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LOGICS WITH TWO TYPES OF INTEGRAL OPERATORS

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Abstract. We prove completeness theorems for absolutely continuous and singular biprobability models of a logic with integrals. Also in both cases, we prove the finite compactness theorem for a set of sentences of the form $\tau \in [r,s]$.

We assume throughout the paper that $\mathcal{A}$ is a countable admissible set with $\omega \in \mathcal{A}$. In [**2**], Keisler introduced a logic $L_{\mathcal{A}\int}$ which has an integral operator which builds terms with bound variables. In our case two types of integral operators $\int_1 \dots dx$ and $\int_2 \dots dx$ are allowed.

A biprobability model for $L_{\mathcal{A}\int_1\int_2}$ logic is a model $\mathfrak{A} = \langle A, R_i, c_j, \mu_1, \mu_2\rangle_{i\in I, j\in J}$, where $\langle A, R_i, c_j\rangle$ is a first-order model without operations and μ_1, μ_2 are probability measures on A. We shall see a difference in semantics for $L^a_{\mathcal{A}\int_1\int_2}$ and $L^s_{\mathcal{A}\int_1\int_2}$ by means of the following definition.

Definition 1. (a) An absolutely continuous biprobability model for $L^a_{\mathcal{A}\int_1\int_2}$ is a biprobability model $\mathfrak{A}$ such that μ_1 is absolutely continuous with respect to μ_2, i. e. $\mu_1 \ll \mu_2$.

(b) A singular biprobability model for $L^s_{\mathcal{A}\int_1\int_2}$ is a biprobability model $\mathfrak{A}$ such that μ_1 is singular with respect to μ_2, i. e. $\mu_1 \perp \mu_2$. $\square$

In both cases, quantifiers are interpreted by

$$\left(\textstyle\int_k \tau(x,\vec{a})\,dx\right)^{\mathfrak{A}} = \int \tau(b,\vec{a})^{\mathfrak{A}}\,d\mu_k(b) \qquad \text{for } k = 1,2\ ,$$

where $\tau(x,\vec{y})$ is a term and $\vec{a} \in A^n$.

Diagonal products $\mu_k^{(n)}$, which are the coresponding restrictions of completions of μ_k^n's $(k = 1,2)$ to σ–algebras generated by the measurable rectangles and the diagonal sets $\{\,\vec{x} \in A^n : x_i = x_j\,\}$, can be replaced by sequences of probability measures on A^n's which satisfy the Fubini theorem. That generalization of a probability structure is relevant for us.

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Definition 2. A graded biprobability model for $L_{\mathcal{A}\int_1\int_2}$ is a model $\mathfrak{A} = \langle A, R_i, c_j, \mu_n^1, \mu_n^2\rangle_{i\in I,\, j\in J,\, n\geq 1}$ such that:

(1) Each μ_n^k is a countably additive probability measure on A^n.
(2) Each n-ary relation R_i is μ_n^k-measurable and the identity relation is μ_2^k-measurable.
(3) $\mu_n^k \times \mu_m^k \subseteq \mu_{n+m}^k$.
(4) Each μ_n^k is preserved under permutation of $\{1, 2, \ldots, n\}$.
(5) $\langle \mu_n^k : n \in \mathbb{N}\rangle$ has the Fubini property: If B is μ_{m+n}^k-measurable, then
  (*a*) For each $\vec{x} \in A^m$, the section $B_{\vec{x}} = \{\vec{y} : B(\vec{x}, \vec{y})\}$ is μ_n^k-measurable.
  (*b*) The function $f(\vec{x}) = \mu_n^k(B_{\vec{x}})$ is μ_m^k-measurable.
  (*c*) $\int f(\vec{x})\, d\mu_m^k = \mu_{m+n}^k(B)$. □

Definition 3. (a) A graded biprobability model for $L^a_{\mathcal{A}\int_1\int_2}$ is a graded biprobability model $\mathfrak{A}$ such that $\mu_n^1 \ll \mu_n^2$ for each $n \in \mathbb{N}$.

(b) A graded biprobability model for $L^s_{\mathcal{A}\int_1\int_2}$ is a graded biprobability model $\mathfrak{A}$ such that $\mu_n^1 \perp \mu_n^2$ for each $n \in \mathbb{N}$. □

1. The logic $L^a_{\mathcal{A}\int_1\int_2}$. Axioms and rules of inference for $L^a_{\mathcal{A}\int_1\int_2}$ are those for $L_{\mathcal{A}\int}$, as listed in [**3**] with both $\int_1$ and $\int_2$ playing the role of $\int$, together with the following axioms:

(A_1) *Axioms of continuity of integral operators*: $(i, j = 1, 2)$

(a) $\bigwedge_n \bigvee_m \bigvee_k \int_i \mathbb{F}_k \left(\int_j \tau(\vec{x}, \vec{y})\, d\vec{x}\right) d\vec{y} < \frac{1}{n}$,

$$\text{where}\quad F_k(s) = \begin{cases} 1, & \text{if}\quad r - 1/m + 1/k \leq s \leq r - 2/k \\ 0, & \text{if}\quad s \leq r - 1/m \quad \text{or} \quad s \geq r - 1/k \\ \text{linear}, & \text{for other cases} \end{cases}$$

is a continuous real function such that $F_k \restriction \mathbb{Q} \in \mathcal{A}$.

(b) $\bigwedge_n \bigvee_m \bigvee_k \int_i \mathbb{G}_k \left(\int_j \tau(\vec{x}, \vec{y})\, d\vec{x}\right) d\vec{y} < \frac{1}{n}$,

$$\text{where}\quad G_k(s) = \begin{cases} 1, & \text{if}\quad r + 2/k \leq s \leq r + 1/m - 1/k \\ 0, & \text{if}\quad s \leq r + 1/k \quad \text{or} \quad s \geq r + 1/m \\ \text{linear}, & \text{for other cases.} \end{cases}$$

(A_2) *Axiom of absolute continuity*:

$\bigwedge_{\varepsilon\in\mathbb{Q}_+} \bigvee_{\delta\in\mathbb{Q}_+} \bigwedge_n \bigwedge_{\tau\in T_n} \left(|\int_2 \tau(\vec{x})\, d\vec{x}| < \delta \implies |\int_1 \tau(\vec{x})\, d\vec{x}| < \varepsilon\right)$,

where $T = \bigcup_n T_n$, T_n is a set of terms with n free variables and $T, T_n \in \mathcal{A}$.

(A_3) $\int_1 \left(\int_2 \tau\, dy\right) dx = \int_2 \left(\int_1 \tau\, dx\right) dy$.

Now we introduce two sorts of auxiliary models.

Definition 4. (a) A weak model for $L^a_{\int_1\int_2}$ is a model $\langle \mathfrak{A}, I_1, I_2\rangle$ where $\mathfrak{A}$ is a first-order model and I_k is what may be called an $\mathcal{A}$-Daniell integral on A, that is, I_k is a positive linear real function on the set of terms with at most one free variable x and parameters from A, i.e.

$$I_k(r) = r, \qquad k = 1,2$$
$$I_k(r\cdot\sigma + s\cdot\tau) = r\cdot I_k(\sigma) + s\cdot I_k(\tau),$$
$$\text{if } \tau(b,\vec{a})^{\mathfrak{A}} \geq 0 \quad \text{for all } b\in A, \text{ then } I_k(\tau(x,\vec{a})) \geq 0.$$

(b) A middle model for $L^a_{\mathcal{A}\int_1\int_2}$ is a weak model $\mathfrak{A}$ such that for each $\varepsilon > 0$ there is $\delta > 0$ such that for each term $\tau(x,\vec{y})$ and $\vec{a}\in A^n$, if $|I_2(\tau(x,\vec{a}))| < \delta$ then $|I_1(\tau(x,\vec{a}))| < \varepsilon$. □

In both cases, for τ a term, define $\tau^{\mathfrak{A}}$ inductively as for biprobability models, except that at the integral step, we define

$$\left(\textstyle\int_k \tau(x,\vec{a})\,dx\right)^{\mathfrak{A}} = I_k(\tau(x,\vec{a}))\ .$$

Lemma 1. (Middle Completeness Theorem for $L^a_{\mathcal{A}\int_1\int_2}$) *Let T be a set of sentences of $L^a_{\mathcal{A}\int_1\int_2}$ such that T is Σ_1-definable over $\mathcal{A}$. Then T is consistent with the axioms of this logic iff it has a middle model in which each theorem of $L^a_{\mathcal{A}\int_1\int_2}$ is true.*

Proof. The soundness is easy to prove because all the axioms represent known properties of integrals (the Generalized Radon-Nikodym Theorem and the Fubini Theorem prove that each function $\tau(x,y)^{\mathfrak{A}}\colon A\times A\to\mathbb{R}$ is compatible with absolutely continuous measures μ_1 and μ_2, i.e.

$$\iint \tau(x,y)^{\mathfrak{A}}\,d\mu_1(x)\,d\mu_2(y) = \iint \tau(x,y)^{\mathfrak{A}}\,d\mu_2(y)\,d\mu_1(x)\ ).$$

A Henkin argument is used to construct a weak model $\langle\mathfrak{A}, I_1, I_2\rangle$ of T in which each theorem of $L^a_{\mathcal{A}\int_1\int_2}$ is true. Let $K = L\cup C$ be the language introduced in this construction, where C is a set of new constant symbols and $C\in\mathcal{A}$. We wish the axiom A_2 to hold for all the terms and that is done by the following construction (see [**9**]).

Let K' be a language with four kinds of variables: $X, Y, Z, \ldots$ are variables for sets, $x, y, z, \ldots$ are variables for urelements, $r, s, t, \ldots$ are variables for reals from $[0,1]\cap\mathcal{A}$, and $U, V, W, \ldots$ are variables for functions $A^n\mapsto\mathbb{R}$, $n\geq 0$. Predicates are: $E^s_n(\vec{x}, X)$ for sets, $n\geq 1$; $E^t_{n+1}(\vec{x}, r, U)$ for terms, $n\geq 0$; $I_k(U,r)$ for $U\colon A^0\to\mathbb{R}$ or $U\colon A^1\to\mathbb{R}$, $k=1,2$; and $\leq$ for reals. Function symbols are $f, g, h, \ldots$ for each continuous real functions $F\colon\mathbb{R}^n\to\mathbb{R}$ such that $F\restriction\mathbb{Q}^n\in\mathcal{A}$. Constant symbols are: X_φ for each formula φ; U_τ for each term τ; and $\overline{r}$ for each real number $r\in[0,1]\cap\mathcal{A}$.

Let S be the following theory of $K'_{\mathcal{A}}$:

1. *Axioms of validity*:

1.1 $(\forall X)\bigwedge_{n<m}\neg(\exists\vec{x},\vec{y})(E^s_m(\vec{x},\vec{y},X)\wedge E^s_n(\vec{x},X))$, where $\{\,\vec{x}\,\}\cap\{\,\vec{y}\,\}=\emptyset$;

1.2 $(\forall U)\bigwedge_{n<m}\neg(\exists\vec{x},\vec{y},r,s)(E^t_{m+1}(\vec{x},\vec{y},r,U)\wedge E^t_{n+1}(\vec{x},s,U))$;

1.3 $(\forall U)(\forall\vec{x},r,s)((E^t_{n+1}(\vec{x},r,U)\wedge E^t_{n+1}(\vec{x},s,U)) \implies r=s)$;

2. *Axioms of extensionality*:

2.1 $(\forall\vec{x})(E_n^s(\vec{x},X) \iff E_n^s(\vec{x},Y)) \iff X=Y$;

2.2 $(\forall\vec{x},r)(E_{n+1}^t(\vec{x},r,U) \iff E_{n+1}^t(\vec{x},r,V)) \iff U=V$;

3. *Axioms of terms*:

3.1 $(\forall\vec{x})(E_{n+1}^t(\vec{x},0,U_\tau) \vee E_{n+1}^t(\vec{x},1,U_\tau))$ if τ is $\mathbf{1}(R(\vec{x}))$;

3.2 $(\forall x,y)(E_{2+1}^t(x,y,0,U_\tau) \vee E_{2+1}^t(x,y,1,U_\tau))$ if τ is $\mathbf{1}(x=y)$;

3.3 $E_{0+1}^t(\overline{r},U_\tau)$ if τ is r;

3.4 $(\forall\vec{x},r)(E_{n+1}^t(\vec{x},r,U_\tau) \iff (\exists\vec{s})(\bigwedge\limits_{i=1}^{k} E_{n+1}^t(\vec{x},s_i,U_{\tau_i}) \wedge$
$\wedge\, f(s_1,\ldots,s_k)=r))$ if τ is $\mathbb{F}(\tau_1,\ldots,\tau_k)$;

3.5 $(\forall\vec{x},r)(E_{n+1}^t(\vec{x},r,U_\tau) \iff (\exists V)((\forall y,s)(E_{1+1}^t(y,s,V) \iff$
$\iff E_{n+1+1}^t(\vec{x},y,s,U_\sigma)) \wedge I_k(V,r)))$ if τ is $\int_k \sigma(\vec{v},v_0)\,dv_0$, $k=1,2$;

4. *Axioms of satisfaction*:

4.1 $(\forall\vec{x})(E_n^s(\vec{x},X_\varphi) \iff (\exists r\geq 0)E_{n+1}^t(\vec{x},r,U_\tau))$ if φ is $\tau\geq 0$;

4.2 $(\forall\vec{x})(E_n^s(\vec{x},X_{\neg\varphi}) \iff \neg E_n^s(\vec{x},X_\varphi))$;

4.3 $(\forall\vec{x})(E_n^s(\vec{x},X_{\wedge\Phi}) \iff \bigwedge\limits_{\varphi\in\Phi} E_n^s(\vec{x},X_\varphi))$;

5. *Axioms of integral operators*:

5.1 $(\forall U)((\bigwedge\limits_{n\geq 2} \neg(\exists\vec{x},r)E_{n+1}^t(\vec{x},r,U)) \iff (\exists_1 s)I_k(U,s))$, $k=1,2$;

5.2 $(\forall r)I_k(U_r,r)$;

5.3 $(\forall U,V,r,s)I_k(r\cdot U+s\cdot V)=r\cdot I_k(U)+s\cdot I_k(V)$, where $I_k(U)=r$ iff $I_k(U,r)$;

5.4 $(\forall U)((\forall x)(\exists r\geq 0)E_{1+1}^t(x,r,U) \implies (\exists s\geq 0)I_k(U,s))$;

6. *Axiom of absolute continuity*:

$(\forall\varepsilon>0)(\exists\delta>0)(\forall U)\,(|I_2(U)|<\delta \implies |I_1(U)|<\varepsilon)$;

7. *Axioms for an Archimedean field*;

8. *Transformations of axioms of* $K^a_{\mathcal{A}\int_1\int_2}$:

$(\forall\vec{x})E_n^s(\vec{x},X_\varphi)$, where φ is an axiom of this logic;

9. *Axioms of realizability of all sentences* φ *of* T :

$(\forall x_0)E_1^s(x_0,X_\varphi)$.

A weak model $\langle\mathfrak{A},I_1,I_2\rangle$ for $K^a_{\mathcal{A}\int_1\int_2}$ can be transformed to a standard model $\mathfrak{B}$ for $K'_{\mathcal{A}}$ by taking: $B_\varphi^{\mathfrak{B}}=\{\,\vec{a}\in A^n:\mathfrak{A}\models\varphi[\vec{a}]\,\}$, $U_\tau^{\mathfrak{B}}(\vec{a})=\tau^{\mathfrak{A}}(\vec{a})$ for $\vec{a}\in A^n$ and $I_k^{\mathfrak{B}}(U_\tau^{\mathfrak{B}})=I_k(\tau)$ for each term τ with at most one free variable. By the Barwise Compactness Theorem (see [**1**]), it can be shown that S has a standard model $\mathfrak{D}$, because S is Σ-definable over $\mathfrak{A}$ and A_2 holds in $\mathfrak{A}$. $\mathfrak{D}$ can be transformed to a middle model $\mathfrak{C}$ of T by taking:

$$R^{\mathfrak{C}}=\{\,\vec{x}\in D^n: E_n^s(\vec{x},X_{\mathbf{1}(R(\vec{x}))=1})\,\} \quad \text{and}$$

$$I_k^{\mathfrak{C}}(\tau(x,\vec{a}))=I_k^{\mathfrak{D}}(U_{\tau(x,\vec{a})}) \quad \text{for } \vec{a}\in D^n \text{ and } k=1,2.$$

This completes the proof of the Middle Completeness Theorem. $\square$

In order to construct an absolutely continuous biprobability model, we need the following lemma.

LEMMA 2. (Loeb [**4**]) *In an ω_1-saturated nonstandard universe, let M be an internal vector lattice of functions from an internal set A into ${}^*\mathbb{R}$ (the set of hyperreal numbers), and let I be an internal positive linear functional on M, such that $\mathbf{1} \in M$ and $I(\mathbf{1}) = 1$. Then there is a complete probability measure μ on A such that for each finitely bounded $\varphi \in M$, the standard part of φ is integrable with respect to μ and its integral is equal to the standard part of $I(\varphi)$.*

THEOREM 1. (Completeness Theorem for $L^a_{\mathcal{A}\int_1\int_2}$) *Let T be a set of sentences of $L^a_{\mathcal{A}\int_1\int_2}$ such that T is Σ_1 on $\mathcal{A}$ and consistent. Then there is an absolutely continuous biprobability model of T.*

Proof. Let $\langle \mathfrak{A}, I_1, I_2\rangle$ be a middle model of T in which each theorem of $L^a_{\mathcal{A}\int_1\int_2}$ is true. The Daniell integral construction from Lemma 2 produces probability measures μ_1, μ_2 on ${}^*A$ such that for each $*$-term $\tau(x)$, the standard part of ${}^*I_k(\tau)$ is the integral $\int \mathrm{st}(\tau(b)^{\mathfrak{A}})\, d\mu_k(b)$ (we define measures μ^k_n on ${}^*A^n$ by using iterated integrals). The absolute continuity in the middle model $\mathfrak{A}$ implies the absolute continuity for all measurable sets. Also, using axiom A_3, it can be shown that $\mu^1_n \ll \mu^2_n$ for each $n \in \mathbb{N}$. This graded biprobability model $\hat{\mathfrak{A}} = \langle {}^*\mathfrak{A}, \mu^1_n, \mu^2_n\rangle$ can be used to produce an absolutely continuous biprobability model of T (see [3]). $\square$

We can look only for a part of $L^a_{\mathcal{A}\int_1\int_2}$ which satisfies the finite compactness property, because this logic cannot satisfy the full compactness (for example, each finite subset of $T = \{\int_1 \mathbf{1}(R(x))\, dx > 0\} \cup \{\int_1 \mathbf{1}(R(x))\, dx \leq \frac{1}{n} : n \in \mathbb{N}\}$, where R is a unary predicate, has a probability model, but not T itself).

THEOREM 2. *Let T be a set of sentences of $L^a_{\mathcal{A}\int_1\int_2}$ of the form $\tau \in [r,s]$. If every finite subset of T has a graded biprobability model, then T has a graded biprobability model.*

Proof. Let us suppose that each finite subset $\Psi \subseteq T$ has a model $\mathfrak{A}_\Psi$. By Lemma 1 we can suppose that $\mathfrak{A}_\Psi$ is a middle model. Take an ultraproduct ${}^*\mathfrak{A}$ such that, for each $\varphi \in T$, almost every $\mathfrak{A}_\Psi$ satisfies φ. Then form a graded biprobability model $\hat{\mathfrak{A}}$ from ${}^*\mathfrak{A}$ by the Daniell integral construction (Lemma 2). It can be shown by induction that every sentence of $L^a_{\mathcal{A}\int_1\int_2}$ of the form $\tau \in [r,s]$ which is true in almost all $\mathfrak{A}_\Psi$ holds in $\hat{\mathfrak{A}}$, too. The absolute continuity condition can be expressed in the middle model by the first-order sentence

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall U)(|I_2(U)| < \delta \implies |I_1(U)| < \varepsilon)\ .$$

By Łos's Theorem and Loeb construction the sentence

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall X)(\mu_2(X) < \delta \implies \mu_1(X) < \varepsilon)$$

holds in $\hat{\mathfrak{A}}$. $\square$

2. The logic $L^s_{\mathcal{A}\int_1\int_2}$. Axioms and rules of inference for the logic $L^s_{\mathcal{A}\int_1\int_2}$ are those of $L_{\mathcal{A}\int}$ (with both $\int_1$ and $\int_2$ in place of $\int$, see [**3**]) together with the axioms of continuity A_1 and

(A_4) *Axiom of singularity*:

$$\bigvee_k \int_i \mathbb{H}_k(\textstyle\int_1 \mathbf{1}(x=y)\,dy, \int_2 \mathbf{1}(x=y)\,dy)\,dx = 0, \quad i=1,2,$$

where $H_k(s,t) = \begin{cases} 1, & \text{if } s \geq \frac{2}{k} \text{ and } t \geq \frac{2}{k} \\ 0, & \text{if } s \leq \frac{1}{k} \text{ or } t \leq \frac{1}{k} \\ \text{linear}, & \text{for other cases} \end{cases}.$

THEOREM 3. (Completeness Theorem for $L^s_{\mathcal{A}\int_1\int_2}$) *A theory T of $L^s_{\mathcal{A}\int_1\int_2}$ is consistent iff T has a singular biprobability model.*

Proof. The proof of soundness is easy. Let $\langle\mathfrak{A}, I_1, I_2\rangle$ be a weak model of T in which each theorem of $L^s_{\mathcal{A}\int_1\int_2}$ is true. Let $\mathcal{F} = \{B \subseteq A : \chi_B \in \operatorname{dom}(I_1) = \operatorname{dom}(I_2)\}$ be an algebra of subsets of A, where $\chi_B(x) = \begin{cases} 1, & \text{if } x \in B \\ 0, & \text{if } x \notin B \end{cases}$. Define finitely additive probability measures ν_1, ν_2 on $\mathcal{F}$ by $\nu_k(B) = I_k(\chi_B)$, $B \in \mathcal{F}$ and $k = 1,2$.

Then, for $a \in A$, the singleton $\{a\}$ belongs to $\mathcal{F}$ because $\chi_{\{a\}} = (x=a)^{\mathfrak{A}}$, the set $B = \{a \in A : \nu_1\{a\} > 0 \ \ \& \ \ \nu_2\{a\} > 0\}$ belongs to $\mathcal{F}$ and $\nu_k(B) = 0$ $k = 1,2$ by A_4.

By construction from [**7**], the measures ν_1, ν_2 can be extended so that $\nu_1 \subseteq \overline{\nu}_1$, $\nu_2 \subseteq \overline{\nu}_2$ and the measures $\overline{\nu}_1, \overline{\nu}_2$ are singular. Then construct a middle biprobability model $\langle\mathfrak{A}, \overline{I}_1, \overline{I}_2\rangle$ of T by

$$\operatorname{dom}(\overline{I}_k) = \operatorname{dom}(I_k) \cup \{\chi_C : C \in \overline{\mathcal{F}} \smallsetminus \mathcal{F}\} \qquad \text{and} \qquad \overline{I}_k(\chi_C) = \overline{\nu}_k(C),$$

for each C from the extension $\overline{\mathcal{F}}$ of $\mathcal{F}$.

By Loeb's construction (Lemma 2) and the construction of the biprobability model from a graded biprobability model (see [**3**]), the singularity of finitely additive measures in the middle model will be preserved in the biprobability model. $\square$

Finally, we prove Finite Compactness Theorem for the singular case.

THEOREM 4. *Let T be a set of sentences of $L^s_{\mathcal{A}\int_1\int_2}$ of the form $\tau \in [r,s]$. If every finite subset of T has a graded biprobability model, then T has a graded biprobability model.*

Proof. As in Theorem 2, our proof is based on the ultraproduct and Daniell integral construction. Now, we can suppose that $\mathfrak{A}_\Psi$ is a weak model for each finite subset $\Psi \subseteq T$. Let $\overline{\mathfrak{A}}_\Psi$ be a middle model as in Theorem 3. Take an ultraproducts ${}^*\mathfrak{A} = \prod \overline{\mathfrak{A}}_\Psi$ such that, for each $\varphi \in T$, almost every $\overline{\mathfrak{A}}_\Psi$ satisfies φ. The condition of singularity can be express in the middle model by the first-order sentence $(\exists f)(\overline{I}_1(f) = 1 \wedge \overline{I}_2 = 0)$. By Los's Theorem and Loeb's construction the sentence $(\exists X)(\mu_1(X) = 1 \wedge \mu_2(X) = 0)$ holds in $\hat{\mathfrak{A}}$. $\square$

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