1 \end{cases}
$$

2. Main results

We first prove a simple observation regarding the graphs from the class $\mathcal{C}$ (see Section 1).

Lemma 2.1. *If $G \in \mathcal{C}$, then $\overline{G} \in \mathcal{C}$ if and only if G is C_4-free.*

Proof. Suppose $G \in \mathcal{C}$. Then (as remarked in Section 1) $\overline{G} \in \mathcal{C}$ if and only if $\overline{G}$ does not contain a graph equal to E or P as an induced subgraph. Since the relation "to be an induced subgraph" is preserved after taking the complement, and since $\overline{E} = C_4$, while $\overline{P} = P$, the proof follows at once. □

Let $\mathcal{F}$ be the set of minimal forbidden induced subgraphs for σ-graphs, and let $\mathcal{F}^*$ be the corresponding set for the σ^*-graphs.

LEMMA 2.2. *If $F \in \mathcal{F}^*$, then F is one of the following graphs*:

1^o $E, \overline{E}, F_1, \overline{F_1}$, *or*

2^o *a C_4-free graph belonging to the class $\mathcal{C}$.*

Proof. Let $\overline{\mathcal{F}} = \{F \mid \overline{F} \in \mathcal{F}\}$. Then, clearly $\mathcal{F}^* \subseteq \mathcal{F} \cup \overline{\mathcal{F}}$. By Proposition 1.1, except for the graphs from $\mathcal{C}$, all other graphs in $\mathcal{F}$ are E, F_1, F_2, F_3, F_4 (see Fig. 2). Now observe that both E and $\overline{E}$ $(= C_4)$ are clearly the members of $\mathcal{F}^*$. The same applies for the graphs F_1 and $\overline{F_1}$ $(= F_4)$. On the other hand, $F_2, \overline{F_2}, F_3, \overline{F_3}$ can be rejected since are not minimal (contain E or $\overline{E}$). All other graphs F belong to $\mathcal{C}$ together with their complements. By Lemma 2.1 they are C_4-free. □

Our next aim is to provide the complete list of graphs from $\mathcal{F}^*$. By Lemma 2.2 (part 2^o), we only need to consider those graphs from $\mathcal{C}$ which are C_4-free. For any such graph, its canonical expression tree does not have two incomparable vertices which are:

(i) both internal;

(ii) one internal and the other leaf of weight at least 2;

(iii) both leaves each of weight at least 2.

Combining these three observations for canonical trees of G and $\overline{G}$ with the fact that their heights are at most three (see Lemma 4.1, [**15**]) we in further, provided $h(T_G) \leq h(T_{\overline{G}}) \leq 3$, consider the following three cases.

Case 1: $G \in \mathcal{C}$ and $h(T_G) \leq 1$

If $h(T_G) = 0$, then G is totally disconnected, while $\overline{G}$ is complete, and thus G is a σ^*-graph. Hence assume $h(T_G) = 1$.

PROPOSITION 2.3. *If $h(T_G) = 1$ $(h(T_G) \leq h(T_{\overline{G}})$, then G is a σ^*-graph if and only if G does not contain (as an induced subgraph) any of the following graphs:*

$$C_4,\ \overline{Q},\ \overline{R_1},\ \overline{R_3},\ \overline{S_2},\ \overline{S_4}.$$

Moreover, G is then one of the following graphs:

$$K_{k,\underbrace{1,\dots,1}_{n}}\ (k \geq 2, n \geq 1),\ K_n \cup mK_1\ (n,m \geq 1),\ K_{2,1,1,1} \cup mK_1\ (m \leq 2),$$

$$K_{2,1,1} \cup mK_1\ (m \geq 1),\ K_{2,1} \cup mK_1\ (m \geq 1),\ K_{k,1} \cup K_1\ (k \geq 4),$$
$$K_{3,1} \cup mK_1\ (m \leq 3).$$

Proof. By assumptions and observations (i)–(iii) the canonical expression trees for G and $\overline{G}$ are as depicted in Fig. 3 (with parameters $k \geq 0$ $(k \neq 1)$, $m \geq 0$ and $n \geq 1$). Notice also that for $k = 0$, we have $n \geq 2$ (since the trees are canonical). We also assume that the vertices of weight zero (other than roots) do not exist neither in T_G, nor in $T_{\overline{G}}$.

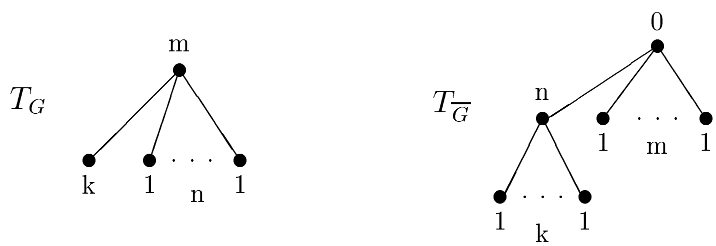


Fig. 3.

We first assume that $m = 0$. But then G is a complete multipartite graph, whereas the non-trivial component of $\overline{G}$ is a complete graph. Hence G is a σ^*-graph.

Assume next that $m > 0$. Then the non-trivial component of G is a complete multipartite graph (for any k and n). Regarding $\overline{G}$ (depending on k) we have:

If $k = 0$, then $\overline{G}$ is a complete multipartite graph, and consequently G is a σ^*-graph.

If $k = 2$, we first deduce that $n \leq 3$ (otherwise, R_1 appears in $\overline{G}$ – what is forbidden). For $n = 3$, we have that $m \leq 2$ (otherwise, S_2 appears in G), and then G is a σ^*-graph. Otherwise, if $n \leq 2$, then m is unrestricted, i.e. $\overline{G}$ is a σ-graph for each $m \geq 1$ (cf. Proposition 1.2). Thus G is a σ^*-graph.

If $k \geq 3$, we first deduce that $n = 1$ (due to Q). For $k = 3$, we have $m \leq 3$ (due to S_4), and then G is a σ^*-graph. Otherwise, if $k \geq 4$, then $m = 1$ (due to R_3). But then (cf. Proposition 1.2) $\overline{G}$ is always a σ-graph, and consequently G a σ^*-graph. □

Case 2: $G \in \mathcal{C}$ and $h(T) = 2$

PROPOSITION 2.4. *If $h(T_G) = 2$ $(h(T_G) \leq h(T_{\overline{G}}))$, then G is a σ^*-graph if and only if G does not contain (as an induced subgraph) any of the following graphs:*

$$C_4,\ Q,\ \overline{Q},\ R_1,\ \overline{R_1},\ R_2,\ \overline{R_2},\ R_3,\ \overline{R_3},\ \overline{S_1},\ S_5,\ \overline{S_5},\ S_{13},\ \overline{S_{13}},\ S_{14},\ \overline{S_{14}},\ \overline{S_{15}}.$$

Moreover, G is then one of the following graphs:

$$(K_{k,1} \cup K_1) \bigtriangledown K_q\ (k \geq 2,\ q \geq 1),\ ((K_k \cup K_1) \bigtriangledown K_1) \cup qK_1\ (k \geq 2,\ q \geq 1),$$
$$(K_{2,1,1,1} \cup K_1) \bigtriangledown K_1,\ ((K_2 \cup 3K_1) \bigtriangledown K_1) \cup K_1,\ (K_{2,1,1} \cup K_1) \bigtriangledown K_q\ (q \leq 2),$$
$$((K_2 \cup 2K_1) \bigtriangledown K_1) \bigtriangledown K_q\ (q \leq 2),\ (K_{2,1} \cup K_1) \bigtriangledown K_q\ (q \leq 2),$$
$$((K_2 \cup K_1) \bigtriangledown K_2) \cup qK_1\ (q \leq 2),\ ((K_{3,1} \cup 2K_1) \bigtriangledown K_1),\ ((K_3 \cup K_1) \bigtriangledown K_2) \cup K_1,$$
$$((K_{2,1} \cup 2K_1) \bigtriangledown K_1) \cup K_1,\ ((K_{2,1,1} \cup K_1) \bigtriangledown K_1) \cup K_1,\ ((K_{2,1} \cup K_1) \bigtriangledown K_1) \cup K_1.$$

Proof. By assumptions and observations (i)–(iii) the canonical expression trees for G and $\overline{G}$ are as depicted in Fig. 4 (with parameters $k \geq 0$ $(k \neq 1)$, $m \geq 1$, $n \geq 1$, $p \geq 0$, $q \geq 1$). Notice also that for $k = 0$, as in Proposition 2.3, we have again that $n \geq 2$. The same applies for vertices of zero weight.

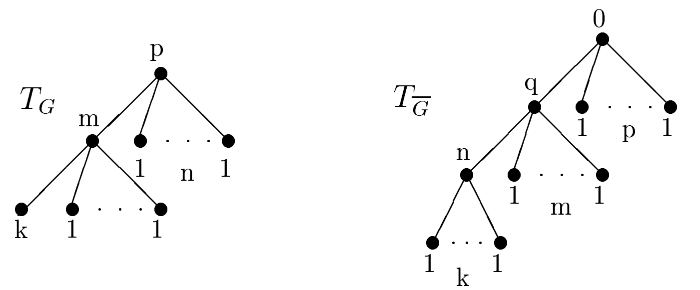


Fig. 4.

We first assume that $p = 0$. Then, by assumptions, we have $k \geq 2$ (note, if $k = 0$ then $h(T_{\overline{G}}) = 1$). Next we first deduce that $m \leq 2$ (otherwise, R_2 appears in G – what is forbidden).

If $m = 1$, then $n \leq 3$ (otherwise, R_1 appears in $\overline{G}$). For $n = 1$, by Proposition 1.2, G and $\overline{G}$ are σ-graphs for any k and q. Thus G is a σ^*-graph. For $2 \leq n \leq 3$, we have $k = 2$ (otherwise, Q appears in $\overline{G}$). In particular, if $n = 2$ then $q \leq 2$ (otherwise, S_{14} appears in G), whereas for $n = 3$, $q = 1$ (otherwise, R_3 appears in G). In the latter two cases, G is a σ^*-graph as can be easily checked (by brute force – say, by using the programming package GRAPH).

If $m = 2$, then we have: $n = 1$ and $q \leq 2$ (otherwise Q, respectively S_5 appears in G); $k \leq 3$ (otherwise, R_3 appears in $\overline{G}$). For $k = 2$ and $q \leq 2$ G is a σ^*-graph as can be easily checked (say, by a brute force). For $k = 3$ we have $q = 1$ (otherwise, S_{13} appears in G). Now again by a brute force we get that G is a σ^*-graph.

We now assume that $p > 0$. Also we can assume that $k \geq 0$ (otherwise, we have that $T_{\overline{G}}$ coincides with T_G from above (with $p = 0$)). Thus we have now only to extend the list of forbidden subgraphs with their complements, and to add to each σ^*-graph so far encountered its complement – without any further analysis.

In the remainder of the proof, let $k \geq 2$. But then we have $k = 2$ (otherwise, Q appears in $\overline{G}$), and in addition $q = 1$ (by the same reason). For other parameters we have: $m \leq 2$ (otherwise, R_2 appears in G; $n \leq 2$ and $p \leq 2$ (otherwise R_1, respectively S_1, appears in $\overline{G}$). Moreover, if $m = 2$, then $n = 1$ or if $n = 2$, then $m = 1$ (Q appears in G). Now if $m = 2$ or $n = 2$, then $p = 1$ (otherwise, R_3 or S_{15} appears in $\overline{G}$). All resulting graphs G are now σ^*-graphs as can be checked as usual (note that their complements are now not included in the list because of heights of their expression trees). □

Case 3: $G \in \mathcal{C}$ and $h(T) = 3$

Proposition 2.5. *If $h(T_G) = 3$ $(h(T_G) \leq h(T_{\overline{G}})$, then G is a σ^*-graph if and only if G does not contain (as an induced subgraph) any of the following graphs:*

$$C_4,\ Q,\ \overline{Q},\ R_2,\ \overline{R_2}, S_{16}\ \overline{S_{16}}.$$

Moreover, G is then one of the following graphs:

$$(((K_{2,1} \cup K_1) \bigtriangledown K_1) \cup K_1) \bigtriangledown K_1,\ ((((K_2 \cup K_1) \bigtriangledown K_1) \cup K_1) \bigtriangledown K_1) \cup K_1.$$

Proof. Now the canonical expression trees for G and $\overline{G}$ are depicted in Fig. 5 with parameters satisfying: $k \geq 0$ $(k \neq 1)$, $m \geq 1$, $n \geq 1$, $p \geq 1$, $q \geq 1$, $s \geq 0$ and $t \geq 1$. In addition, if $k = 0$ then $n \geq 2$, and, since $h(T_{\overline{G}}) \leq 3$, either $k = 0$ or $s = 0$ holds. Here again, some vertices of zero weights in T_G or $T_{\overline{G}}$ do not exist.

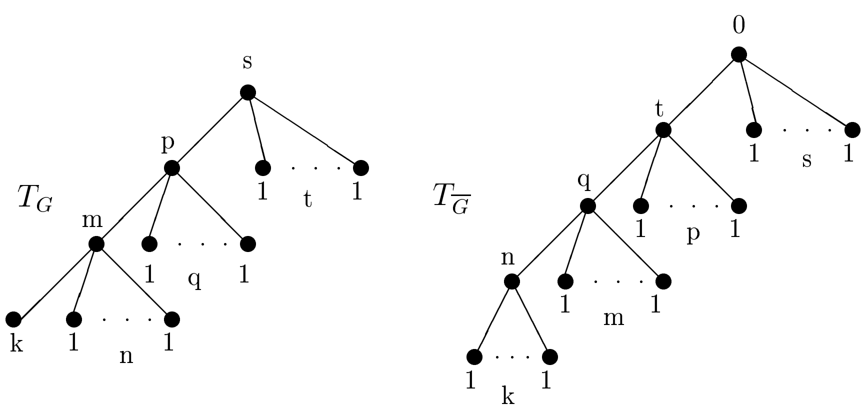


Fig. 5.

Assume first that $s = 0$. Then $k \neq 0$, since otherwise $h(T_{\overline{G}}) = 2$, contrary to the assumptions. Next we have: $m = 1$, $n = 1$, $p = 1$, $t = 1$ (otherwise, R_2, Q, R_2, S_{16} respectively, appear in G – what is forbidden). In addition, we have: $k = 2$, $q = 1$ (otherwise, in both situations, Q appears in $\overline{G}$). Now it is easy, say by a brute force, to see that the obtained graphs G are σ^*-graphs.

Next we assume $s \neq 0$. But then $k = 0$. If so, then $T_{\overline{G}}$ coincides with T_G from above (with $s = 0$). Thus, to finish the proof, we may proceed as in the proof of Proposition 2.4 to avoid further efforts. □

Collecting the above results we immediately get:

THEOREM 2.6. *G is a σ^*-graph if and only if it does not contain as an induced subgraph any of the following graphs:*

1^o E, $\overline{E}$, P;

2^o Q, $\overline{Q}$, R_i, $\overline{R}_i$ $(i \in \{1,2,3\})$;

3^o S_j, $\overline{S}_j$ $(j \in \{1,2,4,5,13,14,15,16\})$.

The complete list of σ^*-graphs is summarized below (notice that some graphs from the above propositions are condensed – to reduce the size of the list).

THEOREM 2.7. *G is a σ^*-graph if and only if G is one of the following graphs:*

$$K_m \cup nK_1 \ (m,n \geq 0),\ \ K_{2,1,1} \cup mK_1,\ K_{2,1} \cup mK_1 \ (m \geq 0),$$

$$K_{3,1} \cup mK_1 \ (m \leq 3),\ \ K_{2,1,1,1} \cup mK_1 \ (m \leq 2),$$

$$((K_{2,1,1} \cup K_1) \bigtriangledown K_1) \cup K_1,\ \ ((K_{2,1} \cup 2K_1) \bigtriangledown K_1) \cup K_1,\ \ ((K_{2,1} \cup K_1) \bigtriangledown K_1) \cup K_1,$$

$$(K_{m,1} \cup K_1) \bigtriangledown K_n \ (m \geq 2,\ n \geq 0),\ \ (K_{2,1,1} \cup K_1) \bigtriangledown K_m,$$

$$(K_{2,1} \cup 2K_1) \bigtriangledown K_m \ (m \leq 2),$$

$$(K_{3,1} \cup 2K_1) \bigtriangledown K_1,\ \ (K_{2,1,1,1} \cup K_1) \bigtriangledown K_1,\ \ (((K_{2,1} \cup K_1) \bigtriangledown K_1) \cup K_1) \bigtriangledown K_1$$

or, the complement of any of them.

From the above theorem it follows that the set of all σ^*-graphs is not finite (since infinite series of graphs exist).

We now deduce one interesting spectral property of σ^*-graphs. For this aim, suppose that G is a σ^*-graph. Thus $\lambda_2(G),\ \lambda_2(\overline{G}) \leq \sigma$. By making use of Courant-Weyl inequalities (see, for example, [**3**, p. 51]) we easily get $\lambda_{n-1}(G), \lambda_{n-1}(\overline{G}) \geq -\sigma - 1$. Thus both graphs G and $\overline{G}$ have all their eigenvalues, except the largest one and the smallest one, in the interval $[-\sigma - 1, \sigma]$.

As we have seen, the (canonical) expression trees are very suitable representation for graphs from $\mathcal{C}$, as many properties of these graphs can be deduced from the structures of these trees. In particular, we have seen how easily we had singled out all graphs from the title. Following these lines of reasoning, we also hope to characterize very soon all σ-graphs in terms of minimal forbidden subgraphs (see [**7**]).

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