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ON NON-QUASIDIAGONAL OPERATORS - II

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Abstract. For any operator T on a Hilbert space, the properties of $\mathrm{qd}(T)$, the modulus of quasidiagonality are studied. The notion of extremely non-quasidiagonal operators is introduced and studied.

An operator T on a Hilbert Space H is said to be quasidiagonal if there exists an increasing sequence $\{P_n\}_{n=1}^{\infty}$ of finite rank orthogonal projections such that $P_n \to I$, the identity operator strongly and $\|TP_n - P_nT\| \to 0$ as $n \to \infty$. The notion of quasidiagonality was introduced 1970 by Halmos [**4**]. Herrero [**5**] defined the notion of modulus of quasidiagonality $\mathrm{qd}(T)$ of any operator T on H as

$$\mathrm{qd}(T) = \varliminf_{\substack{P \epsilon \mathcal{P}(H) \\ P \to I}} \|TP - PT\| = \varliminf_{M \epsilon \tau(H)} \|TP_M - P_MT\|,$$

where $\mathcal{P}(H)$ denotes the directed set of all finite rank (orthogonal) projections on H under the usual ordering, P_M denotes the projection on the closed linear subspace M of H and $\tau(H)$ is the collection of all finite-dimensional closed lenear subspaces of H. From [**4**, p. 902], it follows that T is quasidiagonal if and only if $\mathrm{qd}(T) = 0$. Herrero [**5**, Theorem 6.13] established that $\mathrm{qd}(T)$ is the distance of T from the class of all quasidiagonal operators on H. The purpose of the present paper is to study the properties of $\mathrm{qd}(T)$ and also to introduce and study the notion of extremely non-quasidiagonal operators.

Throughout the paper H denotes an infinite-dimensional complex separable Hilbert space and $\mathcal{B}(H)$, the set of all bounded linear operators on H. $\mathcal{K}(H)$ denotes the ideal of compact operators on H, and π the natural mapping of $\mathcal{B}(H)$ onto the quotient algebra $\mathcal{B}(H)/\mathcal{K}(H)$. The null space and the range of an operator T are denoted by $N(T)$ and $R(T)$ respectively.

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1. THEOREM 1.1. *Let T be in $\mathcal{B}(H)$ with $\|T\| = 1$, and for $0 \leq \lambda \leq 1$, let $E(\lambda)$ denote the spectral projection for $(T^*T)^{1/2}$ that corresponds to the interval $[0, \lambda]$. Then:*

(i) *If $0 \leq \lambda_0 < 1/3$ and dim $R(E(\lambda_0)) = \aleph_0$, the cardinality of the set of natural numbers, then* $\mathrm{qd}(T) \leq (3 - \lambda_0)/4$.

(ii) *If $1/3 \leq \lambda_0 < 1$ and dim $R(E(\lambda_0)) = \aleph_0$, then* $\mathrm{qd}(T) \leq (1 + \lambda_0)/2$.

Proof. (i) Let $T = UA$ be the polar decomposition of T, where U is a partial isometry and A a positive operator. Since $E(\lambda_0)$ reduces A, we have $A = A_1 \oplus A_2$, with $A_1 \in \mathcal{B}((R(E(\lambda_0)))^\perp)$ and $A_2 \in \mathcal{B}(R(E(\lambda_0)))$. Also both A_1 and A_2 are positive operators. The spectral theorem implies that $\|A_2\| \leq \lambda_0$ and $\lambda_0 \leq A_1 \leq 1$.

Denote $V = U(I - E(\lambda_0))$; then V^*V is a projection and hence V is a partial isometry. Also $R(E(\lambda_0))$ is contained in $N(V)$, and since the space H is separable, $\dim R(E(\lambda_0)) = \aleph_0$ implies that $\dim N(V) = N_0$, and therefore by [**2**, Theorem 5 (i)], we have $\mathrm{qd}(V) \leq 1/2$. Now

$$\begin{aligned}
\mathrm{qd}(T) &= \mathrm{qd}(V(1 + \lambda_0)/2 + UA - V(1 + \lambda_0)/2) \\
&= \mathrm{qd}(V(1 + \lambda_0)/2 + U(A - (1 + \lambda_0)/2 \cdot (I - E(\lambda_0)))) \\
&\leq \mathrm{qd}(V(1 + \lambda_0)/2 + \|A - (1 + \lambda_0)/2 \cdot (I - E(\lambda_0))\| \\
&\leq (1 + \lambda_0)/4 + \|(A_1 - (1 + \lambda_0)/2) \oplus A_2\|.
\end{aligned}$$

Since

$$\|A_1 - (1 + \lambda_0)/2\| \leq \mathrm{Sup}_{\lambda_0 \leq \lambda \leq 1} \mid \lambda - (1 + \lambda_0)/2 \mid = (1 - \lambda_0)/2,$$

and $\|A_2\| \leq \lambda_0 \leq (1 - \lambda_0)/2$, we have $\mathrm{qd}(T) \leq (1 + \lambda_0)/4 + (1 - \lambda_0)/2 = (3 - \lambda_0)/4$.

(ii) Proceeding exactly as above, we have

$$\begin{aligned}
\mathrm{qd}(T) \leq&\ \mathrm{qd}((1 - \lambda_0)V) + \|A - (1 - \lambda_0)(I - E(\lambda_0))\| \\
\leq&\ (1 - \lambda_0)/2 + \|(A_1 - (1 - \lambda_0)) \oplus A_2\|.
\end{aligned}$$

Since $\|A_1 - (1 - \lambda_0)\| \leq \mathrm{Sup}_{\lambda_0 \leq \lambda \leq 1} \mid \lambda - (1 - \lambda_0) \mid \leq \lambda_0$ and $\|A_2\| \leq \lambda_0$, we have $\mathrm{qd}(T) \leq (1 - \lambda_0)/2 + \lambda_0 = (1 + \lambda_0)/2$.

COROLLARY 1.2. *Let T be in $\mathcal{B}(H)$ with $\|T\| = 1$, and for $0 \leq \lambda \leq 1$, let $E(\lambda)$ denote the spectral projection for $(T^*T)^{1/2}$ that corresponds to the interval $[0, \lambda]$. Then, if* $\mathrm{qd}(T) > 2/3$, *then there exists $\lambda > 1 - \mathrm{qd}(T)$ such that* $\dim R(E(\lambda)) < \aleph_0$.

Proof. Assume on the contrary that for each $\lambda > 1 - \mathrm{qd}(T)$, $\dim R(E(\lambda)) = \aleph_0$. Since $\mathrm{qd}(T) > 2/3$, we have $1 - \mathrm{qd}(T) < 1/3$, and therefore $\dim R(E(1/3)) = \aleph_0$. Theorem 1.1 (i) implies that $\mathrm{qd}(T) \leq (1 + 1/3)/2 = 2/3$, which is impossible. Hence we have the result.

The following example shows that 2/3 is the best possible lower bound in Corollary 1.2.

Example 1.3. Let U be the unilateral shift of multiplicity one on H, and let $T = U \oplus -1/3$. An easy computation shows that $T - 1/3$ is bounded below by $2/3$. Also since the point spectrum of U^* is the interior of the unit disc, it follows that $\dim N((T-1/3)^*) \neq 0$. Therefore [**1**, Theorem 1.1] implies that $\mathrm{qd}(T) = \mathrm{qd}(T-1/3) \geq 2/3$. Now, when $\mathrm{qd}(T) = 2/3$, we have $1 - \mathrm{qd}(T) = 1/3$. Since $(T^*T)^{1/2} = I \oplus 1/3$, it follows that $\dim R(E(1/3)) = \aleph_0$. Hence for each $\lambda > 1/3$, $\dim R(E(\lambda)) = \aleph_0$. Since $\|T\| = 1$, this example shows that Corollary 1.2 cannot be extended beyond those operators for which $\mathrm{qd}(T) > 2/3\|T\|$.

Remark. If $0 < \epsilon \leq 2/3$, then we can see that there exists a Fredholm operator T_ϵ in $\mathcal{B}(H \oplus H)$ such that $\|T_\epsilon\| = 1$, the index of T_ϵ is negative and $\mathrm{qd}(T_\epsilon) = \epsilon$. For example, if V is the unilateral shift of multiplicity one in $\mathcal{B}(H)$, then we may let T_ϵ to be the operator in $\mathcal{B}(H \oplus H)$ whose matrix is given by

$$T_\epsilon = \begin{bmatrix} 0 & V \\ \epsilon & 0 \end{bmatrix}$$

Also, if $1/2 < \epsilon \leq 2/3$, then since $(T_\epsilon^* T_\epsilon)^{1/2} = \epsilon \oplus I$, there exists $\lambda > 1 - \mathrm{qd}(T_\epsilon)$ such that $\dim R(E(\lambda)) < \aleph_0$. This proves that the converse of Corollary 1.2 is false.

2. In general for any operator T on H, $0 \leq \mathrm{qd}(T) \leq \|T\|$. Motivated by this in this section we introduce and study the extremal case of non-quasidiagonality.

Definition. An operator T on H is called extremely non-quasidiagonal if there exists $M \in \tau(H)$ such that for any $N \supset M$, $N \in \tau(H)$, the equality

$$\|TP_N - P_N T\| = \|T\|$$

holds.

Example 2.1. The unilateral shift of multiplicity one is extremely non-quasidiagonal. In fact, every nonunitary isometry is extremely non-quasidiagonal.

Remark. In [**2**, Theorem 7] we have proved that, if T is an operator on H with finite-dimensional null space, then the following are equivalent:

(i) $\mathrm{qd}(T) = \|\pi(T)\|$

(ii) $T = \|\pi(T)\|V + K$, where V is a non-unitary isometry and K is compact.

Hence, if T is extremely non-quasidiagonal operator with finite-dimensional null-space, then $\mathrm{qd}(T) = \|T\| \geq \|\pi(T)\|$. Also since by [**2**, Theorem 1], we have $\mathrm{qd}(T) \leq \|\pi(T)\|$, it follows that $\mathrm{qd}(T) = \|\pi(T)\|$. Therefore [**2**, Theorem 7] implies that $T = \|T\|V + K$, with V a non-unitary isometry and K compact. We show here that K can be chosen to be of finite rank.

THEOREM 2.2. *Let $T \in \mathcal{B}(H)$ be extremely non-quasidiagonal with $\|T\| = 1$, and let $M \in \tau(H)$ such that $\|TP_N - P_N T\| = 1$, for every $N \in \tau(H)$, $N \supset M$. Then we have $R(I - T^*T) \in \tau(H)$ and $\dim R(I - T^*T) < \dim M$.*

Proof. Let $\{e_j\}_{j=1}^{n}$ be an orthonormal basis in M such that $\{e_j\}_{j=1}^{p}$ is a basis of the subspace $M_0 = \{x \in M : \|(I - P_M)Tx\| = \|x\|\}$. By the hypothesis we can find a unit vector x in M such that $\|x\| = \|(TP_M - P_M T)x\| = \|(I - P_M)Tx\|$, and therefore $M_0 \neq \{0\}$. Also,

$$M_0 \subset \{x \in H : \|Tx\| = \|x\|\} = N(I - T^*T).$$

Therefore, if any x belongs to $(N(I-T^*T))^\perp$, then $\langle e_j, x\rangle = 0$, $1 \le j \le p$. Set $k = \dim R(I-T^*T)$, $n = \dim M$, and assume that $k \ge n$. We can find an orthonormal system $\{f_j\}_{j=1}^p \subset (N(I-T^*T))^\perp$ such that $\langle f_j, e_i\rangle = 0$ for $1 \le j \le p$, $p < i$. As f_j belongs to $(N(I-T^*T))^\perp$, it follows that $\langle f_j, e_i\rangle = 0$ for $1 \le j \le p$, $1 \le i \le n$. Denote $g_j(\lambda) = Te_j + \lambda f_j$, $1 \le j \le p$, and if $L(\lambda)$ is the subspace generated by M and $\{g_j(\lambda)\}_{j=1}^{\;p}$, then $L(\lambda) \in \tau(H)$ and also by the hypothesis we can find $x(\lambda)$ in $L(\lambda)$ with $\|x(\lambda)\| = 1$ and

$$\|(I - P_{L(\lambda)})Tx(\lambda)\| = \|x(\lambda)\| = 1.$$

This implies that $\|Tx(\lambda)\| = 1$, and hence $x(\lambda)$ belongs to

$$N(I-T^*T), \quad P_{L(\lambda)}Tx(\lambda) = 0.$$

Write

$$x(\lambda) = \sum_{j=1}^n \alpha_j e_j + \sum_{j=1}^p \beta_j g_j(\lambda),$$

and choose λ such that

$$\det[\langle Te_i, f_j\rangle + \lambda\delta_{ij}]_{i,j=1}^p \neq 0,$$
$$\det[\bar\lambda\langle Te_i, f_j\rangle + \delta_{ij}]_{i,j=1}^p \neq 0,$$

then we first obtain that

$$\sum_{i=1}^p \beta_i(\langle Te_i, f_j\rangle + \lambda\delta_{ij}) = \langle x(\lambda), f_j\rangle = 0, 1 \le j \le p.$$

This implies that $\beta_i = 0$, $1 \le i \le p$, and hence $x(\lambda)$ belongs to $M \cap N(I-T^*T)$. Since $L(\lambda)^\perp \subset M^\perp$, we can get even more: that $x(\lambda)$ also belongs to M_0. Therefore, $x(\lambda) = \sum_{i=1}^p \alpha_i e_i$. Now using the relation $P_{L(\lambda)}Tx(\lambda) = 0$, we obtain that

$$\sum_{i=1}^p \alpha_i(\bar\lambda\langle Te_i, f_j\rangle + \delta_{ij}) = \langle Tx(\lambda), q_j(\lambda)\rangle = 0, \quad 1 \le j \le p.$$

Thus, $\alpha_i = 0$, $1 \le i \le p$, that is $x(\lambda) = 0$, which is a contradiction to the fact that $\|(I-P_{L(\lambda)})Tx(\lambda)\| = 1 = \|x(\lambda)\|$. Therefore our assumption that $k \ge n$ is wrong, and hence $k < n$.

THEOREM 2.3. *Let T be in $\mathcal{B}(H)$ with $\|T\| = 1$. Then the following are equivalent*

(i) *T is extremely non-quasidiagonal,* (ii) $\dim R(I-T^*T) < \dim R(I-TT^*)$,

(iii) *$T = V + K$, with V a non-unitary isometry and K a finite rank operator.*

Proof. Let $T = UA$ be the polar decomposition of T and let E be the spectral measure of A. Then we have

$$\begin{aligned} M =& R(I-T^*T) = N(T) + R(E((0,1))), \\ N =& R(I-TT^*) = N(T^*) + R(UE((0,1))U^*) \end{aligned}$$

$$\dim R(E((0,1))) = \dim R(UE((0,1))U^*) = \gamma \quad \text{(say)}.$$

(i) $\Rightarrow$ (ii) If T is extremely non-quasidiagonal operator, we have by Theorem 2.2 that $\gamma \leq \dim M < \aleph_0$ and since $\mathrm{qd}(T) = \|T\| > \|T\|/2$, [**2**, Theorem 2] implies that T is not invertible. Therefore, using the fact that $\dim N(T) < \aleph_0$ and T is non-invertible, we get that $\dim N(T) < \dim N(T^*)$. Thus,

$$\dim M = \dim N(T) + \gamma < \dim N(T^*) + \gamma = \dim N.$$

(ii) $\Rightarrow$ (iii) If $\dim M < \dim N$, we have $\gamma \leq \dim M < \aleph_0$, whence

$$\dim N(U) = \dim N(T) < \dim N(T^*) = \dim N(U^*).$$

Let U_0 be a finite rank partial isometry such that $R(U_0^* U_0) = N(U)$ and $R(U_0 U_0^*) \subset N(U^*)$. Then $V = U + U_0$ is a non-unitary isometry.

Also, $T = VA = V + V(I - A) = V + K$, with $K = V(I - A)$. But

$$\begin{aligned}\dim R(K) &= \dim R(I - A) = \dim R((I + A)(I - A)) \\ &= \dim R(I - A^2) = \dim R(I - T^*T) < \aleph_0.\end{aligned}$$

Thus K has finite rank.

(iii) $\Rightarrow$ (i) Let $T = V + K$, with V a non-unitary isometry and K a finite rank operator. Choose $e \in N(V^*)$, $e \neq 0$ and denote by M the subspace generated by e and $R(K)$. Then we obtain that $M \in \tau(H)$ and also

$$\|(V + K)P_N - P_N(V + K)\| = \|VP_N - P_N V\| = 1,$$

for every $N \in \tau(H)$, $N \supset M$ (see [**3**, Theorem 3]). This shows that T is extremely non-quasidiagonal operator.

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