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THEOREMS CONCERNING CERTAIN SPECIAL TENSOR FIELDS ON RIEMANNIAN MANIFOLDS AND THEIR APPLICATIONS

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Abstract. Let M be an n-dimensional Riemannian manifold and F a symmetric $(0,2)$-tensor field on M, which satisfies the condition $R\cdot F=0$. Let, additionally, H, A and B be symmetric $(0,2)$-tensor fields on M. If the tensor B commutes with F (cf. (1.3)) and H satisfies the condition $R\cdot H=Q(A,B)$, then

$$(A_{jk}-\frac{\mathrm{tr}(A)}{\mathrm{tr}(B)}B_{jk})(B_{ir}F^r{}_m-\frac{\mathrm{tr}(B,F)}{\mathrm{tr}(B)}B_{im})=0$$

on the open subset of M on which $\mathrm{tr}(B)\neq 0$. It is also proved that, in certain separately Einstein manifolds, null geodesic collineation and projective collineations reduce to motions.

1. Preliminary results. Let M be an n-dimensional Riemannian manifold with not necessarily definite metric g. We denote by g_{ij}, Γ^h_{ij}, R^h_{ijk} and S_{ij} the local components of the metric g, the Levi Civita connection ∇, The Riemann-Christoffel curvature tensor R and the Ricci tensor S of M, respectively.

For $(0,p)$-tensor T with local components $T_{i_1\dots i_p}$, we define $(0,p+2)$-tensor $R\cdot T$ by

$$\begin{aligned}(R\cdot T)_{i_1\dots i_p mk}&=(\nabla_k\nabla_m-\nabla_m\nabla_k)T_{i_1\dots i_p}\\&=-T_{ri_2\dots i_p}R^r{}_{i_1mk}-\dots-T_{i_1\dots i_{p-1}r}R^r_{i_pmk}.\end{aligned}$$

Moreover, for $(0,2)$-tensors A and B with local components A_{ij} and B_{ij} respectively, define $(0,4)$-tensor $Q(A,B)$ by

$$Q(A,B)_{ijkh}=A_{ih}B_{jk}+A_{jh}B_{ik}-A_{ik}B_{jh}-A_{jk}B_{ih}.$$

LEMMA 1.1. *Let F be a symmetric $(0,2)$-tensor field on M satisfying the condition*

$$R\cdot F=0. \tag{1.1}$$

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If H, A and B are symmetric $(0,2)$-tensor fields on M satisfying the relations

$$R \cdot H = Q(A,B), \tag{1.2}$$

$$B \text{ commutes with } F \text{ (i.e., } B_{ir}F^r{}_j = B_{jr}F^r{}_i), \tag{1.3}$$

then

$$\widetilde{a}B_{jk} - \widetilde{b}A_{jk} = aB_{jr}F^r{}_k - bA_{jr}F^r{}_k, \tag{1.4}$$

$$\widetilde{a}b = a\widetilde{b}, \tag{1.5}$$

where $a = \operatorname{tr}(A) = A_{ij}g^{ij}$, $b = \operatorname{tr}(B) = B_{ij}g^{ij}$, $\widetilde{a} = \operatorname{tr}(A,F) = A_{ij}F^{ij}$, $\widetilde{b} = \operatorname{tr}(B,F) = B_{ij}F^{ij}$, $F^r{}_j = F_{sj}g^{rs}$ *and* $F^{ij} = F_{rs}g^{ri}g^{sj}$, *etc.*.

Proof. At first we note that by virtue of Ricci identity the relations (1.1) and (1.2) can be written in the following forms

$$F_{ir}R^r{}_{jmk} + F_{jr}R^r{}_{imk} = 0, \tag{1.6}$$

$$H_{ir}R^r{}_{jkm} + H_{jr}R^r{}_{ikm} = A_{ik}B_{jm} + A_{jk}B_{im} - A_{im}B_{jk} - A_{jm}B_{ik}. \tag{1.7}$$

We remark also that from (1.7) it follows

$$A_{rm}B^r{}_k = A_{rk}B^r{}_m. \tag{1.8}$$

Transvecting (1.6) with H^{ij}, we have $F^{ir}H_{is}R^s{}_{rkm} = 0$. Transvecting again (1.7) with F^{ij} and applying the above equality,

$$A_{mr}F^{rs}B_{sk} = A_{kr}F^{rs}B_{sm}. \tag{1.9}$$

Moreover, we see from (1.1) that $R^h_{ijr}F^r_k$ is antisymmetric with respect to indices j,k. Therefore the following equality holds good

$$(H_{ir}R^r{}_{jks} + H_{jr}R^r{}_{iks})F^s_m = (H_{ir}R^r{}_{jsm} + H_{jr}R^r{}_{ism})F^s_k.$$

Hence by (1.7)

$$\begin{aligned} &A_{ir}F^r{}_mB_{jk} + A_{jr}F^r{}_mB_{ik} - A_{ik}B_{jr}F^r{}_m - A_{jk}B_{ir}F^r{}_m \\ &= A_{im}B_{jr}F^r{}_k + A_{jm}B_{ir}F^r{}_k - A_{ir}F^r{}_kB_{jm} - A_{rj}F^r{}_kB_{im}. \end{aligned} \tag{1.10}$$

Transvecting this with g^{im} and using (1.3), (1.8) and (1.9), we obtain (1.4). Hence (1.5) follows, completing the proof.

THEOREM 1.2. *Let F be a symmetric $(0,2)$-tensor field on M satisfying the conditon* (1.1). *If H, A and B are symmetric $(0,2)$-tensor fields on M satisfying the conditions* (1.2) *and* (1.3), *then*

$$(A_{jk} - \frac{a}{b}B_{jk})(B_{ir}F^r{}_m - \frac{\widetilde{b}}{b}B_{im}) = 0$$

on the open subset of M on which $b \neq 0$.

Proof. From (1.4) it follows that tensor A commutes with F, that is, $A_{ir}F^r{}_j = A_{rj}F^r{}_i$. Moreover, from (1.4) we derive $A_{jr}F^r{}_k = \frac{1}{b}(aB_{jr}F^r{}_k + \widetilde{b}A_{jk} - \widetilde{a}B_{jk})$, which together with (1.5) substituted to (1.10) gives the equality

$$W_{jm}P_{ik} + W_{im}P_{jk} + W_{jk}P_{im} + W_{ik}P_{jm} = 0, \tag{1.11}$$

where $W_{jm} = B_{jr}F^r{}_m - \frac{\widetilde{b}}{b}B_{jm}$ and $P_{ik} = A_{ik} - \frac{a}{b}B_{ik}$. Now the antisymmetrization of (1.11) with respect to the indices m, j yields an equation which compared with (1.11) leads to

$$W_{im}P_{jk} + W_{jk}P_{im} = 0.$$

Hence, it follows that $W_{im}P_{jk} = 0$, completing the proof.

THEOREM 1.3. *Let F be a symmetric $(0,2)$-tensor field on M satisfying the condition $R \cdot F = 0$. Let H, $\widetilde{H}$ and A be symmetric $(0,2)$-tensor fields on M satisfying the relations* (a) $R \cdot H = Q(A,g)$ *and* (b) $R \cdot \widetilde{H} = Q(A,F)$. *Then $A = 0$ at every point of M at which F is nonsingular and nonproportional to the metric g.*

Proof. We restrict our consideration to a point of M at which F is nonsingular and nonproportional to g. As an immediate consequence of Theorem 1.2, we have by (a)

$$A_{ij} = \frac{a}{n}g_{ij}. \tag{1.12}$$

Moreover, from Lemma 1.1 by (a) it follows that

$$\widetilde{a}n - af = 0, \tag{1.13}$$

and by (b) it follows that

$$\widetilde{a}F_{jk} - \widetilde{f}A_{jk} = aF_{jr}F^r{}_k - fA_{jr}F^r{}_k, \tag{1.14}$$

$$\widetilde{a}f - a\widetilde{f} = 0, \tag{1.15}$$

where $f = \mathrm{tr}(F)$, $\widetilde{f} = \mathrm{tr}(F,F)$, $a = \mathrm{tr}(A)$ and $\widetilde{a} = \mathrm{tr}(A,F)$. Because of (1.12) to prove the theorem it is sufficient to show that $a = 0$.

Consider the case $f = 0$. By (1.13), we have $\widetilde{a} = 0$ and by (1.15) $a = 0$ or $\widetilde{f} = 0$. If $\widetilde{f} = 0$, then from (1.14) and nonsingularity of F, we find $a = 0$.

Let now $f \neq 0$. Comparing the relations (1.13) and (1.15), we have $a(f^2 - n\widetilde{f}) = 0$. In the sequal we assume that $a \neq 0$. Then

$$f^2 - n\widetilde{f} = 0. \tag{1.16}$$

Next, in virtue of Theorem 1.2 by (b), we get $(A_{ij} - \frac{a}{f}F_{ij}(F_{mr}F^r{}_n - \frac{\widetilde{f}}{f}F_{mn}) = 0$. This, because of (1.12) and because F is not proportional to g, can be written as

$a(F_{mr}F^r{}_n - \frac{\widetilde{f}}{f}F_{mn}) = 0$. Hence $F_{mr}F^r{}_n - \frac{\widetilde{f}}{f}F_{mn} = 0$. So, by virtue of (1.14) and (1.15), we get $A_{jr}F^r{}_k = \frac{\widetilde{f}}{f}A_{jk}$. The last equation, together with (1.12) and (1.16) implies $a = 0$. This is a contradiction. Therefore $a = 0$. This completes the proof.

Remark. In the above, we have indead proved that, under our assumptions, $A = 0$ at every point of M at which $F_{ir}F^r{}_j \neq 0$ and F is nonpropotional to g.

THEOREM 1.5. *Let F be a symmetric $(0,2)$-tensor field on M satisfying the condition $R \cdot F = 0$. Let H, $\widetilde{H}$ and A be symmetric $(0,2)$-tensor field on M satisfying the relations* (a) $R \cdot H = Q(A, g)$ *and* (b) $R \cdot \widetilde{H} = Q(\widetilde{A}, g)$, *where $\widetilde{A}_{ij} = A_{ir}F^r{}_j$. Then $A = 0$ at every point of M at which F is nonproportional to the metric g.*

Proof. We restrict our considerations to a point of M at wich F is nonproportional to g. At first we note that by (a) and Theorem 1.2 it follows that

$$A_{ij} = \frac{a}{n}g_{ij}, \ \text{ where } \ a = \mathrm{tr}(A). \tag{1.17}$$

Transvecting now (1.17) with F^i_k we see that $\widetilde{A}$ is symmetric. Next, from Theorem 1.2 by (b), we get

$$A_{ir}F^r{}_j = \frac{a}{n}g_{ij}. \tag{1.18}$$

Substituing now (1.17) into (1.18), we get $a = 0$. This, with the help of (1.17), gives our assertion.

2. Applications. In this section we apply the results obtained in the previous section. Let M be a Riemannian manifold with not necessarily definite metric g and of dimension $n > 2$. For vector field v on M, denote by L_v the Lie derivative with respect to v.

A vector field v on M is said to be a motion if $L_v g = 0$, and affine collineation if $L_v\nabla = 0$ [**9**]. A curvature collineation on M is a vector field v which satisfies the condition $L_vR = 0$. An investigation of this transformation was strongly motivated by the important role of the Riemannian curvature tensor in the theory of general relativity [**3**,**4**].

The assertion of the theorem below is quite obvious.

THEOREM A. *In a non-Ricci-flat Einstein manifold a curvature collineation is a motion.*

Let M be a locally product Riemannian manifold in the sense of Tachibana [**8**]. Then, there exists an atlas of separating coordinate neighborhoods $\{(U,(x^i))\}$ such that in each $(U,(x^i))$ the metric g can be written as

$$g = \sum_{a,b=1}^{p} g_{ab}(x^c)dx^a \otimes dx^b + \sum_{\alpha,\beta=1}^{q} g_{\alpha\beta}(x^\gamma)dx^\alpha \otimes dx^\beta, \ \ p+q=n, \ \ 1 \le p \le n-1.$$

Define an $(0,2)$-tensor field on M by

$$[F_{ij}] = \begin{bmatrix} g_{ab} & 0 \\ 0 & -g_{\alpha\beta} \end{bmatrix}$$

in each $(U,(x^i))$. The tensor field $F$is nonsingular, nonproportional to g, symmetric and parallel.

A locally product Riemannian manifold M is called to be a separately Einstein manifold if its Ricci tensor has the following form

$$S = cg + dF, \tag{2.1}$$

where

$$c = \frac{(ns - f\widetilde{s})}{(n^2 - f^2)}, \quad d = \frac{(n\widetilde{s} - fs)}{(n^2 - f^2)}, \quad f = \mathrm{tr}(F) = p - q, \quad s = \mathrm{tr}(S) \text{ and } \widetilde{s} = \mathrm{tr}(S, F).$$

In a separately Einstein manifold M, $c = \text{const}$ and $d = \text{const}$ if $p > 2$ and $q > 2$ (see [**8**]). Note that a separately Einstein manifold is Ricci-flat if and only if $c = d = 0$. In the case $d = 0$, it reduces to an Einstein one.

It has been proved (cf. [**5**]) that

Theorem B. *In a seprately Einstein manifold with c = const, d = const $\neq 0$ and $c^2 \neq d^2$, a curvature collineation is necessarily a motion.*

According to Katzin and Levine [**4**], a vector field v on M is said to be a null geodesic collineation (NGC) if

$$L_v \Gamma^h_{ij} = g^{hr} A_r g_{ij}, \tag{2.2}$$

where $A_r = \nabla_r p$ and p is a function. For such a transformation, we have

$$L_v R^h{}_{ijk} = A^h{}_k g_{ij} - A^h{}_i g_{jk}, \tag{2.3}$$

where $A_{hk} = \nabla_k \nabla_h p$ and $A^h_k = A_{rk} g^{hr}$. If additionaly $A_{hk} = 0$, then the NGC is said to be special. Note also that a special null geodesic collineation is a curvature collineation.

Theorem 2.1. *Let F be a symmetric $(0,2)$-tensor field on a Riemannian manifold M. Assume additionaly that F is nonproportional to the metric tensor g at every point of M and satisfies the condition $R \cdot F = 0$. Then any NGC on M is special.*

Proof. Applying the Lie derivative to the equation $R \cdot F = 0$ and making use of (2.3), we have $R \cdot \widetilde{H} = Q(\widetilde{A}, g)$, where $\widetilde{H} = L_v F$ and tensor $\widetilde{A}$ have the local components $A_{ir} F^r_j$. Similary, applying the Lie derivative to the equation $R \cdot g = 0$ and using (2.3) we find $R \cdot H = Q(A, g)$, where $H = L_v g$. In our situation, Theorem 1.5 yields $A_{ij} = 0$, which completes the proof.

From Theorem 2.1, we get

Theorem 2.2. *In a locally product Riemannian manifold M any NGC is special.*

Combining Theorems A, B and 2.2, we derive

Theorem 2.3. *In a separately Einstein manifold M with c = const, d = const and $c^2 \neq d^2$, an NGC is necessarily a motion.*

Moreover, from Theorem 2.1, for $F = S$ it follows

Theorem 2.4. *If the Ricci tensor S of a Riemannian manifold M satisfies the relation $R \cdot S = 0$ and if S is nonproportional to g at every point of M, then any NGC on M is special.*

A Riemannian manifold is called semisymmetric [**7**] if the condition $R \cdot R = 0$ is satisfied on M.

As an immediate consequence of Theorem 2.4, we get

Theorem 2.5. *In a semisymmetric manifold M with the Ricci tensor S nonproportional to g at every point, any NGC is special.*

A vector field v on a Riemannian manifold is said to be a projective collineation (PC) if

$$L_v \Gamma^h_{ij} = \delta^h_j A_i + \delta^h_i A_j, \tag{2.4}$$

where the 1-form A is defined by $A_j = (n+1)^{-1} \nabla_j (g^{rs} \nabla_r v_s)$. If $A_j = 0$, then the PC is an affine one. It is well-known that for any PC, we have

$$L_v R^h{}_{ijk} = \delta^h_j A_{ik} - \delta^h_k A_{ij}, \tag{2.5}$$

where $A_{ik} = \nabla_k A_i$. Projective collineation is said to be special, if $A_{ij} = 0$. Note also that a special projective collineation is a curvature collineation.

Theorem 2.6. *Let F be a nonsingular, nonproportional to g at every point of M and summetric $(0,2)$-tensor field satisfying the condition $R \cdot F = 0$ on a Riemannian manifold M. Than any PC on M is special.*

Proof. Applying the Lie derivative to the relations $R \cdot g = 0$ and $R \cdot F = 0$ and using of (2.5), we see that a PC satisfies $R \cdot H = Q(A, g)$ and $R \cdot \widetilde{H} = Q(A, F)$, respectively, where $H = L_v g$ and $\widetilde{H} = L_v F$. In view of Theorem 1.3, we obtain $A_{ij} = 0$, which gives our assertion.

From Theorem 2.6, we find

Theorem 2.7. *In a locally product Riemannian manifold M any PC is special.*

Combining Theorems A, B and 2.7, we derive

Theorem 2.8. *In a separately Einstein manifold M with $c = const$, $d = const$ and $c^2 \neq d^2$, any PC is necessarily a motion.*

Moreover, we prove

Theorem 2.9. *Let the Ricci tensor S of a Riemannian manifold M be nonsingular, nonproportional to the metric g at each point and satisfy the relation $R \cdot S = 0$. Then, any PC on M is an affine collineation.*

Proof. From Theorem 2.6, for $F = S$, we have $\nabla_j A_i = 0$. This, by the Ricci identity, leads to $A_r R^r{}_{ijk} = 0$ and also $A_r S^r{}_k = 0$. Since S is nonsingular, $A_i = 0$. This completes the proof.

As an immediate consequence of Theorem 2.9, we get

COROLLARY 2.10. *Let the Ricci tensor S of a semisymmetric manifold M be nonsingular and nonproportional to the metric g at each point of M. Then, any PC on M is an affine collineation.*

For projective collineation in a locally symmetric or Ricci-symmetric manifolds ($\nabla R = 0$ or $\nabla S = 0$, respectively) see Sumitomo [**6**].

The next theorem can be deduced from Theorem 1.4.

THEOREM 2.11. *Let M be a Riemannian manifold whose Ricci tensor S satisfies the condition $R \cdot S = 0$. Assume additionally that, at each point of M, the scalar curvature $s \neq 0$ and S is nonproportional to the metric g. Then, any PC on M is special.*

COROLLARY 2.12. *Let the Ricci tensor S of a semisymmetric manifold M be nonproportional to the metric g and the scalar curvature $s \neq 0$ at each point of M. Then, any PC on M is special.*

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Added in proof: Theorem 1.2 is a generalization of Grycakś theorem from [**2**]. Certain other generalization of his theorem can also be found in [**1**].

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