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CLASSICAL LOGIC WITH SOME PROBABILITY OPERATORS

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Abstract. We introduce a conservative extension of classical predicate (propositional) logic and prove corresponding completeness (and decidability) theorem.

We study conservative extension of classical first-order predicate logic LP (resp. LPP in the propositional case) wich is complete, with respect to "natural" models, and decidable in the propositional case.

Definition 1. The set of all formulas of LP (LPP) logic is the least set X such that:

(i) Each predicate (propositional) formula φ of L is in X, including a contradiction $\perp$, as well.

(ii) If φ is a sentence of predicate logic (a formula of propositional logic), then $P_r(\varphi) \in X$, where $r \in S$ and S is a finite subset of $[0,1]$ wich contains 0 and 1.

(iii) If $A, B \in X$, A and B are not from language of predicate (propositional) logic, then $\neg A$, $A \wedge B$, $A \vee B$, $A \to B \in X$.

Remark. Infinite S does not make big difference, we only need more complicated list of axioms.

Let us denote predicate (propositional) formulas with $\varphi, \psi, \dots$ and LP (LPP) formulas with $A, B, \dots$. Rules of inferences are MP, generalization for formulas of predicate logic (in the LP case) and the following rule for the sentences of predicate logic (formulas of propositional logic):

$$\frac{\varphi}{P_1(\varphi)}$$

The axioms for LP (LPP) are all the axioms of classical predicate (propositional) logic and the following ones:

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1) $P_0(\varphi)$
2) $P_r(\varphi) \to P_s(\varphi), \quad r \geq s$
3) $(P_r(\varphi) \wedge P_s(\psi) \wedge P_1(\neg\varphi \vee \neg\psi)) \to P_{\min\{1,r+s\}}(\varphi \vee \psi)$
4) $(P_{1-r}(\neg\varphi) \wedge P_{1-s}(\neg\psi)) \to P_{\max\{0,1-(r+s)\}}(\neg\varphi \wedge \neg\psi)$
5) $\neg P_{1-r}(\neg\varphi) \Leftrightarrow P_{r^+}(\varphi), \quad$ where $r^+ = \min\{s \in S | s > r\}$ and $r < 1$

The notions of proof, theorem, etc. are defined in the usual way, but we must take care of limited application of our rules.

In the case of LP logic, let $W_L^{\aleph_0}$ be the set of all nonisomorphic models of predicate logic of the language L with the cardinality $\geq \aleph_0$. Let $[\varphi]_W = \{\mathfrak{A} \in W : \mathfrak{A} \models \varphi\}$ be the spectar of φ and $W \subseteq W_L^{\aleph_0}$.

Definition 2. A model of LP logic is a measure space $\mathcal{W} = \langle W, \{[\varphi]_W : \varphi \in \mathrm{Sent}_L\}, \mu\rangle$ where μ is a finite additive measure and $W \subseteq W_L^{\aleph_0}$.

In the case of LPP situation is much simpler. Let $\tau = \{p_1, p_2, \dots\}$ be a set of the propositional letters and $W \subseteq P(\tau)$.

Definition $2'$. A model for LPP logic is a measure space $\mathcal{W} = \langle W, \{[\varphi]_W : \varphi \in \mathrm{For}_\tau\}\rangle$, where μ is a finite additive measure.

Let us note that, for fixed theory T the model change only if we change measure.

We can define the satisfaction relation in the following way.

Definition 3. If φ is a predicate (propositional) formula, then

$$\begin{array}{llll}
 & \mathcal{W} \models \varphi & \text{iff} & (\forall \mathfrak{A} \in W)\, \mathfrak{A} \models \varphi \\
\text{if } Q = P_r(\varphi), \text{ then} & \mathcal{W} \models Q & \text{iff} & \mu\{\mathfrak{A} \in W : \mathfrak{A} \models \varphi\} \geq r, \\
\text{if } C = (A \wedge B), \text{ then} & \mathcal{W} \models C & \text{iff} & \mathcal{W} \models A \text{ and } \mathcal{W} \models B, \\
\text{if } C = \neg A, \text{ then} & \mathcal{W} \models C & \text{iff} & \mathcal{W} \not\models A.
\end{array}$$

We have the followng theorem.

Completeness theorem. *Let T be a set of formulas of LP (LPP) logic. Then, T is consistent iff T has a model.*

Proof. In order to prove the nontrivial part of our theorem, let us suppose that T is a consistent theory and $st(T)$ be the set of all predicate (propositional) consequences of T. Let $A_1, A_2, \dots$ be an enumeration of all formulas of LP (LPP) wich are not from language of predicate (propositional) logic.

Let $\Sigma_0 = st(T) \cup \{P_1(\varphi) : \varphi \in st(T)\} \cup T \subseteq \Sigma_1 \subseteq \Sigma_2 \subseteq \dots$ be a sequence such that

$$\Sigma_{n+1} = \begin{cases} \Sigma_n \cup \{A_n\}, & \text{if } \Sigma_n \cup \{A_n\} \text{is consistent} \\ \Sigma_n \cup \{\neg A_n\}, & \text{otherwise.} \end{cases}$$

It is easy to show that the theory $\Sigma = \bigcup_{n\in\omega} \Sigma_n$ is consistent.

Let $W = \{\mathfrak{A} : \mathfrak{A} \models st(T)\}$ be a universe and let $\mu\{\mathfrak{A} \in W : \mathfrak{A} \models \varphi\} = \max\{r : P_r(\varphi) \in \Sigma\}$ be a finite additive measure of our model. Then we can prove by induction that $\mathcal{W} \models A$ iff $A \in \Sigma$; specialy, $\mathcal{W} \models T$.

Decidability theorem. *The logic LPP is decidable.*

Proof. If a formula φ is propositional, then obviously it is decidable. If formula A is not propositional, then let $p_1, p_2, \dots, p_n$ be a list of all propositional letters occuring in A and let $Q_1, Q_2, \dots, Q_m$ be the list of all formulas of the type $P_r(\varphi_k)$ occuring in A. It is easy to see that A is a propositional combination β of formulas of the type $P_r(\varphi)$ taken as propositional letters.

Let $\bigvee\{(Q_1^{\varepsilon(1)} \wedge \dots \wedge Q_m^{\varepsilon(m)}) :\ \varepsilon \in {}^{\overline{m}}2,\ A(\varepsilon) = \top\}$ be disjunctive normal form of A, where

$$Q_j^{\varepsilon(i)} = \begin{cases} P_{r_j}(\varphi_{k_j}), & \text{if } \varepsilon(i) = 0 \\ \neg P_{r_j}(\varphi_{k_j}), & \text{if } \varepsilon(i) = 1 \end{cases}$$

and $\overline{m} = \{1, \dots, m\}$.

The formula A is not a contradiction iff some formula $Q_1^{\varepsilon(1)} \wedge \dots \wedge Q_m^{\varepsilon(m)}$ is not a contradiction.

For each $Q_j = P_{r_j}(\varphi_{k_j})$, let $\bigvee\{(p_1^{\tau(1)} \wedge \dots \wedge p_n^{\tau(n)} :\ \tau \in {}^{\overline{n}}2,\ \varphi_{k_j}(\tau) = \top\}$ be disjunctive normal form of φ_{k_j}. Then A is not a contradiction iff there is a valuation $\varepsilon \in {}^{\overline{m}}2$ such that $A(\varepsilon) = \top$ and the following system of equations and inequalities

$$\sum_{\tau \in {}^{\overline{n}}2} \mu(p_1^{\tau(1)} \wedge \dots \wedge p_n^{\tau(n)}) = 1$$

$$\mu(p_1^{(1)} \wedge \dots \wedge p_n^{(n)}) \geq 0 \quad \tau \in {}^{\overline{n}}2$$

$$\sum\{p_1^{\tau(1)} \wedge \dots \wedge p_n^{\tau(n)} :\ \tau \in {}^{\overline{n}}2,\ \varphi_{k_1}(\tau) = \top\} \quad \begin{cases} \geq r_1 & \text{if } \varepsilon(1) = 0 \\ < r_1 & \text{if } \varepsilon(1) = 1 \end{cases}$$

$$\cdots\cdots$$

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$$\sum\{p_1^{\tau(1)} \wedge \dots \wedge p_n^{\tau(n)} :\ \tau \in {}^{\overline{n}}2,\ \varphi_{k_m}(\tau) = \top\} \quad \begin{cases} \geq r_m & \text{if } \varepsilon(m) = 0 \\ < r_m & \text{if } \varepsilon(m) = 1 \end{cases}$$

is consistent. For the sake of simplicity, we write $\mu(\varphi)$ instead of $\mu([\varphi]_W)$.

We can conclude that the problem of decidability is reduced to an easy problem of linear programming, which can be positively solved.

REFERENCES

[1] J. Barwise, *An introduction to first-order logic*, in: *Handbook of Mathematical Logic* (J. Barwise editor), North-Holland, Amsterdam, 1977, pp. 6–46.

[2] H.J. Keisler, *Probability quantifiers*, Chapter 14 of *Model Theoretic Languages* (J. Barwise and S. Feferman, editors), Springer-Verlag, Berlin, 1985.

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