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THE SPECTRAL APPROXIMATION FOR THE SHOCK LAYER PROBLEMS

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Abstract. The paper is concerned with a singularly perturbed boundary value problem, having single turning point. The solution is represented as a sum of the reduced solution and the layer function, which is approximated by the truncated orthogonal series. The domain of the layer function is obtained by the use of numerical layer length, depending on the perturbation parameter and degree of the spectral approximation. The approximate error function, constructed upon the layer subinterval, provides the error estimate. A numerical example is included.

1. Introduction. This paper will be concerned with the singularly perturbed boundary value problem

$$Ly(x) \equiv -\varepsilon^2 y''(x) - 2xp(x)y'(x) + p(x)q(x)y(x) = 0, \quad x \in [-1,1] \tag{1.1}$$

$$y(-1) = A, \qquad y(1) = B, \tag{1.2}$$

where $\varepsilon > 0$ is a small parameter, $p(x)$, $q(x) \in C^2[-1,1]$, $A, B \in R$ and $p(x) > 0$ for $x \in [-1,1]$. It is well known that the solution of the problem (1.1), (1.2) has a single turning point at $x = 0$. The asymptotic behavior of the solution is given in [**1**] by the following theorem:

THEOREM 1. *Let $p(x) > 0$ for all $x \in [-1,1]$ and $q(0) \neq -2n$, $n = 1,2,\dots$. Then, for the solution of the problem* (1.1), (1.2) *it holds*

$$y(x) = \begin{cases} y_L(x) + O(\varepsilon^2) & -1 \le x < 0 \\ O(\varepsilon^k),\ k = q(0)/2 & x = 0 \\ y_D(x) + O(\varepsilon^2), & 0 < x \le 1, \end{cases} \tag{1.3}$$

where $y_L(x)$ is the left solution of the reduced problem

$$-2xp(x)y_L'(x) + p(x)q(x)y_L(x) = 0, \quad y_L(-1) = A, \tag{1.4}$$

and $y_D(x)$ is the right solution of the reduced problem

$$-2xp(x)y_D'(x) + p(x)q(x)y_D(x) = 0, \quad y_D(1) = B. \tag{1.5}$$

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A proof of this theorem is given in [**1**].

From (1.3) we can conclude the following: a) If $q(0) > 0$, then $y(0) \to 0$ when $\varepsilon \to 0$ and $y_L(0) = y_D(0) = y(0)$. This case is known as "corner layer" problem. b) If $q(0) < 0$, then $y(0) \to +\infty$. c) If $q(0) = 0$ and $y(-1) \neq y(1)$, than we have nonuniform convergence and

$$y(0) \to (y(-1) + y(1))/2 = (y_L(0) + y_D(0))/2. \tag{1.6}$$

This case is known as the "shock layer" problem.

Our aim is to construct the approximate solution for the problem (1.1), (1.2) in the case c), that is with the additional assumptions $A \neq B$ and $q(0) = 0$, by using spectral methods.

Transformation of the problem. We are going to search for the solution of the problem (1.1), (1.2) in the form

$$y(x) = \begin{cases} y_L(x) + u(x), & x \in [-1, 0] \\ y_D(x) + v(x), & x \in [0, 1], \end{cases} \tag{2.1}$$

where $y_L(x)$ is left reduced solution, obtained from (1.4), and $u(x)$ satisfies the differential equation

$$\begin{aligned} Lu(x) &\equiv -\varepsilon^2 u''(x) - 2xp(x)u'(x) + p(x)q(x)u(x) \\ &= \varepsilon^2 y_L''(x), \end{aligned} \tag{2.2}$$

$x \in [-1, 0]$ and the boundary conditions

$$\begin{aligned} u(-1) &= y(-1) - y_L(-1) = 0, \\ u(0) = A^0 &= y(0) - y_L(0) = (y_D(0) - y_L(0))/2. \end{aligned} \tag{2.3}$$

(For the right-hand boundary conditions we have used (1.6).) In the same way, $y_D(x)$ is right reduced solution, obtained from (1.5), and $v(x)$ satisfies the differential equation

$$\begin{aligned} Lv(x) &\equiv -\varepsilon^2 v''(x) - 2xp(x)v'(x) + p(x)q(x)v(x) \\ &= \varepsilon^2 y_D''(x), \end{aligned} \tag{2.4}$$

$x \in [0, 1]$ and the boundary conditions

$$v(0) = B^0 = (y_L(0) - y_D(0))/2, \;\; v(1) = y(1) - y_D(1) = 0. \tag{2.5}$$

The first step is to approximate functions $u(x)$ and $v(x)$ by

$$\widetilde{u}(x) = \begin{cases} 0 & x \in [-1, -\delta] \\ w(x) & x \in [-\delta, 0] \end{cases} \qquad \widetilde{v}(x) = \begin{cases} z(x) & x \in [0, \delta] \\ 0 & x \in [\delta, 1] \end{cases} \tag{2.6}$$

where $\delta > 0$ is so called "numerical layer lenght", which will be determined later.

All further investigations will be carried out for the interval $[0,1]$, and the investigations for $x \in [-1,0]$ are similar. The function $z(x)$ represents a solution of the boundary value problem

$$Lz(x) \equiv \; -\varepsilon^2 z''(x) - 2xp(x)z'(x) + p(x)q(x)z(x) = \varepsilon^2 y_D''(x), \quad x \in [0,\delta] \tag{2.7}$$

$$z(0) = B^0, \;\; z(\delta) = 0. \tag{2.8}$$

Our aim is to approximate the layer function $z(x)$ by a truncated orthogonal series according to some orthogonal polynomial basis defined on $[-1,1]$. Intending to construct the spectral approximation for $z(x)$ we must, at first, transform the iterval $[0,\delta]$ into $[-1,1]$ using the substitution $x = \delta(t+1)/2$. Thus (2.7), (2.8) become

$$L_\delta Z(t) \equiv -\mu^2 Z''(t) - 2P(t)Z'(t) + R(t)S(t)Z(t) = G(t), \quad t \in [-1,1] \tag{2.9}$$

$$Z(-1) = B^0, \;\; Z(1) = 0, \tag{2.10}$$

where we denoted

$$\begin{aligned} &Z(t) = z(\delta(t+1)/2), \quad \mu = 2\varepsilon/\delta, \quad P(t) = (t+1)R(t), \\ &R(t) = p(\delta(t+1)/2), \quad S(t) = q(\delta(t+1)/2), \\ &G(t) = \varepsilon^2 y_D''(\delta(t+1)/2). \end{aligned} \tag{2.11}$$

3. Orthogonal projecting. Let σ_n denote the projecting operator such that

$$\sigma_n : \; Z(t) \to Z_n(t) = \sum_{k=0}^{n} a_k Q_k(t), \tag{3.1}$$

where $\{Q_k, \; k = 0, \dots, n\}$ are classical orthogonal polynomials on $[-1,1]$ with respect to the weight function $p(t) = (1-t)^r(1+t)^s, \; r = B(1)/2, \; s = -B(-1)/2$. Here, $B(t)$, is linear function in the differential equation

$$(1-t)^2 Q_k''(t) + B(t)Q_k'(t) + \lambda_k Q_k(t) = 0, \; \lambda_k = k(k-1-B'(0))$$

which determines $Q_k(t)$. The inner product and the norm are defined by

$$(f,g) = \int_{-1}^{1} f(t)g(t)p(t)dt, \quad \|f\|^2 = (f,f).$$

It is well known that all classical orthogonal polynomials satisfy Bonnet's recurrent relation

$$\begin{aligned} &Q_{k+1}(x) - (\alpha_k x + \beta_k)Q_k(x) + \gamma_k Q_{k-1}(x) = 0, \quad k = 1,2,\dots, \\ &Q_0(x) = 1, \quad Q_1(x) = x, \end{aligned} \tag{3.2}$$

where α_k, β_k and γ_k are constants depending on the chosen basis.

So, when speaking of the spectral approximation for the solution $Z(t)$ of the problem (2.9), (2.10), we, in fact, want to find its orthogonal projection in terms of the definition (3.1). If we denote by P^n the space of all real polynomials of degree up to n, then we ask for $Z_n(t) \in P^n$, such that

$$Z_n(-1) = B^0, \quad Z_n(1) = 0, \quad \text{and} \tag{3.3}$$

$$(-\mu^2 Z_n'' - 2PZ_n' + RSZ_n, W) = (G, W) \quad \text{for each } W \in P^{n-2} \tag{3.4}$$

when we use τ - method, or

$$\begin{aligned} &-\mu^2 Z_n''(t_i) - 2P(t_i)Z_n'(t_i) + R(t_i)S(t_i)Z_n(t_i) = G(t_i), \\ &t_i = \cos i\pi/n, \quad i = 1, \dots, n-1 \end{aligned} \tag{3.5}$$

when we use collocation method.

4. Numerical layer length. Numerical layer length has to be determined in such a way that it depends on the caracter of the boundary layer, i.e. parameter ε and on the chosen spectral approximation, i.e. degree n. So we are going to determine the layer interval $[0, \delta]$ upon which the unknown solution, in the neighbourhood of the layer point $x = 0$, resembles some n-th degree parabola that satisfies given boundary conditions. This leeds to the following:

Definition 1. The resemblance function for the "shock layer" problem, upon the subinterval $[0, \delta]$ is $p_n(x) \in P^n$, such that 1° $p_n(x)$ satisfies given boundary conditions. 2° $x = \delta$ is a stationary point for $p_n(x)$. 3° $p_n(x)$ is concave for $y_L(0) > y_D(0)$, and convex for $y_L(0) < y_D(0)$.

LEMMA 1. *The n-th degree parabola*

$$p_n(x) = a(\delta - x)^n/\delta^n, \;\; a = (y_L(0) - y_D(0))/2, \quad n \geq 2 \tag{4.1}$$

is the ressemblance function for the problem (2.7), (2.8).

Proof. We have to verify the conditions in Definition 1.

1° $p_n(0) = a = (y_L(0) - y_D(0))/2 = B^0 = z(0)$, by using (2.5) and (2.8), and $p_n(\delta) = 0 = z(\delta)$.

2° $p_n'(x) = -\frac{an}{\delta}\left(\frac{\delta - x}{\delta}\right)^{n-1}$, $n \geq 2$, which gives $p_n'(\delta) = 0$.

3° From $p_n''(x) = \frac{an(n-1)}{\delta^2}\left(\frac{\delta - x}{\delta}\right)^{n-2}$, $n \geq 2$, $x \in [0, \delta]$ we can see that sgn $p_n''(x) =$ sgn $p_n(x) =$ sgn a, which is positive if $y_L(0) > y_D(0)$, and negative if $y_L(0) < y_D(0)$.

Definition 2. The numerical layer length $\delta = \delta(n, \varepsilon)$ is a positive number for which the resemblance function satisfies given differential equation in the neighbourhood of the layer point.

THEOREM 2. *The numerical layer lenght for the problem* (2.7), (2.8) *is*

$$\delta = \varepsilon \sqrt{\frac{an(n-1)}{2ap(0) - \varepsilon^2 y_D''(0)}} \tag{4.2}$$

Proof. Substituing $p_n(x)$, determined by (4.1), into (2.7) we obtain

$$-\varepsilon^2\frac{an(n-1)}{\delta^2}\Big(\frac{\delta-x}{\delta}\Big)^{n-2}+2xp(x)\frac{an}{\delta}\Big(\frac{\delta-x}{\delta}\Big)^{n-1}+p(x)q(x)a\Big(\frac{\delta-x}{\delta}\Big)^{n}=\varepsilon^2y_D''(x).$$

At the neighbourhood point $x=\delta/n$ of the layer point $x=0$ this will give

$$-\varepsilon^2\frac{an(n-1)}{\delta^2}\Big(1-\frac{1}{n}\Big)^{n-2}+2p\Big(\frac{\delta}{n}\Big)a\Big(1-\frac{1}{n}\Big)^{n-1}+p\Big(\frac{\delta}{n}\Big)q\Big(\frac{\delta}{n}\Big)a\Big(1-\frac{1}{n}\Big)^{n}=\varepsilon^2y_D''\Big(\frac{\delta}{n}\Big).$$

For sufficiently large n we have: $p(\delta/n)\sim p(0)$, $q(\delta/n)\sim q(0)=0$ and $(1-1/n)^{n-i}\sim 1$ for $i=0,1,2$, so $-\varepsilon^2an(n-1)/\delta^2+2p(0)a=\varepsilon^2y_D''(0)$ which gives (4.2). As $p(x)>0$ and $\varepsilon>0$ is sufficiently small, the existence of the square root in (4.2) is always provided.

5. Construction of the spectral approximation When we want to find the appriximate solution on the "shock layer" problem using spectral approximation, at the first step, by the use of Theorem 2, we have to determine the numerical layer length, and, then, to approximate the given problem by using (2.1) - (2.8), so that we finally come to the boundary layer problem (2.9), (2.10). The third step is to determine the coefficients $a_k,\ k=0,\dots,n$ of its spectral solution

$$Z_n(t)=\sum_{k=0}^{n}a_kQ_k(t). \tag{5.1}$$

Remark 1. As the original interval $[0,\delta]$ for the problem (2.7), (2.8) is small, it is sufficient of approximate the coefficients in the differential equation (2.7) by constants obtained for $x=\delta/2$. The error of such an approximation is of order $O(\delta)$, which does not effects significantly the accuracy of the practical calculations. We shall, also, approximate the right-hand side of the differential equation (2.9) by the appropriate orthogonal series

$$G(t)\approx\sum_{k=0}^{n}g_kQ_k(t). \tag{5.2}$$

THEOREM 3. *The coefficients of the spectral solution of the problem* (2.9), (2.10), *obtained using τ-method, represent the solution of the system*

$$\sum_{k=0}^{n}a_kQ_k(-1)=B^0,\quad \sum_{k=0}^{n}a_kQ_k(1)=0 \tag{5.3}$$

$$(-\mu^2c_j+P(0)b_j)\sum_{k=0}^{n}a_k+R(0)'S(0)a_j=q_j,\quad j=0,\dots,n-2, \tag{5.4}$$

where, μ, $P(t)$, $R(t)$ and $S(t)$ are determined by (2.11) *and c_j and b_j are coefficients in the orthogonal representations*

$$Q_k'(t)=\sum_{i=0}^{k-1}b_iQ_i(t),\quad Q_k''(t)=\sum_{i=0}^{k-2}c_iQ_i(t). \tag{5.5}$$

Proof. The derivatives $Q_k'(t)$ and $Q_k''(t)$, being the polynomials of $n-1$ and $n-2$ degree, can be represented exactly as linear combinations of the elements of the chosen orthogonal basis by (5.5). Introducing (5.1) and (5.2) into (3.3) and (3.4), according to Remark 1, we obtain

$$\begin{aligned}-\mu^2\sum_{k=0}^{n} a_k(Q_k'', W) - 2P(0)\sum_{k=0}^{n} a_k(Q_k', W) + R(0)S(0)\sum_{k=0}^{n} a_k(Q_k, W)\\ = \sum_{k=0}^{n} g_k(Q_k, W).\end{aligned}$$

If we make use of (5.5) and choose $W = Q_j$, $j = 0, \dots, n-2$, we shall have

$$\begin{aligned}&-\mu^2\sum_{k=0}^{n} a_k\sum_{i=1}^{k-2} c_i(Q_i, Q_j) - 2P(0)\sum_{k=0}^{n} a_k\sum_{i=1}^{k-1} b_i(Q_i, Q_j)\\ &+ R(0)S(0)\sum_{k=0}^{n} a_k(Q_k, Q_j) = \sum_{k=0}^{n} q_k(Q_k, Q_j), \quad j = 0, 1, \dots, n-2.\end{aligned}$$

According to the orthogonality relation $(Q_i, Q_j) = \|Q_j\|^2\delta_{i,j}$, where $\delta_{i,j}$ is Kronecker δ-symbol, we finaly get

$$-\mu^2\sum_{k=0}^{n} a_k c_j\|Q_j\|^2 - 2P(0)\sum_{k=0}^{n} a_k b_j\|Q_j\|^2 + R(0)S(0)a_j\|Q_j\|^2 = g_j\|Q_j\|^2$$

that can be written down as (5.4). The two more equations (5.3) are obtained directly from the boundary conditions by using (3.3).

THEOREM 4. *The coefficients of the spectral solution of the problem* (2.9), (2.10), *obtained by using collocation method, represent the solution of the system*

$$\sum_{k=0}^{n} f_{k,i}a_k = h_i, \quad i = 0, \dots, n, \tag{5.6}$$

with

$$f_{k,0} = Q_k(-1),\ \ f_{k,n} = Q_k(1),\ \ k = 0, \dots, n,\ \ h_0 = B^0,\ \ h_n = 0 \tag{5.7}$$

$$\begin{aligned}f_{k,i} =& -\mu^2 Q_k''(t_i) - 2P(t_i)Q_k'(t_i) + R(t_i)S(t_i)Q_k(t_i),\\ h_i =& G(t_i), \quad t_i = \cos i\pi/n, \quad i = 1, \dots, n-1.\end{aligned} \tag{5.8}$$

Proof. The coefficients (5.7) of the first and the last equation of the system (5.6) are obtained directly from (3.3), and the coefficients (5.8) of the other $n-1$ equations from (3.5).

Remark 2. The values for $Q_k(t_i)$ and its derivatives are evaluated recurrently by the use of Bonnet's relation (3.2).

Remark 3. In the same way we can find the spectral approximation for $w(x)$, $x \in [-\delta, 0]$. In this way we have constructed the approximate solution to the problem (1.1), (1.2) in the form

$$y_n(x) = \begin{cases} y_L(x) + w_n(x), & x \in [-\delta, 0] \\ y_D(x) + z_n(x), & x \in [0, \delta], \end{cases} \tag{5.9}$$

with

$$z_n(x) = z_n(\delta(t+1)/2) = Z_n(t). \tag{5.10}$$

6. The error estimate. Out of the boundary layer, according to (2.1) and (2.6), using (1.3), we have the error estimates

$$d(x) = |y(x) - y_L(x)| \le C_L \varepsilon^2, \quad x \in [-1, -\delta] \tag{6.1}$$

$$d(x) = |y(x) - y_D(x)| \le C_D \varepsilon^2, \quad x \in [\delta, 1]. \tag{6.2}$$

In order to estimate the error upon the layer subinterval $[0, \delta]$ we have to start from

$$d(x) = |y(x) - y_n(x)| = |v(x) - z_n(x)| \le |v(x) - z(x)| + |z(x) - z_n(x)| \tag{6.3}$$

and prove the following lemma:

LEMMA 2. *For the solutions $v(x)$ and $z(x)$ of the problems* (2.4), (2.5) *and* (2.7), (2.8), *with the additional assumption $q(x) \ge 0$ for $x \in (0, \delta]$, we have*

$$d^0(x) = |v(x) - z(x)| \le |C^0| = |y(\delta) - y_D(\delta)|, \quad x \in (0, \delta]. \tag{6.4}$$

Proof. It is well known that the assumption $xp(x) > 0$ for $x \in (0, 1]$, as ε is sufficiently small, garanties inverse monotonicity of the solutions to the problems (2.4), (2.5) and (2.7), (2.8). Starting from

$$\begin{aligned} &Lv(x) \equiv -\varepsilon^2 v''(x) - 2xp(x)v'(x) + p(x)q(x)v(x) = \varepsilon^2 y_D''(x), \quad x \in (0, \delta] \\ &v(0) = B^0, \quad v(\delta) = y(\delta) - y_D(\delta) \end{aligned}$$

and substracting from it (2.7), (2.8), we obtain

$$\begin{aligned} L(v(x) - z(x)) \equiv -\varepsilon^2 (v(x) - z(x))'' &- 2xp(x)(v(x) - z(x))' \\ &+ p(x)q(x)(v(x) - z(x)) = 0 \\ v(0) - z(0) = 0, \quad v(\delta) - z(\delta) &= y(\delta) - y_D(\delta) = C^0 \end{aligned} \tag{6.5}$$

By the principle of inverse monotonicity this gives

$$v(x) - z(x) \ge 0 \text{ if } C^0 \ge 0 \text{ and } v(x) - z(x) \le 0 \text{ if } C^0 \le 0.$$

If $C^0 > 0$, for the function $S(x) = u(x) - v(x) - C^0$, by using (6.5), we get

$$\begin{aligned} &LS(x) \equiv -\varepsilon^2 S''(x) - 2xp(x)S'(x) + p(x)q(x)S(x) = -p(x)q(x)C^0, \\ &S(0) = -C^0, \ \ S(\delta) = 0. \end{aligned}$$

Again, by the principle of inverse monotonicity we conclude that $S(x) \leq 0$, i.e. $v(x) - z(x) \leq C^0$.

If $C^0 < 0$, by using $-S(x)$ instead of $S(x)$, in the same manner we can conclude $v(x) - z(x) \geq C^0$. These two conclusions together give (6.4).

As there is no exact method for the error estimate in the case of spectral approximations, we are going to use an approximate error estimate proposed in [**4**]. It is known that if $n \to \infty$, then the spectral approximation (5.10) tends to the exact solution $z(x)$ of the problem (2.7), (2.8). Thus, it is necessery to increase n until the values for the coefficients a_k, evaluated for $(n-1)$-st and n-th degree of the spectral approximation, become sufficiently close. Then we can suppose that $z_{2n}(x)$ sufficiently well approximates the exact solution and we can write

$$|z(x) - z_n(x)| \approx |z_{2n}(x) - z_n(x)| = d_n(x) \tag{6.6}$$

This finaly gives:

THEOREM 5. *For the error of the approximation* (5.9) *to the solution of the problem* (1.1), (1.2), *upon the layer subinterval* $(0, \delta]$, *the following estimate holds:*

$$d(x) = |y(x) - y_n(x)| \approx C\varepsilon^2 + d_n(x). \tag{6.7}$$

Proof. We shall use the inequality (6.3). The first term is estimated by (6.4), and this, according to (6.2) gives $|v(x) - z(x)| \leq C_D\varepsilon^2$. The second term is estimated by (6.6), which together gives (6.7).

7. Numerical example We shall construct an approximate solution to the problem

$$-\varepsilon^2 y''(x) - xy'(x) = 0, \quad x \in [-1,1]; \quad y(-1) = 0, \quad y(1) = 2,$$

with the exact solution

$$y(x) = 1 + \frac{\operatorname{erf}\, x/(\varepsilon\sqrt{2})}{\operatorname{erf}\, 1/(\varepsilon\sqrt{2})}$$

Left reduced solution is $y_L(x) = 0$, and right reduced solution is $y_D(x) = 2$. The approximate solution upon the layer interval is constructed in the form

$$y_n(x) = \begin{cases} w_n(x), & -\delta < x < 0 \\ 2 + z_n(x), & 0 < x < \delta \end{cases}$$

where $z_n(x)$ is the spectral approximation to the solution of the problem

$$-\varepsilon^2 z''(x) - xz'(x) = 0, \quad x \in [0, \delta], \quad z(0) = -1, \quad z(\delta) = 0.$$

The division point δ, obtained by (4.2) is $\delta = \varepsilon\sqrt{2n(n-1)}$.

In the following tables we give the values of exact error and the error estimate for several points from the interior layer, for $x > 0$. The results for $x < 0$ are the same.

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