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AN ANALYSIS AND IMPROVEMENT OF THE EL MISTIKAWY AND WERLE SCHEME

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Abstract. The El Mistikawy and Werle scheme (the EMW scheme) is derived as a member of the family of exponential spline difference schemes. Another member of the family (the IEMW scheme), giving a better accuracy then the EMW scheme is analysed. The truncation error of the IEMW scheme for the polynomials of up to the second degree approaches zero as ε approaches zero, which is not the case with the EMW scheme. Some numerical results are also presented.

1. Introduction. Let us consider the following singularly perturbed problem

$$\begin{cases} Ly = \varepsilon y'' + p(x)y' = f(x), & x \in I = [0,1], \\ y(0) = \alpha_0, & y(1) = \alpha_1, \end{cases} \tag{1}$$

where ε is a small positive parameter, α_0 and α_1 are given numbers, $p(x)$ and $f(x)$ are sufficiently smooth functions and $p(x) \geq p > 0$, $p \in R$. By using the exponential spline $e(x)$ from [**4**], $e(x) \in C^1(I)$, as a collocation function, a family of difference schemes is derived in ([**6**]). When $h \to 0$, this family reduces to the one derived in [**7**] via cubic splines. Some relations between those families are given in [**8**]. In Section 2 we shall briefely present the derivation of schemes [**6**]. The well–known Allan-Southwel- Il'in and El Mistikawy-Werle (EMW) schemes are members of that family. In this paper two schemes from that family, which have the second order accuracy (the one called the IEMW scheme) are analysed and compared. Although the second order is the maximal order of the uniform accuracy for this kind of schemes [**1**], the numerical results presented in Tables $2, 4, 6$ and 8 corresponding to the new scheme are much better then the ones presented in Tables $1, 3, 5$ and 7 which correspond to the EMW scheme. Some explanations of that are given in Section 3. Finaly, in Section 4 we shall present numerical examples which support the theoretical results presented in Section 3.

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In [**9**] it was shown that the spline used in the construction of the EMW scheme has the form given in [**4**]. Thus, we know the piecewise form of that spline and some of its properties given in [**4**]. The spline is between the usual cubic spline and the linear splines when the tension parameter respectively approaches ε and ∞. Its B-spline form is not known. The exponential spline given in [**5**] has a B-spline form and approaches a cubic or quadratic spline when the tension parameter approaches zero or $-\infty$, respectively. Although the B-spline form is more convinient for the calculations then the piecewise form , the difference scheme derived in [**5**] (SU scheme) is somewhat more complicated than the IEMW or EMW scheme. When $p(x) = p =$ const., the truncation error of the SU scheme is zero for functions $1, x, x^2$, while the truncation error for the EMW scheme is zero only for 1 and x. The truncation error of the IEMW scheme iz zero for 1 and x, while the truncation error for x^2 has the form $M\varepsilon h$ (M a is constant independent of ε and h). Thus, the latter approaches zero when ε approaches zero. The convergence of the SU scheme is not uniform and the scheme has the second order convergence only for $\varepsilon \ll h$. In that case the truncation error of the IEMW scheme for x^2 becomes negligent. Thus, the scheme SU can not be regarded advantageous in that respect. Moreover, the numerical results indicate the advantage of the IEMW scheme. Namely, the results presented in [**4**], show approximetly the same accuracy for SU and EMW schemes while our numerical results and theoretical analysis show the superiority of the IEMW scheme in comparison to the EMW scheme. The example 1 is treated both in this paper and in [**5**]. Comparing the results in Table 9 for the same h and ε in the corresponding Tables, one can see that the errors in the IEMW are about twice smaller than those in the SU scheme. In [**5**], the convergence is proven only for $p(x) =$ const. Hence, there is some vagueness in the choice of tension parameters. For the IEMW scheme the uniform convergence for $p(x) \neq 0$ is proven and the tension parameters are determined. The extensive problem with the term containing y is considered in [**5**].

2. The derivation of schemes. We are looking for an approximate solution to the Problem (1) in terms of the exponential spline given in [**4**]. The spline $e(x)$ has the form

$$e(x) = e_j(x) = u_j + hm_jt + g_j(ch\mu_jt - 1)/\rho_j + q_j(sh\mu_jt - \mu_j)/\rho_j,$$
$$x \in [x_j, x_{j+1}],$$

where $t = (x - x_j)/h$, $x_j = jh$, $h = 1/(n+1)$, $\mu_j = h\rho_j$, $j = 0(1)n$, ρ_j are tension parameters and $m_j = e'(x_j)$. The values g_j and q_j are determined from the requirement that $e(x) \in C^1(I)$.

From the collocation conditions

$$\varepsilon e''(x) + p^- e'(x) = f^-, \quad x = x_j,\ x = x_{j-1}, \tag{2}$$

$$\varepsilon e''(x) + p^+ e'(x) = f^+, \quad x = x_j,\ x = x_{j+1}, \tag{3}$$

where p^- and f^- are constant approximations to $p(x)$ and $f(x)$, for $x \in [x_{j-1}, x_j]$ and similarly p^+ and f^+ are constant aproximations to $p(x)$ and $f(x)$ on the interval

$[x_j, x_{j+1}]$ for fixed j, we obtain the following family of the difference schemes (see [**6**]):

$$r^- u_{j-1} + r^c u_j + r^+ u_{j+1} = q^- f^- + q^+ f^+, \quad j = 1(1)n-1, \tag{4}$$

$$u_0 = \alpha_0, \quad u_1 = \alpha_1, \tag{5}$$

where

$$R^+ = \mu^+/(1-\exp(-\mu^+)), \quad R^- = \mu^- \exp(-\mu^-)/(1-\exp(-\mu^-)),$$

$$r^+ = \varepsilon R^+/h^2, \quad r^- = \varepsilon R^-/h^2, \quad r^c = -r^- - r^+$$

$$q^- = (1-R^-)/\mu^-, \quad q^+ = (R^+ - 1)/\mu^+.$$

Since $e_j(x) = \operatorname{span}\{1, x, \exp(\rho_j x), \exp(-\rho_j x)\}$, in order to give to function $e(x)$ some properties of the exact solution of the problem (1) [**1**], we choose $\mu^+ = \rho^+ h$, $\mu^- = \rho^- h$, $\rho^+ = p^+/\varepsilon$, $\rho^- = p^-/\varepsilon$.

Determining $p^\pm = (p(x_{j\pm1}) + p(x_j))/2$, $f^\pm = (f(x_{j\pm1}) + f(x_j))/2$, we obtain the EMW scheme. Taking

$$p^\pm = p(x_j \pm h/2), \quad f^\pm = f(x_j \pm h/2), \tag{6}$$

we obtain the IEMW scheme which we shall analyse in detail.

3. The truncation error. THEOREM 1. *Let $y(x) \in C^4(I)$. Let u_j be an approximation to $y(x_j)$, obtained by using the scheme given by* (4), (5) *and* (6) (*the IEMW scheme*). *Then, $|y(x_j) - u_j| \le Mh^2$, where M is a constant, independent of ε and h.*

Proof. The proof follows from the proof for the EMW scheme given in [**1**] and the fact that

$$(p(x_j) + p(x_{j\pm1}))/2 = p(x_j \pm h/2) + O(h^2)$$

and

$$(f(x_j) + f(x_{j\pm1}))/2 = f(x_j \pm h/2) + O(h^2).$$

In the following, wherever it is clear from the context, the j subscripts will be omitted. Here M denotes different constants independent of h and ε.

Although the IEMW and EMW schemes have the same order of uniform accuracy, the IEMW scheme yields better results (see the numerical results). In order to explain this behaviour we consider the truncation errors of those schemes.

Let $h \le \varepsilon$. The truncation error $\tau_j(y) = Ry_j - Q(Ly_j)$ can be written in the form

$$\tau_j(y) = T_{j0} y_j + T_{j1} y_j' + T_{j2} y_j'' + T_{j3} y_j''' + R_{j4}(y),$$

where $T_{j0} = T_{j1} = 0$ for both schemes, IEMW and EMW. Further, for the IEMW scheme,

$$\begin{aligned} T_{j2} &= \frac{h^2}{2}(r^- + r^+) - \varepsilon(q^- + q^+) + \frac{h}{2}(p^- q^- - p^+ q^+) \\ T_{j3} &= \frac{h^3}{6}(r^+ - r^-) + \frac{h}{2}\varepsilon(q^- - q^+) - \frac{h^2}{8}(p^+ q^+ + p^- q^-), \\ R_{j4}(y) &= T_{rj} + T_{qj} \\ T_{rj} &= \frac{R^-\varepsilon}{h^2} R_3(x_j, x_{j-1}, y) + \frac{R^+\varepsilon}{h^2} R_3(x_j, x_{j+1}, y) \\ T_{qj} &= -q^-\varepsilon R_1(x_j, x_{j-1/2}, y'') - q^+\varepsilon R_1(x_j, x_{j+1/2}, y'') \\ &\quad - q^- p^- R_2(x_j, x_{j-1/2}, y') - q^+ p^+ R_2(x_j, x_{j+1/2}, y') \end{aligned}$$

where,

$$R_n(a, b, g) = \frac{1}{n!}\int_a^b (b-s)^n g^{(n+1)}(s) ds = g^{(n+1)}(\xi)\frac{(b-a)^{n+1}}{(n+1)!}, \quad a \le \xi \le b.$$

The corresponding expressions for the EMW scheme can be found in [**1**].

In the case of $h \le \varepsilon$, following some Taylor's expansions, we obtain

$$T_{j2} = \frac{-h^2}{6}(p'(\beta_1) + p'(\beta_2)) + O(\frac{h^3}{\varepsilon}) \tag{7}$$

for the scheme EMW and

$$T_{j2} = \frac{-h^2}{24}(p'(\beta_3) + p'(\beta_4)) + O(\frac{h^3}{\varepsilon}) \tag{8}$$

for the IEMW scheme, where

$$x_{j-1} < \beta_1 < x_j < \beta_2 < x_{j+1}, \quad x_{j-1/2} < \beta_3 < x_j < \beta_4 < x_{j+1/2}.$$

When $p(x) = p =$ const., we have

$$T_{j3} = \frac{-h^2}{6}p + O(\frac{h^3}{\varepsilon}) \tag{9}$$

for the scheme EMW and

$$T_{j3} = \frac{-h^2}{24}p + O(\frac{h^3}{\varepsilon}) \tag{10}$$

for the IEMW scheme. Those facts indicate that the IEMW scheme is four times better then the EMW scheme for $h \le \varepsilon$, which agrees with our numerical results.

Since $r^-, q^- \to 0$ when $\varepsilon \to 0$ the influence of the corresponding expressions for the truncation errors is insignificant. The remainding terms due to the coefficients r^+ and q^+ are dominant. The constants in these expressions are smaller in the case of the IEMW scheme.

The fact that the values denoted by ξ are in intervals $[x_{j-1/2}, x_j]$ for the IEMW scheme is very important and allows better estimates for some exponential expresions. For example,

$$\frac{h^k}{\varepsilon^k}\exp(-\xi p/\varepsilon) \le \exp(-\delta h/\varepsilon),$$

when $x_{j-1/2} < \xi < x_j$, where δ is a positive constant independent of ε and h. We cannot say the same for $x_{j-1} < \xi < x_j$.

Definition 1. The difference scheme has the accuracy of the order s if $\tau_j(y) = 0$ when $y \in P_s$, where P_s is a set of polynomials of degree less than or equal to s.

Definiton 2. The difference scheme has ε-accuracy of the order s if: $\lim_{\varepsilon\to 0}\tau_j(y) = 0$ when $y \in P_s$, where P_s is a set of polynomials of the degree less than or equal to s.

The EMW and IEMW schemes have the first order accuracy, while the SU scheme has the second order accuracy. Besides that, the IEMW scheme has an $\varepsilon-$ accuracy of the second order.

THEOREM 2. *The IEMW scheme has the second order ε-accuracy.*

Proof. One can verify that $T_{j0} = T_{j1} = 0$. T_{j2} can be writen in the form

$$T_{j2} = \varepsilon(R^-/\mu_j^- - R^+/\mu_j^+ + 1/\mu_j^- - 1/\mu_j^+).$$

Denote by $D_\mu R^-(\mu_j^-)_\xi$ the value of the first derivative at the point ξ and $T_{j2} = T_{j2}(p^+, p^-)$, $R^- = R^-(\mu_j^-)$, $R^+ = R^+(\mu_j^+)$. Since $T_{j2}(p^+, p^+) = 0$ and $R^-(\mu^-) = R^-(\mu^+) + (\mu^- - \mu^+)D_\mu R^-(\mu)_\xi$ and $|D_\mu R^-(\mu^-)| \le M$ we have,

$$T_{j2} = T_{j2} - T_{j2}(p^+, p^+),$$

$$T_{j2} = \varepsilon(\mu_j^- - \mu_j^+)D_\mu R_\xi^-/\mu_j^+ + \varepsilon(1/\mu_j^+ - 1/\mu_j^-) - \varepsilon(1/\mu_j^+ - 1/\mu_j^-)R^-(\mu_j^-)$$

and

$$|T_{j2}| \le Mh\varepsilon. \tag{11}$$

Thus, the theorem holds. For the EMW scheme the corresponding result is

$$|T_{j2}| \le Mh^2, \tag{12}$$

which shows the advantage of the IEMW scheme. The numerical results confirm that. Of interest the behaviour of the values T_{j3} for both schemes. Namely, for $p(x) = p =$ const. we obtain

$$\lim_{\varepsilon\to 0} T_{j3} = -h^2/12 \tag{13}$$

for EMW scheme and

$$\lim_{\varepsilon\to 0} T_{j3} = h^2/24 \tag{14}$$

for the IEMW scheme. From (10) and (14) one can see that T_{j3} for the IEMW scheme changes the sign when ε approaches zero, for a fixed h. Since T_{j3} is a continuous function of ε, it becomes zero at a certain point, which may contribute to the error decrease or to the acquirement of convergence with respect to ε. The order of convergence is calculated with respect to h. Because of that, we obtain a negative sign or wrong results in the numerical treatment of the order of convergence when the scheme attains convergence with respect to ε. But, when $\varepsilon \to 0$ we can say in general that the contribution to the error of T_{j3} is twice smaller for the IEMW scheme then for the EMW scheme. Numerical results support that statement. For the SU scheme we have $T_{j2} = 0$, but the scheme attains the second order of convergence only for $\varepsilon \ll h$. In that case, T_{j2} for the IEMW scheme becomes much smaller than the rest of the truncation error, which is $O(h^2)$ for both schemes. Thus, the SU scheme is not beneficial in that sense.

Remark 1: The functions $f(x)$ and $p(x)$ in (1) may depend on ε but they have to be uniformly bounded on ε.

k	n							
	8	16	32	64	128	256	512	
1	1.39(−2)	3.64(−3)	9.08(−4)	2.27(−4)	5.68(−5)	1.42(−5)	3.55(−6)	E_n
			1.94	2.00	2.00	2.00	2.00	Ord
2	1.53(−2)	3.75(−3)	9.33(−4)	2.33(−4)	5.82(−5)	1.46(−5)	3.64(−6)	E_n
			2.03	2.01	2.00	2.00	2.00	Ord
3	1.45(−2)	3.79(−3)	9.68(−4)	2.44(−4)	6.09(−5)	1.52(−5)	3.81(−6)	E_n
			1.95	1.97	1.99	2.00	2.00	Ord
4	9.57(−3)	4.10(−3)	1.00(−3)	2.56(−4)	6.40(−5)	1.60(−5)	4.01(−6)	E_n
			1.26	2.04	1.97	2.00	2.00	Ord
5	4.08(−3)	3.03(−3)	1.09(−3)	2.67(−4)	6.66(−5)	1.67(−5)	4.19(−6)	E_n
			.54	1.51	2.03	2.01	2.00	Ord
6	1.91(−3)	1.35(−3)	8.54(−4)	2.82(−4)	6.94(−5)	1.73(−5)	4.32(−6)	E_n
			.52	.77	1.63	2.03	2.00	Ord
7	1.37 (−3)	5.92(−4)	3.92(−4)	2.27(−4)	7.17(−5)	1.77(−5)	4.41(−6)	E_n
			1.10	.64	.88	1.69	2.03	Ord
8	1.23(−3)	3.99(−3)	1.66(−4)	1.06(−4)	5.86(−5)	1.81(−5)	4.47(−6)	E_n
			1.57	1.16	.70	.94	1.72	Ord
9	1.20(−3)	3.51(−4)	1.08(−4)	4.40(−5)	2.75(−5)	1.49(−5)	4.56(−6)	E_n
			1.75	1.65	1.20	.74	.97	Ord
10	1.20(−3)	3.40(−4)	9.38(−5)	2.81(−5)	1.13(−5)	7.00(−6)	3.75(−6)	E_n
			1.80	1.84	1.68	1.21	.75	Ord
11	1.19(−3)	3.38(−4)	9.04(−5)	2.42(−5)	7.16(−6)	2.87(−6)	1.77(−6)	E_n
			1.82	1.90	1.88	1.67	1.22	Ord
12	1.19(−3)	3.37(−4)	8.96(−5)	2.33(−5)	6.15(−6)	1.81(−6)	7.24(−7)	E_n
			1.82	1.91	1.94	1.90	1.70	Ord
13	1.19(−3)	3.37(−4)	8.94(−5)	2.31(−5)	5.90(−6)	1.55(−6)	4.54(−7)	E_n
			1.82	1.91	1.95	1.96	1.91	Ord
14	1.19(−3)	3.37(−4)	8.94(−5)	2.30(−5)	5.85(−6)	1.49(−6)	3.89(−7)	E_n
			1.82	1.91	1.96	1.97	1.97	Ord
15	1.19(−3)	3.37(−4)	8.94(−5)	2.30(−5)	5.84(−6)	1.47(−6)	3.73(−7)	E_n
			1.82	1.91	1.96	1.98	1.98	Ord

Table 1 (Example 1)

k	n							
	8	16	32	64	128	256	512	
1	3.32(−3)	9.00(−4)	2.26(−4)	5.68(−5)	1.42(−6)	3.55(−6)	8.88(−7)	E_n
			1.87	1.99	1.99	2.00	2.00	Ord
2	3.61(−3)	9.24(−4)	2.32(−4)	5.82(−5)	1.46(−5)	3.64(−6)	9.10(−7)	E_n
			1.96	1.99	2.00	2.00	2.00	Ord
3	3.23(−3)	9.22(−4)	2.40(−4)	6.08(−5)	1.52(−5)	3.81(−6)	9.52(−7)	E_n
			1.79	1.94	1.98	2.00	2.00	Ord
4	1.63(−3)	9.40(−4)	2.44(−4)	6.37(−5)	1.60(−5)	4.01(−6)	1.00(−6)	E_n
			.74	1.93	1.93	1.99	2.00	Ord
5	3.09(−4)	5.53(−4)	2.52(−4)	6.54(−5)	1.66(−5)	4.18(−6)	1.04(−6)	E_n
			-.91	1.09	1.93	1.98	1.98	Ord
6	5.66(−5)	1.18(−4)	1.60(−4)	6.54(−5)	1.70(−5)	4.30(−6)	1.08(−6)	E_n
			-1.01	-.53	1.25	1.93	1.98	Ord
7	4.54(−5)	1.77(−5)	3.66(−5)	4.31(−5)	1.67(−5)	4.33(−6)	1.10(−6)	E_n
			1.51	-1.03	-.33	1.33	1.93	Ord
8	4.86(−5)	1.23(−5)	5.03(−6)	1.02(−5)	1.12(−5)	4.22(−6)	1.10(−6)	E_n
			2.03	1.44	-1.02	-.23	1.36	Ord
9	5.11(−5)	1.31(−5)	3.19(−6)	1.35(−6)	2.70(−6)	2.85(−6)	1.06(−6)	E_n
			1.95	2.10	1.38	-1.01	-.18	Ord
10	5.26(−5)	1.38(−5)	3.40(−6)	8.12(−7)	3.57(−7)	6.93(−7)	7.20(−7)	E_n
			1.91	2.02	2.13	1.36	-1.00	Ord
11	5.33(−5)	1.43(−5)	3.59(−6)	8.65(−7)	2.05(−7)	8.90(−8)	1.76(−7)	E_n
			1.89	1.98	2.05	2.15	1.34	Ord
12	5.38(−5)	1.45(−5)	3.71(−6)	9.15(−7)	2.18(−7)	5.15(−8)	2.25(−8)	E_n
			1.88	1.96	2.01	2.06	2.16	Ord
13	5.40(−5)	1.46(−5)	3.78(−6)	9.46(−7)	2.31(−7)	5.48(−8)	1.29(−8)	E_n
			1.88	1.95	1.99	2.02	2.07	Ord
14	5.41(−5)	1.47(−5)	3.81(−6)	9.63(−7)	2.39(−7)	5.80(−8)	1.37(−8)	E_n
			1.88	1.94	1.98	2.00	2.03	Ord
15	5.41(−5)	1.47(−5)	3.83(−6)	9.72(−7)	2.43(−7)	6.00(−8)	1.45(−8)	E_n
			1.88	1.94	1.98	1.99	2.01	Ord

Table 2 (Example 1)

4. Numerical results. In this section we shall present the results of some numerical examples using EMW, IEMW and SU schemes. We denote by E_n the maximum of $|y(x_j) - u_j|, j = 0(1)n+1$. Here $[u_0, u_1, \ldots, u_{n+1}]^\top$ is the corresponding numerical solution to the system (4) and (5). Also, we define, in the usual way, the order of convergence (Ord) for two succesive values of n with respective errors E_n and E_{2n}

$$\mathrm{Ord} = \frac{\log E_n - \log E_{n_2}}{\log n_2 - \log n},$$

where $n_2 = 2n$. Different values of $\varepsilon = 2^{-k}$ and n are considered. Tables $1, 3, 5$ and 7 present the numerical solution obtained by using the EMW scheme. Tables $2, 4, 6$ and 8 present the corresponding results obtained from the IEMW scheme.

The first example was taken from [**1**].

k	n							
	8	16	32	64	128	256	512	
1	2.22(−2)	5.53(−3)	1.40(−3)	3.49(−4)	8.74(−5)	2.18(−5)	5.46(−6)	E_n
			2.00	1.98	2.00	2.00	2.00	Ord
2	2.42(−2)	6.15(−3)	1.56(−3)	3.91(−4)	9.78(−5)	2.44(−5)	6.11(−6)	E_n
			1.97	1.98	2.00	2.00	2.00	Ord
3	2.66(−2)	6.73(−3)	1.70(−3)	4.31(−4)	1.08(−4)	2.70(−5)	6.74(−6)	E_n
			1.98	1.98	1.98	2.00	2.00	Ord
4	2.34(−2)	7.22(−3)	1.83(−3)	4.63(−4)	1.16(−4)	2.91(−5)	7.28(−6)	E_n
			1.70	1.98	1.98	2.00	2.00	Ord
5	1.80(−2)	6.76(−3)	1.87(−3)	4.76(−4)	1.22(−4)	3.07(−5)	7.69(−6)	E_n
			1.43	1.86	1.98	1.96	2.00	Ord
6	1.55(−2)	5.19(−3)	1.82 (−3)	4.78(−4)	1.25(−4)	3.16(−5)	7.95(−6)	E_n
			1.57	1.53	1.93	1.94	1.98	Ord
7	1.48(−2)	4.35(−3)	1.40(−3)	4.71(−4)	1.21(−4)	3.19(−5)	8.07(−6)	E_n
			1.76	1.64	1.58	1.97	1.92	Ord
8	1.47 (−2)	4.12(−3)	1.15(−3)	3.62(−4)	1.20(−4)	3.03(−5)	8.07(−6)	E_n
			1.83	1.83	1.66	1.61	1.99	Ord
9	1.46(−2)	4.06(−3)	1.08(−3)	2.96(−4)	9.23(−5)	3.03(−5)	7.64(−6)	E_n
			1.85	1.90	1.86	1.68	1.63	Ord
10	1.46(−2)	4.04(−3)	1.06(−3)	2.77(−4)	7.50(−5)	2.33(−5)	6.71(−6)	E_n
			1.85	1.92	1.94	1.88	1.69	Ord
11	1.46(−2)	4.04(−3)	1.06(−3)	2.73(−4)	7.01(−5)	1.89(−5)	5.85(−6)	E_n
			1.85	1.93	1.96	1.96	1.88	Ord
12	1.46(−2)	4.04(−3)	1.06(−3)	2.73(−4)	6.90(−5)	1.76(−5)	4.74(−6)	E_n
			1.85	1.93	1.97	1.98	1.96	Ord
13	1.46(−2)	4.04(−3)	1.06(−3)	2.71(−4)	6.90(−5)	1.73(−5)	4.42(−6)	E_n
			1.85	1.93	1.97	1.98	1.98	Ord
14	1.46(−2)	4.04(−3)	1.06(−3)	2.71(−4)	6.90(−5)	1.73(−5)	4.35(−6)	E_n
			1.85	1.93	1.97	1.98	1.99	Ord
15	1.46(−2)	4.04(−3)	1.06(−3)	2.71(−4)	6.90(−5)	1.72(−5)	4.33(−6)	E_n
			1.85	1.93	1.97	1.98	1.99	Ord

Table 3 (Example 2)

Example 1.

$$\begin{cases} Ly = \varepsilon y'' + (x+1)^3 y' = f(x,\varepsilon), & x \in I = (0,1), \\ y(0) = \alpha_0, & y(1) = \alpha_1, \end{cases} \tag{15}$$

The exact solution has the form

$$y(x) = \frac{1}{p(x)} \exp(-\varepsilon^{-1} \int_0^x p(\xi)d\xi) + \exp(-\frac{x}{2}), \quad p(x) = (x+1)^3.$$

The solution determines $f(x,\varepsilon), \alpha_0$ and α_1 . The derivatives of $f(x,\varepsilon)$ and α_1 are bounded functions of ε. The same problem was treated in [**5**]. Comparing Table 1 with Table 2, we can see that the new scheme has much better results then the EMW scheme, especially when ε is much smaller then h. The exact solution has only exponential terms and constants in the remainder terms are smaller since the length of the interval for the Taylor's expansions is $h/2$. The existence of the unknown values ξ, β_i etc., does not permit a complete comparison of those expressions. A better behaviour of the scheme, presented in Table 2, for small values of ε results in the fast decrease of Ord. The results from IEMW and SU schemes are compared in Table 9.

k	n							
	8	16	32	64	128	256	512	
1	5.27(−3)	1.37(−3)	3.48(−4)	8.73(−5)	2.18(−5)	5.46(−6)	1.36(−6)	E_n
			1.94	1.98	2.00	2.00	2.00	Ord
2	5.49(−3)	1.49(−3)	3.88(−4)	9.75(−5)	2.44(−5)	6.11(−6)	1.53(−6)	E_n
			1.86	1.93	2.00	2.00	2.00	Ord
3	5.20(−3)	1.54(−3)	4.16(−4)	1.07(−4)	2.69(−5)	6.74(−6)	1.69(−6)	E_n
			1.70	1.86	1.95	1.99	2.00	Ord
4	2.95(−3)	1.47(−3)	4.25(−4)	1.13(−4)	2.89(−5)	7.27(−6)	1.82(−6)	E_n
			.84	1.74	1.89	1.96	1.99	Ord
5	8.29(−4)	9.38(−4)	3.88(−4)	1.11(−4)	3.00(−5)	7.64(−6)	1.92(−6)	E_n
			-.66	1.14	1.76	1.87	1.97	Ord
7	1.97(−4)	2.82(−4)	2.64(−4)	9.97(−5)	2.90(−5)	7.74(−6)	1.98(−6)	E_n
			-1.28	-.32	1.28	1.75	1.88	Ord.
8	8.04 (−5)	6.34(−5)	8.26(−5)	6.98(−5)	2.53(−5)	7.43(−6)	1.98(−6)	E_n
			1.09	-1.26	-.14	1.35	1.73	Ord
9	5.73(−5)	2.37(−5)	1.81(−5)	2.24(−5)	1.80(−5)	6.37(−6)	1.88(−6)	E_n
			1.97	.97	-1.22	-.04	1.39	Ord
10	5.33(−5)	1.60(−5)	6.466(−6)	4.84(−6)	5.83(−6)	4.56(−6)	1.60(−6)	E_n
			1.94	1.90	2.06	.88	-1.20	Ord
11	5.31(−5)	1.46(−5)	4.22(−6)	1.69(−6)	1.25(−6)	1.49(−6)	1.15(−6)	E_n
			1.91	2.01	2.11	.84	-1.19	Ord
12	5.35(−5)	1.45(−5)	3.80(−6)	1.08(−6)	4.32(−7)	3.19(−7)	3.76(−7)	E_n
			1.89	1.97	2.04	2.13	.81	Ord
13	5.38(−5)	1.46(−5)	3.76(−6)	9.76(−7)	2.75(−7)	1.09(−7)	8.03(−8)	E_n
			1.88	1.96	2.01	2.06	2.14	Ord
13	5.40(−5)	1.46(−5)	3.79(−6)	9.60(−7)	2.45(−7)	6.92(−8)	2.74(−8)	E_n
			1.88	1.95	1.99	2.02	2.07	Ord
14	5.41(−5)	1.47(−5)	3.81(−6)	9.67(−7)	2.42(−7)	6.16(−8)	1.74(−8)	E_n
			1.88	1.94	1.98	2.00	2.03	Ord
14	5.41(−5)	1.47(−5)	3.83(−6)	9.73(−7)	2.44(−7)	6.09(−8)	1.45(−8)	E_n
			1.85	1.93	1.97	1.98	1.99	Ord

Table 4 (Example 2)

Example 2. The problem (15) has the exact solution

$$y(x) = \frac{1}{p(x)} \exp(-\varepsilon^{-1} \int_0^x p(\xi) d\xi) + \exp(-\frac{x}{2}) + x^2, \quad p(x) = (x+1)^3.$$

Comparing Table 1 with Table 3 and then Table 2 with Table 3 one can see that the polynomial term of the second order in the exact solution ruins, to a large extent, the results obtained from the EMW scheme, but from the IEMW scheme the results are slightly worse when $h \leq \varepsilon$, while the results are the same when $\varepsilon \ll h$. This is the consequence of the ε-accuracy of the IEMW scheme. Also, the results in Table 3 and Table 4 show the important advantage superiority of the IEMW scheme.

k	n							
	8	16	32	64	128	256	512	
1	3.38(-2)	8.51(-3)	2.14(-3)	5.37(-4)	1.34(-4)	3.36(-5)	8.39(-6)	E_n
			1.99	1.99	2.00	2.00	2.00	Ord
3	4.08(-2)	1.06(-3)	2.74(-3)	6.94(-4)	1.74(-4)	4.35(-5)	1.09(-5)	E_n
			1.93	1.95	1.98	2.00	2.00	Ord
5	3.43(-2)	1.07(-3)	2.81(-3)	7.50(-4)	1.93(-4)	4.86(-5)	1.22(-5)	E_n
			1.68	1.94	1.89	1.96	1.98	Ord
7	2.99(-2)	8.19(-3)	2.40(-3)	7.12(-4)	1.86(-4)	4.89(-5)	1.25(-5)	E_n
			1.87	1.77	1.76	1.94	1.91	Ord
9	2.92(-2)	7.65(-3)	1.99(-3)	5.31(-4)	1.54(-4)	4.52(-5)	1.18(-5)	E_n
			1.94	1.95	1.90	1.79	1.78	Ord
11	2.91(-2)	7.55(-3)	1.92 (-3)	4.89(-4)	1.26(-4)	3.35(-5)	9.71(-6)	E_n
			1.95	1.97	1.98	1.96	1.91	Ord
13	2.90(-2)	7.53(-3)	1.91(-3)	4.82(-4)	1.22(-4)	3.07(-5)	7.88(-6)	E_n
			1.95	1.98	1.99	1.99	1.99	Ord
15	2.90 (-2)	7.53(-3)	1.91(-3)	4.81(-4)	1.21(-4)	3.03(-5)	7.61(-6)	E_n
			1.95	1.98	1.99	1.99	2.00	Ord

Table 5 (Example 3)

k	n							
	8	16	32	64	128	256	512	
1	7.84(-3)	2.08(-3)	5.344(-4)	1.34(-4)	3.35(-5)	8.39(-6)	2.10(-6)	E_n
			1.89	1.96	1.99	2.00	2.00	Ord
3	6.73(-3)	2.30(-3)	6.55(-4)	1.71(-4)	4.33(-5)	1.09(-5)	2.72(-6)	E_n
			1.43	1.76	1.92	1.98	1.99	Ord
5	7.83(-3)	1.03(-3)	4.89(-4)	1.65(-4)	4.63(-5)	1.20(-5)	3.04(-6)	E_n
			1.68	1.94	1.89	1.96	1.98	Ord
7	2.60(-3)	4.62(-4)	4.90(-5)	7.69(-5)	3.22(-5)	1.08(-5)	3.02(-6)	E_n
			1.87	1.77	1.76	1.94	1.91	Ord
9	3.17(-3)	7.88(-4)	1.72(-4)	2.88(-5)	3.06(-6)	5.02(-6)	2.07(-6)	E_n
			1.94	1.95	1.90	1.79	1.78	Ord
11	3.31(-3)	8.72(-4)	2.17 (-4)	5.11(-5)	1.09(-5)	1.80(-6)	1.91(-7)	E_n
			1.95	1.97	1.98	1.96	1.91	Ord
13	3.35(-3)	8.94(-4)	2.29(-4)	5.71(-5)	1.39(-5)	3.22(-6)	6.81(-7)	E_n
			1.95	1.98	1.99	1.99	1.99	Ord
15	3.36 (-3)	8.99(-4)	2.32(-4)	5.86(-5)	1.46(-5)	3.61(-6)	8.71(-7)	E_n
			1.95	1.98	1.99	1.99	2.00	Ord

Table 6 (Example 3)

Example 3. The problem (15) has the exact solution

$$[y(x) = \frac{1}{p(x)} \exp(-\varepsilon^{-1} \int_0^x p(\xi) d\xi) + \exp(-\frac{x}{2}) + x^3, \; p(x) = (x+1)^3.]$$

The errors in Table 6, when $h \leq \varepsilon$, are about four times smaller than the corresponding errors in Table 5. This is in agreement with relations (7), (8), (9) and (10). The corresponding errors for $\varepsilon < h$ are about ten times smaller in Table 6 than in Table 5. This can be explained by (11), (12), (13), (14) and (10), i.e. by the ε-accuracy and the change of the sign of T_{j3} when $\varepsilon \to 0$. In the above examples the exact solutions have only polynomial and exponential terms. The following example has a more general form.

k	n							
	8	16	32	64	128	256	512	
1	5.61(−3)	1.44(−3)	3.64(−4)	9.05(−5)	2.19(−5)	4.78(−6)	1.26(−6)	E_n
			1.89	1.98	2.00	2.00	2.00	Ord
3	9.98(−3)	2.19(−3)	5.55(−4)	1.38(−4)	3.52(−5)	9.51(−6)	3.12(−6)	E_n
			2.22	1.99	2.02	2.00	2.00	Ord
5	1.46(−2)	5.60(−3)	1.42(−3)	3.66(−4)	9.24(−4)	2.40(−5)	6.91(−6)	E_n
			1.38	1.98	1.95	2.00	2.00	Ord
7	9.73(−3)	4.43(−3)	1.57(−3)	4.63(−4)	1.22(−4)	3.20(−5)	8.99(−6)	E_n
			1.14	1.49	1.74	1.92	1.97	Ord
9	7.76(−3)	3.52(−3)	1.20(−3)	3.75(−4)	1.14(−4)	3.28(−5)	9.64(−6)	E_n
			1.15	1.57	1.57	1.72	1.80	Ord
11	7.20(−3)	3.24(−3)	1.07 (−3)	3.14(−4)	8.96(−5)	2.64(−5)	8.53(−6)	E_n
			1.15	1.61	1.80	1.88	1.88	Ord
13	7.06(−3)	3.17(−3)	1.03(−3)	2.95(−4)	8.07(−5)	2.25(−5)	7.01(−6)	E_n
			1.15	1.62	1.81	1.89	1.92	Ord
15	7.02 (−3)	3.15(−3)	1.02(−3)	2.90(−4)	7.83(−5)	2.13(−5)	6.45(−6)	E_n
			1.15	1.63	1.82	1.91	1.95	Ord

Table 7 (Example 4)

k	n							
	8	16	32	64	128	256	512	
1	1.47(−3)	3.64(−4)	9.06(−5)	4.78(−6)	1.26(−6)	1.27(−6)	1.26(−6)	E_n
			2.01	2.00	2.00	2.00	2.00	Ord
3	2.74(−3)	5.68(−4)	1.41(−4)	3.53(−5)	9.52(−6)	3.12(−6)	1.57(−6)	E_n
			2.33	2.04	2.03	2.01	2.00	Ord
5	2.78(−3)	1.28(−3)	3.47(−4)	9.20(−5)	2.40(−5)	6.91(−6)	2.64(−6)	E_n
			1.06	1.85	1.92	1.99	2.00	Ord
7	9.90(−4)	4.86(−5)	1.71(−4) 9	9.27(−5)	2.97(−5)	8.83(−6)	3.18(−6)	E_n
			2.66	.94	.41	1.59	1.88	Ord
9	2.62(−3)	8.50(−4)	1.80(−4)	1.29(−5)	1.13(−5)	7.19(−6)	3.21(−6)	E_n
			1.55	2.06	2.83	2.54	.04	Ord
11	3.09(−3)	1.11(−3)	3.13(−4)	7.32(−5)	1.26(−5)	1.26(−6)	1.88(−6)	E_n
			1.45	1.79	2.01	2.26	2.87	Ord
13	3.21(−3)	1.18(−3)	3.49(−4)	9.16(−5)	2.14(−5)	3.77(−6)	1.27(−6)	E_n
			1.43	1.75	1.90	2.00	2.11	Ord
15	3.24(−3)	1.20(−3)	3.59(−4)	9.64(−5)	7.83(−5)	2.13(−6)	6.45(−6)	E_n
			1.43	1.74	1.88	1.95	2.00	Ord

Table 8 (Example 4)

Example 4. [**10**] Problem (1) with $[p(x) = (2 + 2\varepsilon(1 + x))/(1 + x)^2]$ and $[y(x) = \cos(\pi x/(1 + x)) + (\exp(-1/\varepsilon) - \exp(-2x/(\varepsilon(1 + x))))/(1 - \exp(-1/\varepsilon))]$

The results in Table 8 are about four times better than those shown in Table 7 for $h \le \varepsilon$, which agrees with relations (7), (8) and (9), (10). In the case when

$\varepsilon < h$, the results in Table 8 are about two times better than those in Table 7 which agrees with (13) and (14).

	n							
	32	64	128	256	512	1024	2048	
SU scheme								
$\varepsilon = h^{1.5}$	7.4(−5)	6.1(−6)	1.4(−7)	6.0(−8)	2.3(−8)	6.7(−9)	[illegible].8(−9)	E_n
$\varepsilon = h^2$	7.0(−6)	1.9(−6)	4.8(−7)	1.2(−7)	3.1(−8)	7.8(−9)	1.9([illegible])	E_n
IEMW scheme								
$\varepsilon = h^{1.5}$	1.2(−5)	1.3(−6)	1.2(−7)	5.1(−8)	1.3(−8)	3.4(−9)	8.9(−10)	E_n
$\varepsilon = h^2$	3.4(−6)	9.1(−7)	2.4(−7)	6.1(−8)	1.5(−8)	3.9(−9)	9.7(−10)	E_n

Table 9 (Example 1)

Table 9 contains the results from [**5**] for the SU scheme and ours obtained from the IEMW scheme.

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