PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 54 (68), 1993, 55–57

ITERATION OF ULTRAPRODUCTS OF LOCALLY CONVEX SPACES

José Antonio Facenda Aguirre

Abstract. Let $\{E_{ij}: (i,j) \in I \times J\}$ be a family of locally convex Hausdorff spaces. We study the relation between the spaces $P_{(i,j)\in I\times J}(E_{ij})$ and $P_{j\in J}(P_{i\in I}(E_{ij}))$. In particular, we prove that the ultraproduct of an ultraproduct is again an ultraproduct.

In the following, $\mathbf{N}$ will denote the set of all positive integers, μ the first uncountable measurable cardinal, $|X|$ the cardinal of the set X, I and J sets of indices such that $|I| \geq \mu$ and $|J| \geq \mu$, $\mathcal{F}$ and $\mathcal{G}$ countably complete ultrafilters on I and J respectively and $\{E_{ij}: (i,j) \in I \times J\}$ a family of locally convex Hausdorff spaces.

An ultrafilter $\mathcal{F}$ is said to be countably complete if for each sequence $\{A_n\}_{n\in\mathbf{N}} \subset \mathcal{F}$ such that $\bigcup_{n\in\mathbf{N}} A_n \in \mathcal{F}$ there exists $m \in \mathbf{N}$ such that $A_m \in \mathcal{F}$ (or equivalently, if $A_n \notin \mathcal{F}$, $\forall n \in \mathbf{N}$ then $\bigcup_{n\in\mathbf{N}} A_n \notin \mathcal{F}$).

In [**2**] the ultraproduct is defined of a family $\{E_i: i \in I\}$ of locally convex Hausdorff spaces such that $|I| \geq \mu$, with respect to a countably complete ultrafilter $\mathcal{F}$ on I, whose construction we recall again for the sake of completeness: consider the equivalence relation in the product space $\prod_{i\in I} E_i$ given by:

$$\{x_i: i \in I\} =_{\mathcal{F}} \{y_i: i \in I\} \iff \{i \in I: x_i = y_i\} \in \mathcal{F}$$

Let p_i be an arbitrary continuous seminorm on the space E_i, $(i \in I)$, and $[\{x_i: i \in I\}]$ the equivalence classes of the quotient space. We call the ultraproduct of the family $\{E_i: i \in I\}$ with respect to the ultrafilter $\mathcal{F}$ the above mentioned quotient space endowed with the topology defined by the family of seminorms:

$$\bar{p}([\{x_i: i \in I\}]) = \lim_{\mathcal{F}} p_i(x_i)$$

This limit always exists because $I = \bigcup_{n\in\mathbf{N}} \{i \in I: p_i(x_i) \in [n, n+1]\} \in \mathcal{F}$, and since $\mathcal{F}$ is countably complete there exists $m \in \mathbf{N}$ such that $\{i \in I: p_i(x_i) \in$

AMS Subject Classification (1980): Primary 46 A 05, Secondary 46 Q 05

$[m, m+1]\} \in \mathcal{F}$. Furthermore, it is easy to prove that $\bar{p}$ is a seminorm. We denote the ultraproduct space by $P_{i\in I}(E_i)$. Since E_i, $i \in I$, are Hausdorff spaces, we obtain that the ultraproduct space is also a Hausdorff space. If all the spaces E_i ($i \in I$) are equal to a certain space E, then we refer to their ultraproduct with respect to $\mathcal{F}$ as to the ultrapower of E with respect to $\mathcal{F}$ which we shall denote by $P_{\mathcal{F}}(E)$.

Finally, we denote by $\mathcal{F} \times \mathcal{G}$, the family of all subsets $K \subset I \times J$ such that

$$\{j \in J : \{i \in I : (i,j) \in K\} \in \mathcal{F}\} \in \mathcal{G}$$

It is an ultrafilter on $I \times J$ [**1**, page 156]. We prove that it is countably complete. Let $\{A_n\}_{n\in\mathbf{N}} \subset \mathcal{F} \times \mathcal{G}$ be an arbitrary sequence. It follows that $\{j \in J : \{i \in I : (i,j) \in A_n\} \in \mathcal{F}\} \in \mathcal{G}$, for all $n \in \mathbf{N}$. Since $\{i \in I : (i,j) \in \bigcup_{n\in\mathbf{N}} A_n\} = \bigcup_{n\in\mathbf{N}}\{i \in I : (i,j) \in A_n\} \in \mathcal{F}$, because $\mathcal{F}$ is countably complete, then

$$\{j \in J : \{i \in I : (i,j) \in \bigcup_{n\in\mathbf{N}} A_n\} \in \mathcal{F}\} = \bigcup_{n\in\mathbf{N}} \{j \in J : \{i \in I : (i,j) \in A_n\} \in \mathcal{F}\},$$

which belongs to $\mathcal{G}$, because it is countably complete.

Now, we prove the iteration theorem:

Theorem. *Let $\{E_{ij} : (i,j) \in I \times J\}$ be a family of locally convex Hausdorff spaces, $|I| \geq \mu$, $|J| \geq \mu$ and let $\mathcal{F}$, $\mathcal{G}$ be countably complete ultrafilters on I and J respectively. Then the ultraproduct spaces $P_{(i,j)\in I\times J}(E_{ij})$ and $P_{j\in J}\left(P_{i\in I}(E_{ij})\right)$ are isomorphic.*

Proof. Let $\tilde{x} = [\{x_{ij} : (i,j) \in I \times J\}] \in P_{(i,j)\in I\times J}(E_{ij})$ and let $\{x_{ij} : (i,j) \in I \times J\}$ be an element of this class. Denote by $\pi_j(\tilde{x})$ the class in the space $P_{i\in I}(E_{ij})$, corresponding to the element $\{x_{ij} : i \in I\} \in \prod_{i\in I} E_{ij}$, and by $[\pi_j(\tilde{x})]$ the class in the space $P_{j\in J}\left(P_{i\in I}(E_{ij})\right)$ associated to the element $\{\pi_j(\tilde{x}) : j \in J\} \in \prod_{j\in J}\left(P_{i\in I}(E_{ij})\right)$.

We prove that the mapping

$$\pi : P_{(i,j)\in I\times J}(E_{ij}) \longrightarrow P_{j\in J}\left(P_{i\in I}(E_{ij})\right); \qquad \pi(\tilde{x}) = [\pi_j(\tilde{x})]$$

is the desired isomorphism.

Clearly π is linear and onto. π is one to one: let $\pi(\tilde{x})$ be the null class of the ultraproduct $P_{j\in J}\left(P_{i\in I}(E_{ij})\right)$. So $\{j \in J : \pi_j(\tilde{x}) = 0\} \in \mathcal{G}$, that is to say,

$$\{j \in J : \{i \in I : x_{ij} = 0\} \in \mathcal{F}\} \in \mathcal{G} \tag{1}$$

If we call $K = \{(i,j) \in I \times J : x_{ij} = 0\}$, then, by (1), $K \in \mathcal{F} \times \mathcal{G}$ and so $\tilde{x} = 0$.

π is bicontinuous: the proof of the continuity of π and π^{-1} is similar, we only see that of π. Let $\bar{p} = \lim_{\mathcal{G}} \bar{p}_j$ be a continuous seminorm on $P_{j\in J}\left(P_{i\in I}(E_{ij})\right)$, where $\bar{p}_j = \lim_{\mathcal{F}} p_{ij}$. Let $[\pi_j(\tilde{x})]$ be an arbitrary element of the iterated ultraproduct, $\bar{q} = \lim_{\mathcal{F}\times\mathcal{G}} p_{ij}$ continuous seminorm on $P_{(i,j)\in I\times J}(E_{ij})$ and $\alpha = \lim_{\mathcal{F}\times\mathcal{G}} p_{ij}(x_{ij})$. This means that

$$\forall \varepsilon > 0, \quad K_\varepsilon = \{(i,j) \in I \times J : |p_{ij}(x_{ij}) - \alpha| < \varepsilon\} \in \mathcal{F} \times \mathcal{G}$$

Therefore, for all $j \in J_0 \in \mathcal{G}$, we have $\{i \in I: |p_{ij}(x_{ij}) - \alpha| < \varepsilon\} \in \mathcal{F}$, where $J_0 = \{j \in J: \{i \in I: (i,j) \in K_\varepsilon\} \in \mathcal{F}\}$.

Then $|\bar{p}_j([\{x_{ij}: i \in I]\}) - \alpha| \leq \varepsilon$, $\forall j \in J_0$. It follows that $\lim_{\mathcal{G}} \lim_{\mathcal{F}} p_{ij}(x_{ij}) = \alpha$. So $\bar{p}(\pi(\tilde{x})) \leq \bar{q}(\tilde{x})$ and π is continuous, as required.

COROLLARY. *Let $\mathcal{F}$, $\mathcal{G}$ be countably complete ultrafilters on I and J respectively and let $\{E_i: i \in I\}$ be a family of locally convex Hausdorff spaces. Then the ultrapower $P_{\mathcal{G}}(P_{i\in I}(E_i))$ of the ultraproduct $P_{i\in I}(E_i)$ is isomorphic to the ultraproduct $P_{(i,j)\in I\times J}(E_{ij})$ where $E_{ij} = E_i$, for all $j \in J$. In particular the ultrapower of an ultrapower is itself an ultrapower.*

Proof. For each $j \in J$, $P_{i\in I}(E_{ij})$ is isomorphic to $P_{i\in I}(E_i)$.

Remarks. 1. In fact we have proved that for all $\bar{p}$, continuous seminorn on the iterated ultraproduct $P_{j\in J}(P_{i\in I}(E_{ij}))$, there exists $\bar{q}$, continuous seminorm on $P_{(i,j)\in I\times J}(E_{ij})$, such that $\bar{p}(\pi(\tilde{x})) = \bar{q}(\tilde{x})$ (and reciprocally).

2. A result of this type is known within the field of normed spaces ([**3**, page 101], [**4**, Proposition 2.1]).

REFERENCES

[1] W. W. Comfort, S. Negrepontis, *The Theory of Ultrafilters.* Springer-Verlag, Berlin, 1974.

[2] J. A. Facenda Aguirre, *Ultraproducts of locally convex spaces*, Rev. Roumaine Math. Pures Appl. **32** (1987), 99-106.

[3] B. Sims, *"Ultra"-techniques in Banach space theory"*, Queen's Papers in Pure and Applied Mathematics no. **60** (1982), Queen's University.

[4] J. Stern, *Ultrapowers and local properties of Banach spaces*, Trans. Amer. Math. Soc. **240** (1978), 231-252.

Facultad de Matemáticas
Dpto. de Análisis Matemático
Apdo. de Correos 1160
41080 Sevilla, SPAIN

(Received 25 10 1993)
(Revised 29 03 1993)