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OPTIMAL PROBLEMS CONCERNING INTERPOLATION METHODS OF SOLUTION OF EQUATIONS

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Abstract. We consider optimality problems regarding the order of convergence of the iterative methods which are obtained by inverse interpolation of Lagrange-Hermite type. A similar problem for a class of Steffensen-type methods is solved.

Introduction

The inverse interpolation problem, connected with equation solution was essentially approached in [**5**, **6**, **7**, **4**]. In [**4**] it was shown that the most known iteration methods, as: Newton's method, Chebyshev's method, chord method, as well as various generalizations of them, are all obtained by means of Lagrange-Hermite-type inverse interpolations. A classification of these methods according to their order of convergence is not beyond interest. In this paper we propose to solve two extremum problems concerning the order of convergence of the iteration methods obtained by the Lagrange-Hermite-type inverse interpolation and that of the Steffensen-type method which follows from the preceding ones.

The Lagrange-Hermite-type inverse interpolation polynomial with $n+1$ nodes ($n \in \mathbf{N}$), each node having a given multiplication order, leads to a class of $(n+1)!$ iteration methods. Out of this class of methods, we propose to determine the method having the highest order of convergence. In the last part of the paper we solve the same problem for a class of Steffensen-type iteration methods.

1. The inverse interpolation problems and the determination of optimal iteration method

Let $f : I \to \mathbf{R}$ be a function, where I is an interval of the real axis. Denote by E a set of $n+1$ distinct points from the interval I, namely $E = \{x_1, x_2, \dots, x_{n+1}\}$, where $x_i \neq x_j$ for $i \neq j$, $i, j = 1, 2, \dots, n+1$.

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Consider the natural numbers $\alpha_1, \alpha_2, \dots, \alpha_{n+1}$ and suppose that they satisfy

$$\alpha_1 + \alpha_2 + \dots + \alpha_{n+1} = m + 1, \qquad m \in \mathbf{N}. \tag{1.1}$$

It is a well-known fact that the numbers y_i^j, $j = 0, 1, \dots,$ $\alpha_i = 1$, $i = 1, 2, \dots, n+1$ being given, there exists a unique polynomial P, of degree at most m, which satisfies the relations

$$P^{(j)}(x_i) = y_i^j, \qquad j = 0, 1, \dots, \alpha_i - 1, \quad i = 1, 2, \dots, n+1. \tag{1.2}$$

The polynomial P satisfying the conditions (1.2) has the form:

$$P(x) = \sum_{i=1}^{n+1} \sum_{j=0}^{\alpha_i - 1} \sum_{k=0}^{\alpha_i - j - 1} y_i^j \frac{1}{k!\, j!} \left[\frac{(x - x_i)^{\alpha_i}}{\omega(x)} \right]_{x = x_i}^{(k)} \frac{\omega(x)}{(x - x_i)^{\alpha_i - j - k}}, \tag{1.3}$$

where:

$$\omega(x) = \prod_{i=1}^{n+1} (x - x_i)^{\alpha_i}. \tag{1.4}$$

If we suppose that the function f admits derivatives up to the $(m+1)$-th order on the interval I, and if we put $y_i^j = f^{(j)}(x_i)$, $j = 0, 1, \dots, \alpha_i - 1$, $i = 1, 2, \dots, n+1$, in (1.3), then P is the Hermite interpolating polynomial on the nodes x_i, $i = 1, 2, \dots, n+1$, associated with the function f, and the following equality holds:

$$f(x) - P(x) = \frac{f^{(n+1)}(\xi)}{(m+1)!} \omega(x), \tag{1.5}$$

with $\xi \in I$.

We shall suppose, in what follows, that $f'(x) \neq 0$ for every $x \in I$ and denote $F = f(I)$. It follows that $f : I \to F$ is bijective, hence there exists $f^{-1} : F \to I$ and the function f^{-1} admits derivatives up to the $(m+1)$-th order, for every $x \in F$. The k-th order derivative, where $k \leq m + 1$, can be determined by means of the following formula [**7**]:

$$\begin{aligned} &(f^{-1})^{(k)}(y_0) \\ &= \sum \frac{(2k - 2 - i_1)!\, (-1)^{k-1+i_1}}{i_2!\, i_3! \, \dots \, i_k!\, (f'(x_0))^{2k-1}} \left(\frac{f'(x_0)}{1!} \right)^{i_1} \left(\frac{f''(x_0)}{2!} \right)^{i_2} \cdot \ldots \cdot \left(\frac{f^{(k)}(x_0)}{k!} \right)^{i_k}, \end{aligned} \tag{1.6}$$

where the above sum is extended over all integral and nonnegative solutions of the system

$$\begin{aligned} i_2 + 2i_3 + \dots + (k-1) i_k &= k - 1; \\ i_1 + i_2 + \dots + i_k &= k - 1. \end{aligned} \tag{1.7}$$

If we suppose that the equation:

$$f(x) = 0 \tag{1.8}$$

admits a root $\bar{x} \in I$, then the hypothesis $f'(x) \neq 0$ for every $x \in I$ leads to the conclusion that this root is unique.

From $f(\bar{x}) = 0$ it follows that $\bar{x} = f^{-1}(0)$ and the problem of the approximate solution of the equation (1.8) reduces to the determination of an approximation for $f^{-1}(0)$. Let φ be a function which approximates the function f^{-1}, at least in a neighbourhood V_0 of the point $y = 0$; thus an approximation formula of the form:

$$f^{-1}(y) = \varphi(y) + R[f^{-1}, y] \tag{1.9}$$

holds on the set V_0.

Neglecting the function $R[f^{-1}, y]$ and putting $y = 0$ into (1.9), we obtain the following approximation for $\bar{x}$:

$$\bar{x} \approx \varphi(0), \tag{1.10}$$

with an approximation error given by the inequality

$$|\bar{x} - \varphi(0)| \leq |R[f^{-1}, 0]|. \tag{1.11}$$

There are two natural conditions to be imposed on the function φ:

a) to approximate the function $f^{-1}(y)$ as well as possible, i.e. the number $R[f^{-1}, 0]$ must be as small as possible;

b) to have some simplicity properties for the computation of its values.

The functions which agree with the second condition are (or rather seem so) polynomials, since the computation of their values reduces to elementary arithmetic operations.

Obviously, if we succeed in constructing the best approximating polynomial for f^{-1} on the set V_0, then the first condition will be satisfied too.

In what follows we shall not approach the problem from this point of view; therefore the function will be replaced by the Hermite interpolating polynomial associated to the function f^{-1}, on the set F, called Hermite inverse interpolating polynomial. For this purpose, in the interpolating polynomial (1.3) we consider, as interpolating nodes, the values $y_i = f(x_i)$, $i = 1, 2, \ldots, n+1$, while for y_i^j, $j = 0, 1, \ldots, \alpha_i - 1$, $i = 1, 2, \ldots, n+1$, we consider $y_i^{(0)} = x_i$, $i = 1, 2, \ldots, n+1$, respectively $y_i^{(j)} = (f^{-1})^{(j)}(y_i)$, $i = 1, 2, \ldots, n+1$, $j = 1, 2, \ldots, \alpha_i - 1$. In this way the polynomial (1.3) acquires the form:

$$P(y) = \sum_{i=1}^{n+1} \sum_{j=0}^{\alpha_i - 1} \sum_{k=0}^{\alpha_i - j - 1} (f^{-1})^{(j)}(y_i) \, \frac{1}{k!\, j!} \left[\frac{(y - y_i)^{\alpha_i}}{\omega(y)} \right]_{y=y_i}^{(k)} \frac{\omega(y)}{(y - y_i)^{\alpha_i - j - k}}, \tag{1.12}$$

where:

$$\omega(y) = \prod_{i=1}^{n+1} (y - y_i)^{\alpha_i}. \tag{1.13}$$

From (1.5) it follows that

$$f^{-1}(y) - P(y) = \frac{[f^{-1}(\eta)]^{(m+1)}}{(m+1)!}\omega(y), \qquad \text{where } \eta \in F. \tag{1.14}$$

Taking into account the fact that $\bar{x} = f^{-1}(0)$, from (1.14) one obtains:

$$\bar{x} - P(0) = \frac{[f^{-1}(\eta_1)]^{(m+1)}}{(m+1)!}\omega(0) \tag{1.15}$$

where η_1 is a point lying in the shortest interval which contains 0, $f(x_1)$, $f(x_2)$, $\dots$, $f(x_{n+1})$. Denoting this interval by F_1 and putting:

$$M_{m+1} = \sup_{y \in F_1} |[f^{-1}(y)]^{(m+1)}|, \tag{1.15$'$}$$

it results, from (1.15),

$$|\bar{x} - P(0)| \leq \frac{M_{m+1}}{(m+1)!}|\omega(0)| \tag{1.16}$$

and, since $y_i = f(x_i)$, $i = 1, 2, \dots, n+1$,

$$|\bar{x} - P(0)| \leq \frac{M_{m+1}}{(m+1)!}|f(x_1)|^{\alpha_1} \cdot |f(x_2)|^{\alpha_2} \cdot \dots \cdot |f(x_{n+1})|^{\alpha_{n+1}}. \tag{1.17}$$

From (1.17) it follows that if $x_1, x_2, \dots, x_{n+1}$ are chosen sufficiently close to $\bar{x}$, then the values $f(x_1)$, $f(x_2)$, $\dots$, $f(x_{n+1})$ are real numbers close to zero; so much more the product $\prod_{i=1}^{m+1} |f(x_i)|^{\alpha_i}$ will be close to zero. This remark allows us to consider the value $P(0)$ as an approximation for $\bar{x}$ the root of the equation (1.8).

If $P(0)$ is not a good enough approximation for $\bar{x}$, then we can construct, by iterations, a sequence of approximations $(x_k)_{k\geq 1}$, which, under certain conditions, will be convergent, and $\lim x_k = \bar{x}$.

More precisely, let $x_1, x_2, \dots, x_{n+1}$ be $n+1$ approximations of the root $\bar{x}$ of the given equation (1.8). We denote $x_{n+2} = P(0)$ and replace one of the $n+1$ nodes $x_1, x_2, \dots, x_{n+1}$, then continue the iteration process by the above described procedure.

The problem which arises is to determine the interpolating node which should be replaced at each iteration step in order to obtain a sequence of approximations $(x_k)_{k\geq 1}$ with a maximum order of convergence. To solve the problem, let us consider the following equations:

$$P(t) = t^{n+1} - a_{n+1}t^n - a_n t^{n-1} - \dots - a_2 t - a_1 = 0, \tag{1.18}$$

$$Q(t) = t^{n+1} - a_1 t^n - a_2 t^{n-1} - \dots - a_n t - a_{n+1} = 0, \tag{1.19}$$

$$R(t) = t^{n+1} - a_{i_1} t^n - a_{i_2} t^{n-1} - \dots - a_{i_n} t - a_{i_{n+1}} = 0. \tag{1.20}$$

where $a_1, a_2, \dots, a_{n+1}$ are real number satisfying the conditions:

$$a_1 + a_2 + \dots + a_n > 1, \qquad a_i \geq 0, \quad i = 1, 2, \dots, n+1, \tag{1.21}$$

$$a_{n+1} \geq a_n \geq \dots \geq a_2 \geq a_1, \tag{1.22}$$

while $i_1, i_2, \dots, i_{n+1}$ is an arbitrary permutation of the numbers $1, 2, \dots, n+1$.

The following lemma holds:

LEMMA 1. *If* $a_1, a_2, \dots, a_{n+1}$ *satisfy the conditions* (1.21), *then any of the equations* (1.20) *has a single positive and supraunitary root. If, in addition, the condition* (1.22) *is also satisfied and we denote by* a, b, c, *respectively, the positive roots of the equations* (1.18)–(1.20), *then*

$$1 < b \leq c \leq a, \tag{1.23}$$

i.e. the equation (1.18) *admits the greatest root.*

Proof. Consider one of the $(n+1)!$ equations of the form (1.20) and denote by s the greatest natural number for which $a_{i_s} \neq 0$. We have $a_{i_{s+1}} = a_{i_{s+2}} = \ldots = a_{i_{n+1}} = 0$, and consider the function $\psi(t) = R(t)/t^{n-s+1}$. We have $\psi(1) = 1 - a_{i_1} - \cdots - a_{i_s} < 0$ and $\lim_{t\to\infty} \psi(t) = +\infty$. Accordingly, the equation $\psi(t) = 0$ has at least one supraunitary root and therefore $R(t) = 0$ has at least one supraunitary root.

The uniqueness of this root follows easily if we consider the function $f(t) = -t^{n+1}R(1/t)$ which satisfies the condition $f'(t) > 0$, for $t > 0$.

In order to prove the inequalities (1.23) it is sufficient to show that $R(b) \leq 0$ and $R(a) \geq 0$. Indeed, we have:

$$\begin{aligned} R(b) - Q(b) &= (a_1 - a_{i_1})b^n + (a_2 - a_{i_2})b^{n-1} + \cdots + (a_n - a_{i_n})b + a_{n+1} - a_{i_{n+1}} \\ &= (b-1)[(a_1 - a_{i_1})b^{n-1} + (a_1 + a_2 - a_{i_1} - a_{i_2})b^{n-2} + \cdots \\ &\quad + (a_1 + a_2 + \cdots + a_{n-1} - a_{i_1} - a_{i_2} - \cdots - a_{i_{n-1}})b \\ &\quad + a_1 + a_2 + \cdots + a_n - a_{i_1} - a_{i_2} - \cdots - a_{i_n}] \leq 0, \end{aligned}$$

since $b > 1$ and $a_1 + a_2 + \cdots + a_s - a_{i_1} - a_{i_2} - \cdots - a_{i_s} \leq 0$, $s = 1, 2, \dots, n$.

The inequality $R(a) \geq 0$ can be proved analogously.

Consider now the permutation $i_1, i_2, \dots, i_{n+1}$ of the numbers $1, 2, \dots, n+1$ for which the natural numbers $\alpha_1, \alpha_2, \dots, \alpha_{n+1}$ satisfying the equality (1.1), can be increasingly ordered, namely:

$$\alpha_{i_1} \leq \alpha_{i_2} \leq \ldots \leq \alpha_{i_n} \leq \alpha_{i_{n+1}}. \tag{1.24}$$

We renumber, accordingly, the elements of the set E, i.e. we consider:

$$E = \{x_{i_1}, x_{i_2}, \dots, x_{i_{n+1}}\}$$

For the sake of clearness we shall set:

$$a_s = \alpha_{i_s}, \qquad s = 1, 2, \dots, n+1 \tag{1.25}$$

and

$$u_s = x_{i_s}, \qquad s = 1, 2, \dots, n+1, \tag{1.26}$$

and denote by $H(y_1; a_1, y_2; a_2, \dots, y_{n+1}; a_{n+1}; f^{-1} \mid x)$ the Hermite interpolating polynomial, corresponding to the nodes $y_i = f(u_i)$, $i = 1, 2, \dots, n+1$, having the multiplicities $a_1, a_2, \dots, a_{n+1}$ respectively.

From (1.17) we obtain

$$\left|\bar{x}-H(y_1;a_1,y_2;a_2,\dots,y_{n+1};a_{n+1};f^{-1}\mid 0)\right| \le \frac{M_{m+1}}{(m+1)!}\prod_{i=1}^{n+1}|f(x_i)|^{\alpha_i} \tag{1.27}$$

where M_{m+1} has the same meaning as in (1.15).

Let $u_1,u_2,\dots,u_{n+1}$ be the $n+1$ initial approximations of the root $\bar{x}$ of the equation (1.8). We construct the sequence $(u_p)_{p\ge 1}$ by means of the following iterative procedure:

$$\begin{cases} u_{n+2}=H(y_1;a_1,y_2;a_2,\dots,y_{n+1};a_{n+1};f^{-1}\mid 0),\ \dots, \\ u_{n+s+1}=H(y_s,a_1,y_{s+1},a_2;\dots,y_{s+n};a_{n+1};f^{-1}\mid 0),\quad s=2,3,\dots . \end{cases} \tag{1.28}$$

Denoting $M=\sup_{y\in F}|[f^{-1}(y)]^{(m+1)}|$ and $\beta=\sup_{x\in I}|f'(x)|$, we obtain from (1.27) and (1.28):

$$|\bar{x}-x_{n+s+1}|\le\frac{M\beta^{m+1}}{(m+1)!}\prod_{i=1}^{n+1}|\bar{x}-u_{s+i-1}|^{a_i},\qquad s=1,2,\dots . \tag{1.29}$$

Denoting $\rho_i=\beta\sqrt[m]{\beta M/((m+1)!)}\,|\bar{x}-u_i|$, $i=1,2,\dots$, we obtain from (1.29):

$$\rho_{n+s+1}\le\prod_{i=1}^{n+1}\rho_{s+i-1}^{a_i},\qquad s=1,2,\dots . \tag{1.30}$$

Suppose that there exists a $d\in\mathbf{R}$, $0<d<1$, such that:

$$\rho_i\le d^{\omega^i},\qquad \text{for } i=1,2,\dots,n+1,$$

where ω is the positive root of the equation (1.18). That number is called the convergence order of the method (1.28).

If we suppose that:

$$\rho_i\le d^{\omega^i},\qquad i=n+2,n+3,\dots,n+s$$

then, taking into account (1.30) and the fact that ω is a root of the equation (1.18), we derive the inequality:

$$\rho_{n+s+1}\le d^{\omega^{n+s+1}}, \tag{1.31}$$

which is valid for every $s=1,2,\dots$.

Taking into account the expression for ρ_{n+s+1}, we obtain the following inequality for the error evaluation:

$$|\bar{x}-u_{n+s+1}|\le\frac{1}{\beta\sqrt[m]{\beta M/((m+1)!)}}\cdot d^{\omega^{n+s+1}},\qquad s=1,2,\dots, \tag{1.32}$$

from which it follows that $\lim_{n\to\infty}u_p=\bar{x}$.

Observe that both the error evaluation and the convergence speed of the method (1.28) depend on the root ω of the equation (1.18). The greater ω is, the better the upper limit obtained for the error is.

Consider all $(n+1)!$ permutations of the set $\{1, 2, \dots, n+1\}$. To each permutation $i_1, i_2, \dots, i_{n+1}$ it corresponds an iterative method of the form:

$$\text{(1.33)} \qquad \begin{cases} x_{n+2} = H(y_{i_1}; \alpha_{i_1}, y_{i_2}; \alpha_{i_2}, \dots, y_{i_{n+1}}; \alpha_{i_{n+1}}; f \mid 0); \\ x_{n+s+2} = H(y_{i_1+s}; \alpha_{i_1}, y_{i_2+s}; \alpha_{i_2}, \dots, y_{i_{n+1}+s}; \alpha_{i_{n+1}}; f \mid 0), \\ \qquad s = 1, 2, \dots . \end{cases}$$

All together we have $(n+1)!$ iterative methods.

Taking into account Lemma 1 and the results proved so far, we can state the following theorem:

THEOREM 1. *Out of the* $(n+1)!$ *iterative methods of the form* (1.33), *with the greatest convergence order* (*namely these which provide the best upper limit for the absolute value of the error*) *is that determined by the permutation* $i_1, i_2, \dots, i_{n+1}$, *which orders increasingly the numbers* $\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_{n+1}}$, *namely* $\alpha_{i_1} \le \alpha_{i_2} \le \dots \le \alpha_{i_{n+1}}$.

2. Some particular cases

In what follows we shall discuss some particular cases.

Case $n = 0$. From (1.12) one obtains the Taylor inverse interpolating polynomial:

$$\text{(2.1)} \qquad T(y) = x_1 + \frac{[f^{-1}(y_1)]'}{1!}(y - y_1) + \dots + \frac{[f^{-1}(y_1)]^{(\alpha_1 - 1)}}{(\alpha_1 - 1)!}(y - y_1)^{\alpha_1 - 1}$$

while, from (1–6), we obtain the following expressions for the successive derivatives $[f^{-1}(y)]^{(k)}$, $k = 1, 2, 3, 4$:

$$\text{(2.2)} \qquad [f^{-1}(y)]' = \frac{1}{f'(x)},$$

$$\text{(2.3)} \qquad [f^{-1}(y)]'' = -\frac{f''(x)}{[f'(x)]^3},$$

$$\text{(2.4)} \qquad [f^{-1}(y)]''' = -\frac{f'''(x)f'(x) - 3[f''(x)]^2}{[f'(x)]^5},$$

$$\text{(2.5)} \qquad [f^{-1}(y)]^{(4)} = \frac{-[f'(x)]^2 f^{(4)}(x) + 10 f'(x) f''(x) f'''(x) - 15[f''(x)]^3}{[f'(x)]^7}.$$

From (2.2) and (2.1) for $\alpha_1 = 2$ we obtain:

$$T(y) = x_1 + \frac{1}{f'(x_1)}(y - f(x_1)),$$

which, for $y = 0$, leads to the approximation x_2 of $\bar{x}$ given by the expression

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}, \tag{2.6}$$

i.e. to the Newton's method.

From (2.2), (2.3) and (2.1) for $\alpha_1 = 3$ we obtain Chebyshev's method, i.e.:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} - \frac{1}{2}\frac{f''(x_1)f^2(x_1)}{[f'(x_1)]^3}. \tag{2.7}$$

Lastly, from (2.2), (2.3), (2.4) and (2.1) for $\alpha_1 = 4$ we obtain:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} - \frac{1}{2}\frac{f''(x_1)f^2(x_1)}{[f'(x_1)]^3} + \frac{f'''(x)f'(x_1) - 3[f''(x_1)]^2}{6[f'(x_1)]^5}. \tag{2.8}$$

From the above methods one obtains by iterations the corresponding sequence of approximations, which has the order of convergence 2, 3 and 4 respectively.

As one may notice from (2.5) and (1.6), for $\alpha_1 \geq 5$ the expressions for the derivatives $[f^{-1}(y)]^{(k)}$, $k \geq 4$, have a more complex form. That is why the methods following from (2.1) in these cases are also complex.

Case $n = 1$**.** In this case, from (1.12) it follows:

$$P(y) = \sum_{i=1}^{2}\sum_{j=0}^{\alpha_i-1}\sum_{k=0}^{\alpha_i-j-1}[f^{-1}(y_i)]^{(j)}\,\frac{1}{k!\,j!}\left[\frac{(y-y_i)^{\alpha_i}}{\omega(y)}\right]^{(k)}_{y=y_i}\frac{\omega(y)}{(y-y_i)^{\alpha_i-j-k}} \tag{2.9}$$

where:

$$\omega(y) = (y-y_1)^{\alpha_1}\cdot(y-y_2)^{\alpha_2}. \tag{2.10}$$

From (2.9) one obtains two iterative methods; namely denoting as above by $H(y_1;\alpha_1,y_2;\alpha_2;f^{-1} \mid y)$ the Hermite inverse interpolating polynomial (2.9), we find:

$$\begin{cases} x_3 = H(y_1;\alpha_1,y_2;\alpha_2;f^{-1} \mid 0), \quad x_1,x_2 \in I,\ y_1 = f(x_1),\ y_2 = f(x_2), \\ x_{n+1} = H(y_{n-1};\alpha_1,y_n;\alpha_2;f^{-1} \mid 0), \quad n = 3,4,\dots, \end{cases} \tag{2.11}$$

or

$$\begin{cases} x_3 = H(y_1;\alpha_2 : y_2;\alpha_1;f^{-1} \mid 0), \quad x_1,x_2 \in I,\ y_1 = f(x_1),\ y_2 = f(x_2), \\ x_{n+1} = H(y_{n-1};\alpha_2,y_n;\alpha_1;f^{-1} \mid 0), \quad n = 3,4,\dots. \end{cases} \tag{2.12}$$

The characteristic equations which provide the convergence orders for the two methods are:

$$t^2 - \alpha_2 t - \alpha_1 = 0 \tag{2.13}$$

for the method (2.11), and:

$$t^2 - \alpha_1 t - \alpha_2 = 0 \tag{2.14}$$

for the method (2.12).

If we denote by ω_1 and ω_2, respectively, the positive roots of equations (2.13) and (2.14), then it is clear that $\alpha_2 \geq \alpha_1$ implies $\omega_2 \leq \omega_1$; so, the method with optimal convergence order is the method (2.11).

Now, we shall briefly discuss some particular cases.

From (2.9), for $\alpha_1 = \alpha_2 = 1$, we obtain

$$P_1(y) = (y_1 - y_2)^{-1}[(y - y_2)f^{-1}(y_1) - (y - y_1)f^{-1}(y_2)] \tag{2.15}$$

wherefrom, taking into account the fact that $f^{-1}(y_1) = x_1$ and $f^{-1}(y_2) = x_2$, we find for $y = 0$

$$x_3 = x_1 - \frac{x_2 - x_1}{f(x_2) - f(x_1)} f(x_1) = x_1 - \frac{f(x_1)}{[x_1, x_2; f]}, \tag{2.16}$$

where $[x_1, x_2; f]$ stands for the first order divided difference of the function f on the nodes x_1 and x_2, and

$$x_{n+1} = x_{n-1} - \frac{f(x_{n-1})}{[x_{n-1}, x_n; f]}, \qquad n = 3, 4, \dots, \tag{2.17}$$

which is the chord method. In this case, since $\alpha_1 = \alpha_2$, the above method has the same convergence order as the other one, which follows from (2.12), i.e.:

$$x_{n+1} = x_n - \frac{f(x_n)}{[x_{n-1}, x_n; f]}, \qquad n = 2, 3, \dots. \tag{2.18}$$

The order of convergence of the method (2.17) is $\omega_1 = \frac{1}{2}(1 + \sqrt{5})$.

Now we shall discuss the case $\alpha_1 = 1$, $\alpha_2 = 2$. In this particular case, we obtain from (2.9) the following iterative methods:

$$\begin{aligned} x_{n+2} = x_n &- \frac{x_{n+1} - x_n}{f(x_{n+1}) - f(x_n)} f(x_n) \\ &+ \frac{f(x_{n+1}) - f(x_n) - (x_{n+1} - x_n) f'(x_{n+1})}{[f(x_{n+1}) - f(x_n)]^2 f'(x_{n+1})} f(x_n) \cdot f(x_{n+1}), \\ &\qquad n = 1, 2, \dots, \quad x_1, x_2 \in I \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} x_{n+2} = x_{n+1} &- \frac{x_n - x_{n+1}}{f(x_n) - f(x_{n+1})} f(x_{n+1}) \\ &+ \frac{f(x_n) - f(x_{n+1}) - (x_n - x_{n+1}) f'(x_n)}{[f(x_n) - f(x_{n+1})]^2 f'(x_n)} f(x_n) \cdot f(x_{n+1}). \end{aligned} \tag{2.20}$$

Solving the corresponding characteristic equations, we find the convergence orders $\omega_1 = 1 + \sqrt{2}$ for the method (2.19) and $\omega_2 = 2$ for the method (2.20).

As we showed above, the Hermite inverse interpolating polynomial leads to a large class of iterative methods. The convergence order of each method depends on the number of interpolating nodes, the order of multiplicity of these ones, and, essentially, on the interpolating node replaced at each iteration step by that calculated at the precedent one.

As Steffensen notices, in the case of the method (2.17), the convergence order of this method can be increased if at each iteration step the element x_n depends in a certain manner on x_{n-1}. More exactly, if we consider a function $\varphi : I \to \mathbf{R}$ having the property $\varphi(\bar{x}) = \bar{x}$, where $\bar{x}$ is the root of the equation (1.8), and if we put $x_n = \varphi(x_{n-1})$ into (2.17), then we obtain the sequence $(x_n)_{n \geq 1}$ generated by Steffensen's method:

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{[x_{n-1}, \varphi(x_{n-1}); f]}, \qquad n = 2, 3, \ldots ,$$

which has, as it is well known, the convergence order 2.

3. Steffensen-type optimal methods

Now let $\varphi_i : I \to \mathbf{R}$, $i = 1, 2, \ldots, n+1$, be $n+1$ functions satisfying the following conditions:

a) $\varphi_i(\bar{x}) = \bar{x}$, $i = 1, 2, \ldots, n+1$, where $f(\bar{x}) = 0$;

b) there exist real numbers $\rho_i \geq 0$ and $p_i > 1$, $i = 1, 2, \ldots, n+1$ such that the functions φ_i and f satisfy the relations:

$$|f(\varphi_i(x))| \leq \rho_i |f(x)|^{p_i}, \qquad i = 1, 2, \ldots, n+1. \tag{3.1}$$

Denote by $u_0 \in I$ an initial approximation of the root $\bar{x}$ of the equation (1.8). We construct $n+1$ interpolating nodes x_i^1, $i = 1, 2, \ldots, n+1$, as follows:

$$x_1^1 = \varphi_1(u_0); \qquad x_{i+1}^1 = \varphi_{i+1}(x_i^1), \quad i = 1, 2, \ldots, n. \tag{3.2}$$

Denote $y_i^1 = f(x_i^1)$, $i = 1, 2, \ldots, n+1$, and consider the natural numbers $\alpha_1, \alpha_2, \ldots, \alpha_{n+1}$ satisfying:

$$\alpha_1 + \alpha_2 + \cdots + \alpha_{n+1} = m + 1. \tag{3.3}$$

Using the interpolating nodes y_i^1, $i = 1, 2, \ldots, n+1$, and the Hermite interpolating polynomial on these nodes, with the orders of multiplicity $\alpha_1, \alpha_2, \ldots, \alpha_{n+1}$, respectively, we obtain, for $\bar{x}$ the following approximations:

$$u_1 = H(y_1^1; \alpha_1, y_2^1; \alpha_2, \ldots, y_{n+1}^1; \alpha_{n+1}; f \mid 0), \tag{3.4}$$

with the error evaluation given by the inequality:

$$|\bar{x} - u_1| \leq \frac{M}{(m+1)!} |\omega(0)|, \tag{3.5}$$

where

$$|\omega(0)| = |f(x_1^1)|^{\alpha_1} \cdot |f(x_2^1)|^{\alpha_2} \cdot \ldots \cdot |f(x_{n+1}^1)|^{\alpha_{n+1}}. \tag{3.6}$$

Now, using the property b) of the function φ_i, we derive

$$\begin{aligned} |f(x_1^1)| &= |f(\varphi_1(u_0))| \le \rho_1 |f(u_0)|^{p_1}, \\ |f(x_2^1)| &= |f(\varphi_2(x_1^1))| \le \rho_2 |f(x_1^1)|^{p_2} \le \rho_2 \rho_1^{p_2} |f(u_0)|^{p_1 p_2}, \end{aligned}$$

and generally,

$$|f(x_{i+1}^1)| \le \rho_{i+1} \rho_i^{p_{i+1}} \rho_{i-1}^{p_i p_{i+1}} \cdot \ldots \cdot \rho_1^{p_2 p_3 \cdot \ldots \cdot p_{i+1}} |f(u_0)|^{p_1 p_2 \cdot \ldots \cdot p_{i+1}}, \quad i = 0, 1, \ldots, n. \tag{3.7}$$

If we denote

$$\alpha = \sum_{i=1}^{n+1} \alpha_i \prod_{j=1}^{i} p_j \tag{3.8}$$

and

$$\rho = \prod_{i=1}^{n+1} \rho_i^{\alpha_i + \sum_{j=i+1}^{n+1} \alpha_j \prod_{k=i+1}^{j} p_k}, \tag{3.9}$$

having in view (3.6) and (3.7), we obtain:

$$|\omega(0)| \le \rho |f(x_0)|^{\alpha} \tag{3.10}$$

wherefrom, taking into account (3.5), we derive

$$|\bar{x} - u_1| \le \frac{M\rho}{(m+1)!} |f(u_0)|^{\alpha}. \tag{3.11}$$

If we consider now some element u_{k-1} that we have constructed by iterations, then the interpolating nodes x_i, $i = 1, 2, \ldots, n+1$ corresponding to the next step are obtained, as in the case of the first step, by using the relations:

$$x_1^k = \varphi_1(u_{k-1}), \qquad x_{i+1}^k = \varphi_{i+1}(x_i^k), \quad i = 1, 2, \ldots, n; \quad k \ge 2. \tag{3.12}$$

Constructing the element u_k, as for the first step, we obtain:

$$u_k = H(y_1^k; \alpha_1, y_2^k; \alpha_2, \ldots, y_{n+1}^k; \alpha_{n+1}; f \mid 0), \qquad k \ge 2, \tag{3.13}$$

where $y_i^k = f(x_i^k)$, $i = 1, 2, \ldots, n+1$, which infers the following inequality

$$|\bar{x} - u_k| \le \frac{\rho M}{(m+1)!} |f(u_{k-1})|^{\alpha}, \qquad k = 2, 3, \ldots. \tag{3.14}$$

Denoting $\beta = \sup_{x \in I} |f'(x)|$, the inequalities (3.14) take the form

$$|\bar{x} - u_k| \le \frac{M\rho\beta^{\alpha}}{(m+1)!} |\bar{x} - u_{k-1}|^{\alpha}, \qquad k = 1, 2, \ldots,$$

wherefrom we derive

$$|\bar{x}-u_k| \le C^{1/(1-\alpha)}\left(C^{1/(1-\alpha)}|\bar{x}-u_0|\right)^{\alpha^k}, \tag{3.15}$$

where $C = M\rho\beta^{\alpha}/((m+1)!)$.

If we assume that

$$C^{1/(1-\alpha)}|\bar{x}-u_0| < 1, \tag{3.16}$$

then, from (3.15), it follows

$$\lim_{k\to\infty} u_k = \bar{x}. \tag{3.17}$$

Let $(k_1, k_2, \dots, k_{n+1})$ and $(j_1, j_2, \dots, j_{n+1})$ be two arbitrary permutations of numbers $1, 2, \dots, n+1$. Also denote

$$H(y) = H(y^1_{k_1}, \alpha_{j_1}; y^1_{k_2}, \alpha_{j_2}; \dots; y^1_{k_{n+1}}; \alpha_{j_{n+1}}; f \mid y)$$

the Hermite inverse inerpolating polynomial having the interpolating nodes y_{k_i} with the orders of multiplicity α_{j_i}, $i = 1, 2, \dots, n+1$.

With the above denotations, let us consider the following class of iterative methods

$$u_s = H(y^s_{k_1}; \alpha_{j_1}, y^s_{k_2}; \alpha_{j_2}, \dots, y^s_{k_{n+1}}; \alpha_{j_{n+1}}; f \mid 0), \qquad s = 1, 2, \dots \tag{3.18}$$

where

$$y^s_{k_i} = f(x^s_{k_i}), \qquad i = 1, 2, \dots, n+1; \quad s = 1, 2, \dots,$$

and

$$\begin{aligned} &x^s_{k_1} = \varphi_{k_1}(u_{s-1}), \\ &x^s_{k_i} = \varphi_{k_i}(x^s_{k_{i-1}}), \qquad i = 2, 3, \dots, n+1; \quad s = 1, 2, \dots, \end{aligned} \tag{3.19}$$

u_0 being the given initial approximation.

To each couple of permutations $(k_1, k_2, \dots, k_{n+1})$ and $(j_1, j_2, \dots, j_{n+1})$ of the numbers $1, 2, \dots, n+1$ there corresponds an iterative method of the form (3.18). All together we have again $(n+1)!$ iterative methods of this form.

We shall attempt to determine, out of the $(n+1)!$ iterative methods, that one for which the number α given by (3.8) is maximum.

With this goal in view, we shall first prove the following lemma:

Lemma 2. *Let $p_1, p_2, \dots, p_{n+1}$ with $\alpha_1, \alpha_2, \dots, \alpha_{n+1}$, with $p_i \ge 1$, $\alpha_1 \ge 1$, $i = 1, 2, \dots, n+1$, be real numbers satisfying*

$$p_1 \ge p_2 \ge \dots \ge p_{n+1}; \qquad \alpha_1 \le \alpha_2 \le \dots \le \alpha_{n+1}. \tag{3.20}$$

Out of all numbers of the form

$$\alpha = \alpha_{j_1} p_{k_1} + \alpha_{j_2} p_{k_1} p_{k_2} + \dots + \alpha_{j_{n+1}} p_{k_1} p_{k_2} \cdot \ldots \cdot p_{k_{n+1}}, \tag{3.21}$$

where $(j_1, j_2, \dots, j_{n+1})$ and $(k_1, k_2, \dots, k_{n+1})$ are arbitrary permutations of the set $(1, 2, \dots, n+1)$, the greatest one is the number

$$\alpha_{\max} = \alpha_1 p_1 + \alpha_2 p_1 p_2 + \dots + \alpha_{n+1} p_1 p_2 \cdot \ldots \cdot p_{n+1}. \tag{3.22}$$

Proof. From the first set of inequalities (3.20) it follows that the inequality:

$$\alpha_{j_1} p_{k_1} + \alpha_{j_2} p_{k_1} p_{k_2} + \cdots + \alpha_{j_{n+1}} p_{k_1} p_{k_2} \cdot \ldots \cdot p_{k_{n+1}} \leq \alpha_{j_1} p_1 + \alpha_{j_2} p_1 p_2 + \cdots + \alpha_{j_{n+1}} p_1 p_2 \cdot \ldots \cdot p_{n+1} \tag{3.23}$$

holds for any two permutations $(j_1, j_2, \ldots, j_{n+1})$ and $(k_1, k_2, \ldots, k_{n+1})$ of the numbers $(1, 2, \ldots, n+1)$.

Let us denote

$$b_i = p_1 p_2 \cdot \ldots \cdot p_i, \qquad i = 1, 2, \ldots, n+1, \tag{3.24}$$

and let us attempt to prove the inequality

$$\alpha_{j_1} b_1 + \alpha_{j_2} b_2 + \cdots + \alpha_{j_{n+1}} b_{n+1} \leq \alpha_1 b_1 + \alpha_2 b_2 + \cdots + \alpha_{n+1} b_{n+1} \tag{3.25}$$

for every permutation $(j_1, j_2, \ldots, j_{n+1})$. We shall do that by induction. For $n = 0$ the inequality (3.25) is obvious, since $n + 1 = 1$ and hence $\alpha_{j_1} = \alpha_1$. Suppose now that the inequality is true for n pairs of numbers (α_1, b_1), (α_2, b_2), $\ldots$, (α_n, b_n), namely

$$\alpha_{j_1} b_1 + \alpha_{j_2} b_2 + \cdots + \alpha_{j_n} b_n \leq \alpha_1 b_1 + \cdots + \alpha_n b_n, \tag{3.26}$$

where $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n$ and $b_1 \leq b_2 \leq \ldots \leq b_n$. Using the inequalities $b_1 \leq b_2 \leq \ldots \leq b_n \leq b_{n+1}$ and $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n \leq \alpha_{n+1}$, as well as the induction hypothesis (3.26), and assuming that $j_1 = i$, $1 \leq i \leq n$, we have

$$\begin{aligned}
&\alpha_{j_1} b_1 + \alpha_{j_2} b_2 + \cdots + \alpha_{j_{n+1}} b_{n+1} \\
&= b_1(\alpha_{j_1} + \alpha_{j_2} + \cdots + \alpha_{j_{n+1}}) + (b_2 - b_1)\alpha_{j_2} + (b_3 - b_1)\alpha_{j_3} + \cdots \\
&\qquad + (b_{n+1} - b_1)\alpha_{j_{n+1}} \\
&\leq b_1(\alpha_1 + \alpha_2 + \cdots + \alpha_{n+1}) + (b_2 - b_1)\alpha_1 + (b_3 - b_1)\alpha_2 + \cdots \\
&\qquad + (b_i - b_1)\alpha_{i-1} + (b_{i+1} - b_1)\alpha_{i+1} + \cdots + (b_{n+1} - b_1)\alpha_{n+1} \\
&\leq b_1(\alpha_1 + \alpha_2 + \cdots + \alpha_{n+1}) + (b_2 - b_1)\alpha_2 + (b_3 - b_1)\alpha_3 + \cdots \\
&\qquad + (b_i - b_1)\alpha_i + (b_{i+1} - b_1)\alpha_{i+1} + \cdots + (b_{n+1} - b_1)\alpha_{n+1} \\
&= b_1\alpha_1 + b_2\alpha_2 + \cdots + b_{n+1}\alpha_{n+1}, \qquad \text{q.e.d.}
\end{aligned}$$

The above lemma leads to the following theorem:

THEOREM 2. *Out of all* $(n+1)!$ *iterative methods of the form* (3.12)–(3.13), *the one for which the maximum convergence order given by* (3.8) *is achieved, is the method determined by the order of the numbers* p_i, α_i, $i = 1, 2, \ldots, n+1$, *given by the inequalities* (3.20).

The proof of this theorem follows immediately from Lemma 2 and (3.8).

REFERENCES

[1] C. Iancu, I. Păvăloiu, *La résolution des équations par interpolation inverse de type Hermite*, Mathematica — Revue d'analyse numerique et de Théorie de l'approximation **26** (**49**) (2) (1984), 115–123.

[**2**] C. Iancu, I. Păvăloiu, I. Sert, *Methodes itératives optimales de type Steffensen obtenues par interpolation inverse*, Research Seminar on Functional Analysis and Numerical Methods, Preprint **1** (1983), 81–88.

[**3**] M. A. Ostrowski, *Solution of Equations and Systems of Equations*, Academic Press, New York and London, 1980.

[**4**] I. Păvăloiu, *Equation Solution by Interpolation*, Dacia, Cluj-Napoca, 1981, (in Romanian).

[**5**] J. F. Steffensen, *Interpolation*, Chelsea Publ., New York, 1950.

[**6**] J. F. Traub, *Iterative Methods for the Solution of Equations*, Prentice-Hall, Englewood Cliffs, N.J., 1964.

[**7**] B. A. Turowicz, *Sur les derivées d'ordre superieur d'une fonction inverse*, Ann. Polon. Math. **8** (1960), 265–269.

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