PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 52 (66), 1992, 77–85

ON BEST SIMULTANEOUS APPROXIMATION

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Abstract. For nonempty subsets F and K of a nonempty set V and a real valued function f on $X \times X$ the notion of f-best simultaneous approximation to F from K is introduced as an extension of the known notion of best simultaneous approximation in normed linear spaces. The concept of uniformly quasi-convex function on a vector space is also introduced. Sufficient conditions for the existence and uniqueness of f-best simultaneous approximation are obtained.

The concept of best simultaneous approximation was studied by several authors in normed linear spaces. In [**1**] the concept was extended to locally convex spaces. In this paper we extend the notion to arbitrary sets and study it on arbitrary sets, vector spaces, topological spaces and topological vector spaces.

Definition 1. Let X be a nonempty set and f be a real valued function on $X \times X$. Let F and K be nonempty subsets of X. For x in X, we define $f_F(x) = \sup\{f(x,y) \mid y \in F\}$. We define $f_F(K) = \inf\{f_F(x) \mid x \in K\}$ and $P_K^f(F) = \{x \in K \mid f_F(x) = f_F(K)\}$. An element of $P_K^f(F)$ is called an f-best simultaneous approximation to F from K. An element z of F is called an f-farthest point of an element x in X from F if $f(x,z) = f_F(x)$. F is said to be f-antiproximinal if every element in X admits an f-farthest point from F. F is said to be f-antiproximinal with respect to K if every element in K admits an f-farthest point from F. An element z of K is called an f-nearest point of an element y in X from K if $f(z,y) = \inf\{f(x,y) \mid x \in K\}$. K is said to be f-proximinal if every element in X admits an f-nearest point from K. K is said to be f-proximinal with respect to F if every element in F admits an f-nearest point from K.

Remark 1. When X is a normed linear space over the field of real numbers and $f(x,y) = \|x-y\|$ for all x, y in X, the notions introduced above coincide with the corresponding notions that already exist in literature.

Notation. When X is a vector space over the field of real numbers $\mathbf{R}$, f is a real valued function on X, F and K are subsets of X and $x \in X$, we write $f_F(x)$ for

AMS Subject Classification (1980): Primary 41 A 28, 41 A 50, 41 A 52, 54 H 99, 46 A 99

$g_F(x)$, $f_F(K)$ for $g_F(K)$ and $P_K^f(F)$ for $P_K^g(F)$, where $g : X \times X \to \mathbf{R}$ is defined by $g(x, y) = f(x - y)$ for all x, y in X. Similarly we speak of f-farthest point, f-nearest point, f-antiproximinal, f-proximinal etc. for the corresponding notions associated with g. When X is a normed linear space and f stands for the norm on X, we generally drop f from the terminology. Thus we speak of farthest point for f-farthest point and so on. We also write $P_K(F)$ for $P_K^f(F)$.

THEOREM 1. *Let X be a topological space, F and K be nonempty subsets of X and f be a nonnegative real valued function on $X \times X$. Suppose that $f(\cdot, y)$ is lower semicontinuous on X for each y in F. Suppose that $\{k_n\}$ admits a convergent subnet with limit in K whenever $\{k_n\}$ is a sequence in K such that $f(k_n, \cdot)$ is uniformly bounded on F. Then $P_K^f(F)$ is nonempty. If $f_F(K) < +\infty$, then $P_K^f(F)$ is countably compact.*

Proof. Since f is a nonnegative real valued function, we have $0 \leq f_F(K) \leq +\infty$. If $f_F(K) = +\infty$, then $P_K^f(F) = K \neq \varnothing$. Suppose that $f_F(K) < +\infty$. Then there exists a sequence $\{k_n\}$ in K such that $\{f_F(k_n)\}$ decreases to $f_F(K)$. We have $0 \leq f(k_n, y) \leq f_F(k_n)$ for all $n = 1, 2, \ldots$ and for all y in F. Hence $\{f(k_n, \cdot)\}$ is uniformly bounded on F. By hypothesis it follows that $\{k_n\}$ has a convergent subnet with limit, say, k in K. By hypothesis, $f(\cdot, y)$ is lower semicontinuous on X for each y in F. Hence for any positive real number ε and for any y in F, $\{x \in X \mid f(k, y) - \varepsilon < f(x, y)\}$ is an open set containing k. Hence for each y in F, $f(k, y) - \varepsilon < f(k_n, y)$ for infinitely many n. Hence for each y in F, $f(k, y) - \varepsilon \leq f_F(K)$. Since this is true for any positive real number ε, we must have $f_F(k) = f_F(K)$. Hence $k \in P_K^f(F)$. A perusal of the above proof shows that when $f_F(K) < +\infty$, every sequence in $P_K^f(F)$ admits a convergent subnet with limit in $P_K^f(F)$. Hence $P_K^f(F)$ is countably compact when $P_K^f(F) < +\infty$.

The following theorem is evident from Theorem 1:

THEOREM 2. *Let X be a topological vector space $(T.V.S.)$ over $\mathbf{R}$, F and K be nonempty subsets of X, and f be a nonnegative real valued lower semicontinuous function on X. Suppose that $\{k_n\}$ has a convergent subnet with limit in K whenever $\{k_n\}$ is a sequence in K such that* $\sup\{f(k_n - y) \mid n = 1, 2, \ldots,\ y \in F\}$ *is finite. Then $P_K^f(F)$ is nonempty. If $f_F(K) < +\infty$, then $P_K^f(F)$ is countably compact.*

Definitions 2. Let X be a T.V.S. over $\mathbf{R}$, f be a nonnegative real valued function on X and S be a subset of X. For a real number r, let $A(r) = \{x \in X \mid f(x) \leq r\}$. We say that S is (i) f-boundedly compact if for every real number r every net in $S \cap A(r)$ has a convergent subnet with limit in S, (ii) f-boundedly countably compact if for every real number r every sequence in $S \cap A(r)$ has a convergent subnet with limit in S, (iii) f-boundedly weakly countably compact if for every real number r every sequence in $S \cap A(r)$ has a weakly convergent subnet with weak limit in S, and (iv) f-boundedly weakly compact if for every real number r every net in $S \cap A(r)$ has a weakly convergent subnet with limit in S.

Remark 2. Let X be a T.V.S. over $\mathbf{R}$, f be a nonnegative real valued function on X and S be a subset of X. Every translate of S in X is f-boundedly weakly countably compact or f-boundedly countably compact or f-boundedly compact according as S is weakly countably compact or countably compact or compact.

From Theorem 2 we have the following:

COROLLARY 1. *Let X be a T.V.S. over R and f be a nonnegative real valued, lower semicontinuous function on X. Let F and K be nonempty subsets of X. Suppose that the translate of K by some element of F is f-boundedly countably compact. Then $P_K^f(F)$ is a nonempty, countably compact set.*

The following definitions are known:

Definitions 3. Let X be a midpoint convex subset of a vector space over $\mathbf{R}$. A real valued function f on X is said to be (i) quasi-convex on X if $f\big((x+y)/2\big) \leq \max\{f(x), f(y)\}$ for all x, y in X, (ii) strictly quasi-convex on X if $f\big((x+y)/2\big) < \max\{f(x), f(y)\}$ for all distinct x, y in X.

PROPOSITION 1. *Let X be a locally convex T.V.S. over $\mathbf{R}$ and f be a lower semicontinuous, quasi-convex function on X. Then f is lower semicontinuous with respect to the weak topology on X.*

Proof. Let r be a real number and $A(r) = \{x \in X \mid f(x) \leq r\}$. Since f is lower semicontinuous on X, $A(r)$ is a closed subset of X. Since f is quasi-convex on X, the midpoint of any two points of $A(r)$ belongs to $A(r)$. Hence $A(r)$ is a closed, convex subset of X. Since X is locally convex, it follows that $A(r)$ is weakly closed. Hence f is lower semicontinuous on X with respect to the weak topology on X.

In view of Proposition 1 we have the following corollary of Theorem 2:

COROLLARY 2. *Let X be a locally convex T.V.S. over $\mathbf{R}$, F and K be nonempty subsets of X and f be a nonnegative real valued, lower semicontinuous, quasi-convex function on X. Suppose that $\{k_n\}$ has a weakly convergent subnet with weak limit in K whenever $\{k_n\}$ is a sequence in K such that $\sup\{f(k_n - y) \mid n = 1, 2, \dots,\ y \in F\}$ is finite. Then $P_K^f(F)$ is nonempty. If $f_F(K) < +\infty$, then $P_K^f(F)$ is weakly countably compact.*

From Corollary 2 we have the following:

COROLLARY 3. *Let X be a locally convex T.V.S. over $\mathbf{R}$, F and K be nonempty subsets of X and f be a nonnegative real valued, lower semicontinuous, quasi-convex function on X. Suppose that the translate of K by some element of F is f-boundedly weakly countably compact. Then $P_K^f(F)$ is a nonempty, weakly countably compact set.*

From Corollary 3 we have the following:

Corollary 4 (Theorem 1 of [**2**]). *Let X be a normed linear space; F be a nonempty, bounded subset of X and K be a nonempty, boundedly weakly sequentially compact subset of X. Then the set of all best simultaneous approximations to F from K is nonempty and weakly sequentially compact.*

Definition 4. A topological space X is said to be locally countably compact if for each x in X there exists a neighbourhood of x whose closure is countably compact.

Proposition 2. *Let X be a T.V.S. over $\mathbf{R}$ and K be a closed, bounded, star-shaped, locally countably compact subset of X. Then K is countably compact.*

Proof. Since K is star-shaped, there exists x in K such that $\lambda x+(1-\lambda)y \in K$ whenever $y \in K$ and $0 \leq \lambda \leq 1$. Since K is locally countably compact, there exists a neighbourhood V of x such that $K \cap \overline{V}$ is countably compact, where bar stands for closure in X. Let $W = V - x$. Then W is a neighbourhood of zero and $\overline{V} = x + \overline{W}$. We have $(K-x)\cap\overline{W} = (K\cap\overline{V}) - x$. Hence $(K-x)\cap\overline{W}$ is countably compact. Since K is bounded, so is $K-x$. Hence, there exists a real number $t > 1$ such that $K - x \subseteq tW$. If $y \in K - x$, then $y/t \in (K-x)\cap W$ since $t > 1$ and $K - x$ is star-shaped with centre at zero. Hence $K - x \subseteq t\big((K-x)\cap W\big) \subseteq t\big((K-x)\cap\overline{W}\big)$. Since $(K-x)\cap\overline{W}$ is countably compact, so is $t((K-x)\cap\overline{W})$. Since $K-x$ is a closed subset of $t\big((K-x)\cap\overline{W}\big)$, it follows that $K-x$ is countably compact. Hence K is countably compact.

The following Proposition can be established along the lines of Proposition 2:

Proposition 3. *Let X be a T.V.S. over $\mathbf{R}$ and K be a closed, bounded, star-shaped, locally compact subset of X. Then K is compact.*

In view of Proposition 2 we have the following corollary of Corollary 1:

Corollary 5. *Let X be a T.V.S. over $\mathbf{R}$ and f be a nonnegative real valued lower semicontinuous function on X such that $f(tx) \leq f(x)$ for all x in X and for all t in $[0,1]$. Suppose that for each nonnegative real number r, $\{x \in X \mid f(x) \leq r\}$ is bounded and locally countably compact. Let F and K be nonempty subsets of X. Suppose that K is closed. Then $P_K^f(F)$ is nonempty. If $f_F(K) < +\infty$, then $P_K^f(F)$ is countably compact.*

Proof. For a nonnegative real number r, let $A(r) = \{x \in X \mid f(x) \leq r\}$. Since $f(tx) \leq f(x)$ for all x in X and for all t in $[0,1]$, it follows that $A(r)$ is star-shaped with centre at zero. Since f is lower semicontinuous on X, $A(r)$ is closed. By hypothesis $A(r)$ is bounded and locally countably compact. In view of Proposition 2 it follows that $A(r)$ is countably compact. Let $\{k_n\}$ be a sequence in K such that $\{f(k_n - z)\}$ is bounded for some z in F. Then there exists a nonnegative real number s such that $f(k_n - z) \leq s$ for all $n = 1, 2, \ldots$. Hence $\{k_n - z\}$ is a sequence in $A(s)$. Since $A(s)$ is countably compact, it follows that $\{k_n - z\}$ has a convergent subnet. Hence $\{k_n\}$ has a convergent subnet with limit, say, k. Since K is closed, $k \in K$. From Corollary 1 it now follows that $P_K^f(F)$ is nonempty and that, when $f_F(K) < +\infty$, it is countably compact.

Again from Corollary 1 we have the following:

COROLLARY 6. *Let X be a T.V.S. over $\mathbf{R}$ and f be a nonnegative, lower semicontinuous, quasi-convex function on X. Let F and K be nonempty subsets of X. Suppose that K is closed, star-shaped and that there is a z in F such that the set $(K - z) \cap \{x \in X \mid f(x) \leq r\}$ is bounded and locally countably compact for every positive real number r. Then $P_K^f(F)$ is nonempty. If $f_F(K) < +\infty$, then $P_K^f(F)$ is countably compact.*

Proof. For a nonnegative real number r, let $A(r) = \{x \in X \mid f(x) \leq r\}$. Since f is lower semicontinuous and quasi-convex, $A(r)$ is a closed, convex subset of X. Since K is star-shaped, there exists x_0 in K such that $lx_0 + (1 - l)x \in K$ for all x in K and for all l in $[0, 1]$. By hypothesis there exists an element z in F such that $(K - z) \cap A(r)$ is bounded and locally countably compact for any positive real number r. Let s be a positive real number such that $s \geq f(x_0 - z)$. Then the set $(K - z) \cap A(s)$ is star-shaped with centre at $x_0 - z$. Since both K and $A(s)$ are closed, $(K - z) \cap A(s)$ is closed. Now from Proposition 2 it follows that $(K - z) \cap A(s)$ is countably compact. Hence $K - z$ is f-boundedly countably compact. Now the corollary is evident from Corollary 1.

Remark 3. Corollaries 1, 3, 5 and 6 remain valid if the word "countably" is deleted from them.

Remark 4. Theorems 2.1 and 2.2 of [**1**] are corollaries of Corollary 1 as well as Corollary 3. This shows that many conditions in Theorems 2.1 and 2.2 of [**1**] are redundant. The first and second parts of Theorem 2.3 of [**1**] are corollaries of Corollary 5 and Corollary 6 respectively.

THEOREM 3. *Let X be a nonempty set and f be a real valued function on $X \times X$. Let F and K be nonempty subsets of X. Suppose that there exist x^* in K and y^* in F such that y^* is an f-farthest point of x^* from F and x^* is an f-nearest point of y^* from K. Then $f_F(K) = f(x^*, y^*)$ and $x^* \in P_K^f(F)$.*

Proof. Let $x \in K$. Then $f_F(x) \geq f(x, y^*) \geq f(x^*, y^*) = f_F(x^*)$. Hence $f_F(K) = f_F(x^*)$ and $x^* \in P_K^f(F)$.

The following is the vector space analogue of Theorem 3:

THEOREM 4. *Let X be a vector space over $\mathbf{R}$ and f be a real valued function on X. Let F and K be nonempty subsets of X. Suppose that there exist x^* in K and y^* in F such that y^* is an f-farthest point of x^* from F and x^* is an f-nearest point of y^* from K. Then $f_F(K) = f(x^* - y^*)$ and $x^* \in P_K^f(F)$.*

Remark 5. Let F and K be nonempty subsets of a normed linear space X over $\mathbf{R}$. Let $x_1, x_2, \ldots, x_m$ be points of K and $y_1, y_2, \ldots, y_m$ be points of F such that for each i in $\{1,\dots,m\}$, y_i is a farthest point of x_i from F and x_{i+1} is a nearest point of y_i from K, where $x_{m+1}=x_1$. In view of Theorem 4 it is natural to ask whether $\{x_1,x_2,\dots,x_m\}\cap P_K(F)\neq\varnothing$. The following examples show that this need not be true even when $m=2$, $X=\mathbf{R}^2$, F and K are disjoint compact sets and K is convex. While in Example 1 F has exactly two elements, in Example 2 F is convex.

Example 1. Let $X=\mathbf{R}^2$, $F=\{(-2,0),(2,0)\}$ and K be the convex hull of $\bigl\{(0,0),\bigl(-\frac{3}{2},-\frac{\sqrt{3}}{2}\bigr),\bigl(\frac{3}{2},-\frac{\sqrt{3}}{2}\bigr)\bigr\}$. Let f denote the Euclidean norm of $\mathbf{R}^2$. It can be seen that $P_K^f(F)=\{(0,0)\}$. Set $x_1=\bigl(-\frac{3}{2},-\frac{\sqrt{3}}{2}\bigr)$, $x_2=\bigl(\frac{3}{2},-\frac{\sqrt{3}}{2}\bigr)$, $y_1=(2,0)$ and $y_2=(-2,0)$. Then y_1, y_2 are the farthest points of x_1, x_2 from F and x_2, x_1 are the nearest points of y_1, y_2 from K. But neither x_1 nor x_2 belongs to $P_K^f(F)$. In fact, even the convex hull of $\{x_1,x_2\}$ is disjoint with $P_K^f(F)$.

Example 2. Let $X=\mathbf{R}^2$, $F=$ convex hull of $\{(-2,0),(2,0)\}$ and $K=$ convex hull of $\bigl\{\bigl(-\frac{3}{4},-\frac{\sqrt{3}}{4}\bigr),\bigl(-\frac{3}{2},-\frac{\sqrt{3}}{2}\bigr),\bigl(\frac{3}{2},-\frac{\sqrt{3}}{2}\bigr),\bigl(\frac{3}{4},-\frac{\sqrt{3}}{4}\bigr)\bigr\}$. Let f denote the Euclidean norm of $\mathbf{R}^2$. We see that F and K are compact, convex subsets of $\mathbf{R}^2$ and that they are disjoint. We note that $P_K^f(F)=\bigl\{\bigl(0,-\frac{\sqrt{3}}{4}\bigr)\bigr\}$. Choosing x_1,x_2,y_1,y_2 as in Example 1 we see that the observations made about them in Example 1 are also true here. For each x in K the set of all farthest points of x from F is a subset of $\{y_1,y_2\}$. We have $\inf\{\|x-y_1\|\mid x\in K\}=1=\inf\{\|x-y_2\|\mid x\in K\}$. But $f_F(K)=\sqrt{67}/4>2$.

Remark 6. Example 2 shows that the equation '$f_F(K)=\inf\{f(y_0-x)\mid x\in K\}$' given in the proof of Theorem 2.6 of [**1**] is false. It is not known whether the conclusions of the theorem are true or false (vide M.R. of [**1**]).

THEOREM 5. *Let X be a vector space over $\mathbf{R}$ and f be a quasi-convex function on X such that $f(-x)=f(x)$ for all x in X. Let F and K be nonempty subsets of X. Suppose that there exist x^* in K and y^* in F such that $f_F(x^*)=f(x^*-y^*)$ and $2x^*-y^*\in F$. Then $x^*\in P_K^f(F)$.*

Proof. For k in K we have

$$\begin{aligned} f(x^*-y^*)&=f\left(\frac{(2x^*-y^*-k)+(k-y^*)}{2}\right)\leq\max\bigl\{f(2x^*-y^*-k),f(k-y^*)\bigr\}\\ &=\max\bigl\{f(k-(2x^*-y^*)),f(k-y^*)\bigr\}\leq f_F(k).\end{aligned}$$

Hence $f_F(x^*)\leq f_F(k)$, so $x^*\in P_K^f(F)$.

THEOREM 6. *Let X be a vector space over $\mathbf{R}$ and f be a quasi-convex function on X such that $f(-x)=f(x)$ for all x in X. Suppose that there exist y_1,y_2 in F such that $(y_1+y_2)/2\in K$ and $f_F(x)=\max\{f(x-y_1),f(x-y_2)\}$ for all x in K. Then $(y_1+y_2)/2\in P_K^f(F)$. If further f is strictly quasi-convex, then $P_K^f(F)$ is a singleton.*

Proof. Set $z = (y_1 + y_2)/2$. We have

$$f_F(z) = \max\{f(z - y_1), f(z - y_2)\} = f\big((y_1 - y_2)/2\big).$$

For k in K we have

$$\begin{aligned} f\left(\frac{y_1 - y_2}{2}\right) &= f\left(\frac{(y_1 - k) + (k - y_2)}{2}\right) \leq \max\{f(y_1 - k), f(k - y_2)\} \\ &= \max\{f(k - y_1), f(k - y_2)\} = f_F(k). \end{aligned}$$

Hence $f_F(z) \leq f_F(k)$ for all k in K, so $z \in P_K^f(F)$. When f is strictly quasi-convex, for $k \neq z$ we have

$$f\left(\frac{(y_1 - k) + (k - y_2)}{2}\right) < \max\{f(y_1 - k), f(k - y_2)\}$$

so that $f_F(z) < f_F(k)$. Hence in this case $P_K^f(F) = \{z\}$.

Remark 7. In Theorem 6 if the condition '$(y_1 + y_2)/2 \in K$' is deleted from the hypothesis, then it is natural to ask whether an f-nearest point of $(y_1 + y_2)/2$ from K can be an element of $P_K^f(F)$. Example 3 shows that this need not be true even when $X = \mathbf{R}^2$, f is the Euclidean norm on $\mathbf{R}^2$, and F and K are compact, convex subsets of $\mathbf{R}^2$.

Example 3. Let $X = \mathbf{R}^2$, $F =$ convex hull of $\{(1,0), (0,1)\}$ and $K =$ convex hull of $\big\{(0,0), \big(\frac{1}{2}, 0\big), \big(\frac{1}{2}, \frac{1}{4}\big)\big\}$. Let f be the Euclidean norm of $\mathbf{R}^2$. We note that $\big(\frac{2}{5}, \frac{1}{5}\big)$ is the nearest point of $(0,1)$ from K and $(0,1)$ is the farthest point of $\big(\frac{2}{5}, \frac{1}{5}\big)$ from F. It can be seen that $P_K(F) = \big\{\big(\frac{2}{5}, \frac{1}{5}\big)\big\}$. For each x in K the set of all farthest points of x from F is a subset of $\{y_1, y_2\}$, where $y_1 = (1,0)$ and $y_2 = (0,1)$. The nearest point of $(y_1 + y_2)/2$ from K is $\big(\frac{1}{2}, \frac{1}{4}\big)$.

We shall now obtain a couple of theorems which prescribe conditions under which $P_K^f(F)$ can have at most one element.

THEOREM 7. *Let X be a vector space over $\mathbf{R}$, f be a strictly quasi-convex function on X, and F and K be nonempty subsets of X. Suppose that K is midpoint convex and F is f-antiproximinal with respect to K. Then $P_K^f(F)$ has at most one element.*

Proof. If possible, suppose that $P_K^f(F)$ has more than one element. Let x_1, x_2 be distinct elements of $P_K^f(F)$. Since K is midpoint convex, $(x_1 + x_2)/2 \in K$. Since each point of K has an f-farthest point from F, there exists z in F such that $f_F\big((x_1 + x_2)/2\big) = f\big((x_1 + x_2)/2 - z\big)$. Since $x_1 \neq x_2$, we have $x_1 - z \neq x_2 - z$. Hence from strict quasi-convexity of f we have

$$\begin{aligned} f\left(\frac{x_1 + x_2}{2} - z\right) &= f\left(\frac{(x_1 - z) + (x_2 - z)}{2}\right) < \max\big\{f(x_1 - z), f(x_2 - z)\big\} \\ &\leq \max\big\{f_F(x_1), f_F(x_2)\big\} = f_F(K). \end{aligned}$$

Thus we have $f_F\big((x_1 + x_2)/2\big) < f_F(K)$ which is a contradiction. Hence the theorem.

Remark 8. In proving Theorem 7 we adopted the line of argument given in Theorem 3.1 of [**1**] which was observed to be false in the M.R. of [**1**].

We shall now introduce the concept of uniformly quasi-convex function as a generalization of uniformly convex norm.

Definition 5. A nonnegative real valued function f on a vector space X over $\mathbf{R}$ is said to be uniformly quasi-convex if for any positive real numbers r and ε there corresponds a real number α (depending on r and ε) in $(0,1)$ such that $f\big((x+y)/2\big) \leq \alpha r$ whenever x, y are elements of X such that $f(x) \leq r$, $f(y) \leq r$ and $\max\{f(x-y), f(y-x)\} \geq \varepsilon$.

THEOREM 8. *Let X be a vector space over $\mathbf{R}$ and f be a uniformly quasi-convex function on X such that $f(x) \neq 0$ for $x \neq 0$. Let F and K be nonempty subsets of X. Suppose that K is midpoint convex and $f_F(K) < +\infty$. Then $P_K^f(F)$ contains at most one element.*

Proof. If possible, suppose that there are two distinct elements x_1, x_2 in $P_K^f(F)$. Since K is midpoint convex, $(x_1+x_2)/2 \in K$. Since f takes positive values at nonzero points of X, it follows that either $f_F(x_1)$ or $f_F(x_2)$ is positive. Since both x_1 and x_2 are in $P_K^f(F)$, we must have $f_F(x_1) = f_F(K) = f_F(x_2)$. Hence $f_F(K)$ is positive. Since $f_F(K) < +\infty$, $r = f_F(K)$ is a positive real number. Set $\varepsilon = \max\{f(x_1-x_2), f(x_2-x_1)\}$. We note that ε is also a positive real number. Since f is uniformly quasi-convex, there exists an α in $(0,1)$ such that $f\big((x+y)/2\big) \leq \alpha r$ whenever x, y are elements of X such that $f(x) \leq r$, $f(y) \leq r$ and $\max\{f(x-y), f(y-x)\} \geq \varepsilon$. For all z in F we have $f(x_1-z) \leq r$ and $f(x_2-z) \leq r$. Hence $f\big((x_1+x_2)/2 - z\big) \leq \alpha r$ for all z in F. Hence $f_F\big((x_1+x_2)/2\big) \leq \alpha r < r = f_F(K)$. This is a contradiction. Hence $P_K^f(F)$ contains at most one element.

In view of Theorem 7 it is of interest to know some sufficient conditions for the existence of f-farthest points. So we give the following:

PROPOSITION 4. *Let X be a T.V.S. over $\mathbf{R}$ and f be a real valued upper semicontinuous function on X. Let F be a nonempty, countably compact subset of X. Let x be an element of X such that $f_F(x) < +\infty$. Then x has an f-farthest point from F.*

The following is a generalization of Proposition 4 to topological spaces:

PROPOSITION 5. *Let X be a topological space, f be a real valued function on $X \times X$ and F be a nonempty, countably compact subset of X. Let $x \in X$ be such that $f_F(x) < +\infty$ and $f(x,\cdot)$ is upper semicontinuous on X. Then x admits an f-farthest point from F.*

Proof. For a positive integer n, let $r_n = f_F(x) - 1/n$ and $U_n = \{y \in X \mid f(x,y) < r_n\}$. Since $f(x,\cdot)$ is upper semicontinuous on X, for each positive integer n, U_n is an open subset of X. We note that $U_n \subseteq U_m$ if $n \leq m$. If possible, suppose that there is no z in F such that $f(x,z) = f_F(x)$. Then $\{U_n \mid n = 1, 2, \dots\}$ is a countable open cover of F. Since F is countably compact, it follows that there exists a positive integer N such that $F \subseteq U_N$. Hence $f_F(x) \leq r_N$. This is a contradiction. Hence x has an f-farthest point from F.

Out of heuristic interest we shall now state without proof a proposition which lays down sufficient conditions for the existence of f-nearest points.

PROPOSITION 6. *Let X be a topological space, f be a real valued function on $X \times X$ and K be a nonempty, countably compact subset of X. Let $y \in X$ be such that $f(\,\cdot\,, y)$ is lower semicontinuous on X and bounded below on K. Then y admits an f-nearest point from K.*

The following is the vector space analogue of Proposition 6:

PROPOSITION 7. *Let X be a T.V.S. over* $\mathbf{R}$, *f be a real valued lower semicontinuous function on X and K be a nonempty, countably compact subset of X. Let $y \in X$ be such that f is bounded below on $K - y$. Then y admits an f-nearest point from K.*

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(Received 25 05 1990)