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ON A CLASS OF p-VALENT ANALYTIC FUNCTIONS DEFINED BY RUSCHEWEYH DERIVATIVE

K. S. Padmanabhan and **M. Jayamala**

Introduction. Let $A(p)$ denote the class of functions of the form $f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k}$, $(p \in \mathbf{N} = \{1, 2, 3, \dots\})$ which are analytic in the unit disk $E = \{z : |z| < 1\}$. We denote by $f * g(z)$ the Hadamard product of two functions $f(z)$ and $g(z)$ in $A(p)$. Following Goel and Sohi [**2**] we put,

$$D^{n+p-1} f(z) = \frac{z^p}{(1-z)^{n+p}} * f(z), \qquad (n > -p)$$

for the $(n+p-1)$-th order Ruscheweyh derivative of $f(z) \in A(p)$. Let h be convex univalent in E with $h(0) = 1$.

Definition 1. We say that a function $f(z) \in A(p)$ is in $T_{n,p}(h)$ if and only if $\big(D^{n+p} f(z)\big)'/(pz^{p-1}) \prec h(z)$, $z \in E$.

We will prove that $T_{n,p}(h) \subset T_{n-1,p}(h)$ and that f is preserved under the Bernardi integral operator under certain conditions. Finally some coefficient estimates for the class will be also obtained.

We require the following theorems which provide a method for finding the best dominant for certain differential subordinations.

THEOREM A [**1**]. *Let β and γ be complex constants and let h be convex (univalent) in E with $h(0) = 1$ and $\operatorname{Re}[\beta h(z) + \gamma] > 0$. If $p(z) = 1 + p_1 z + \cdots$ is analytic in E, then $p(z) + \dfrac{zp'(z)}{\beta p(z) + \gamma} \prec h(z)$ implies $p(z) \prec h(z)$.*

THEOREM B [**1**]. *Let β and γ be complex constants and let h be convex in E with $h(0) = 1$ and $\operatorname{Re}[\beta h(z) + \gamma] > 0$. Let $p(z) = 1 + p_1 z + \cdots$ be analytic in E and let it satisfy the differential subordination*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z). \tag{1}$$

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If the differential equation $q(z) + \dfrac{zp'(z)}{\beta q(z) + \gamma} = h(z)$ *with* $q(0) = 1$ *has a univalent solution* $q(z)$*, then* $p(z) \prec q(z) \prec h(z)$ *and* $q(z)$ *is the best dominant of* (1).

Corollary A [**1**]. *Let* $p(z)$ *be analytic in* E *and let it satisfy the differential subordination,*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec \frac{1-(1-2\delta)z}{1+z} \equiv h(z) \qquad \text{with } \beta > 0 \text{ and } -\mathrm{Re}(\gamma/\beta) \leq \delta < 1.$$

Then the differential equation

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z), \qquad q(0) = 1,$$

has a univalent solution $q(z)$*. In addition* $p(z) \prec q(z) \prec h(z)$ *and* $q(z)$ *is the best dominant.*

Theorem 1. *If* $(n+p) > 0$*, then* $T_{n,p}(h) \subset T_{n-1,p}(h)$*, where* h *is a convex univalent function in* E *with* $h(0) = 1$.

Proof. Set $g = (D^{n+p-1}f)'/(pz^{p-1})$. Taking logarithmic derivatives and multiplying by z we get

$$\frac{zg'(z)}{g(z)} = z\,\frac{(D^{n+p-1}f)''}{(D^{n+p-1}f)'} - (p-1). \tag{2}$$

Using the fact

$$z(D^{n+p-1}f)' = (n+p)D^{n+p}f - nD^{n+p-1}f \tag{3}$$

(2) can be reduced to

$$\frac{zg'(z)}{(n+p)} + g(z) = \frac{(D^{n+p}f(z))'}{pz^{p-1}} \prec h(z), \tag{4}$$

since $f \in T_{n,p}(h)$. Now if $(n+p) > 0$, we can conclude, by Theorem A, that $g(z) \prec h(z)$, that is $f \in T_{n-1,p}(h)$.

Choosing $n = -p+1$, we get the inclusion relations, $T_{n,p}(h) \subset T_{n-1,p}(h) \subset \ldots \subset T_{-p+1,p}(h)$. So $f \in T_{-p+1,p}(h)$ implies $\dfrac{(D^0 f)'}{pz^{p-1}} \prec h$ or $\dfrac{f'(z)}{pz^{p-1}} \prec h$. If $\mathrm{Re}\, h > 0$, it follows that f is p-valent, by a result due to Umezawa [**6**]. Hence we have the following

Corollary 1. *Let* $f \in T_{n,p}(h)$*,* $n+p > 0$*, where* $\mathrm{Re}\, h > 0$*,* $h(0) = 1$ *and* h *is univalently convex in* E*. Then* f *is* p*-valent.*

Remarks. However, we observe that h need not be the best dominant for g in Theorem 1. We proceed to find the best dominant for g using Theorem B. In fact if q is the best dominant for g in (4), then q should satisfy,

$$zq'/(n+p) + q = h \tag{5}$$

and q should be univalent. Hence, $q' + (n+p)z^{-1}q = (n+p)z^{-1}h$. Solving, we get the best dominant,

$$q(z) = \frac{(n+p)}{z^{n+p}} \int_0^z h(z) z^{n+p-1}\, dz; \tag{6}$$

h is univalently convex with $h(0) = 1$. We show that q is also univalently convex. Set $Q = 1 + zq''/q'$. Taking logarithmic derivative and multiplying by z we get,

$$\frac{zQ'}{Q-1} = 1 + \frac{zq'''}{q''} - \frac{zq''}{q'}.$$

$$Q + \frac{zQ'}{Q-1} = 2 + \frac{zq'''}{q''}. \tag{7}$$

From (5) we get,

$$\frac{zh''}{h'} = \frac{Q + \dfrac{zQ'}{Q-1} + (n+p)}{1 + \dfrac{(n+p+1)}{Q-1}}.$$

Using (7) this reduces to

$$1 + \frac{zh''}{h'} = Q + \frac{zQ'}{Q + (n+p)}.$$

Since h is convex, we have $(1 + zh''/h') \prec (1-z)/(1+z)$, therefore

$$\left(1 + \frac{zq''}{q'}\right) \prec \frac{1-z}{1+z}, \qquad \text{if} \quad \operatorname{Re}\left(\frac{1-z}{1+z} + (n+p)\right) > 0,$$

which is true; hence q is convex univalent.

Set $h(z) = [1 + (2\alpha - 1)z]/(1+z)$, $0 \le \alpha < 1$, in (6) so that $\operatorname{Re} h(z) > \alpha$. The best dominant q for g in this case is given by,

$$q(z) = \frac{(n+p)}{z^{n+p}} \int_0^z z^{n+p-1} \frac{1 + (2\alpha - 1)z}{1+z}\, dz.$$

By integration we get,

$$q(z) = 1 - \frac{2(1-\alpha)(n+p)}{z^{n+p}} \left[\log(1+z) - \left(z - \frac{z^2}{2} + \cdots - \frac{z^{n+p}}{n+p}\right)\right], \tag{8}$$

when $(n+p)$ is even,

$$q(z) = 1 + \frac{2(1-\alpha)(n+p)}{z^{n+p}} \left[\log(1+z) - \left(z - \frac{z^2}{2} + \cdots + \frac{z^{n+p}}{n+p}\right)\right], \tag{9}$$

when $(n+p)$ is odd.

So we obtain the following

Corollary 2. *Set* $p = 1$, $n = 0$ *and* $\alpha = 0$ *in* (9). *Then the best dominant* $q(z)$ *reduces to* $q(z) = 2z^{-1}\log(1+z) - 1$.

We note that $\operatorname{Re} q(e^{i\theta}) = \operatorname{Re} q(e^{-i\theta})$ and $\operatorname{Im} q(e^{i\theta}) = -\operatorname{Im} q(e^{-i\theta})$. Furthermore $\operatorname{Re} q(-1) = +\infty$. Hence the curve given by the set of points $q(e^{i\theta})$, $0 \le \theta \le 2\pi$ is symmetrical about the real axis and since q is convex, $\operatorname{Re} q(e^{i\theta})$ is minimum at $\theta = 0$ and the minimum value is $\operatorname{Re} q(1) = 2\log 2 - 1 = .38\ldots$

Theorem 2. *If* $f(z) \in T_{n-1,p}(h)$, *then*

$$F(z) = \frac{c+p}{z^c}\int_0^z t^{c-1} f(t)\,dt \in T_{n-1,p}(h) \qquad \text{for } c+p > 0. \tag{10}$$

Proof. From (10) we have,

$$z(D^{n+p-1}F)' = (c+p)(D^{n+p-1}f) - c(D^{n+p-1}F). \tag{11}$$

Differentiating (11) we get,

$$z(D^{n+p-1}f)'' + (D^{n+p-1}F)' = (c+p)(D^{n+p-1}f)' - c(D^{n+p-1}F)'. \tag{12}$$

Set $G = (D^{n+p-1}F)'/(pz^{p-1})$. Taking logarithmic derivatives and multiplying by z we get,

$$\frac{zG'(z)}{G(z)} = \frac{z(D^{n+p-1}F)''}{(D^{n+p-1}F)'} - (p-1).$$

Using (12) this reduces to,

$$\frac{zG'(z)}{(c+p)} + G(z) = \frac{(D^{n+p-1}f)'}{pz^{p-1}} \prec h. \tag{13}$$

Since $f \in T_{n-1,p}(h)$. If $(c+p) > 0$, we conclude, by Theorem A, that $G \prec h$, that is $(D^{n+p-1}F)'/(pz^{p-1}) \prec h$. If q is the best dominant for G in (13), then q should satisfy $zg'(z)/(c+p) + q(z) = h(z)$. Solving it we get the best dominant

$$q(z) = \frac{(c+p)}{z^{c+p}}\int_0^z h(z) z^{c+p-1}\,dz.$$

If we choose

$$h(z) = \frac{1+(2\alpha-1)z}{1+z}, \qquad 0 \le \alpha < 1,$$

then the best dominant in this case is given by,

$$q(z) = 1 - \frac{2(1-\alpha)(c+p)}{z^{c+p}}\left[\log(1+z) - \left(z - \frac{z^2}{2} + \cdots - \frac{z^{c+p}}{c+p}\right)\right], \quad \text{when } (c+p) \text{ is even}, \tag{14}$$

$$q(z) = 1 + \frac{2(1-\alpha)(c+p)}{z^{c+p}}\left[\log(1+z) - \left(z - \frac{z^2}{2} + \cdots + \frac{z^{c+p}}{c+p}\right)\right], \quad \text{when } (c+p) \text{ is odd}. \tag{15}$$

Evidently $q(z)$ is convex, since h is so.

COROLLARY 3. *Taking* $\alpha = 0$, $p = 1$ *and* $c = 1$, (14) *reduces to* $q(z) = 4z^{-1} - 4z^{-2}\log(1+z) - 1$. *Here again we can show that* $\operatorname{Re} q(e^{i\theta}) = \operatorname{Re} q(e^{-i\theta})$, $\operatorname{Im} q(e^{i\theta}) = -\operatorname{Im} q(e^{-i\theta})$ *and* $\operatorname{Re} q(-1) = +\infty$. *Hence the curve extends to* ∞ *and since the curve is convex, it is minimum at* $\theta = 0$ *and the minimum value is* $\operatorname{Re} q(1) = 3 - 4\log 2 = .227\ldots$.

This is an improvement of the results of Goel and Sohi [**3**, Remarks (i) and (ii)].

THEOREM 3. *The class* $T_{n,p}(h)$ *is closed with respect to convex combination, where* h *is univalently convex and* $h(0) = 1$.

Proof. Let $f, g \in T_{n,p}(h)$. Therefore $(D^{n+p}f)'/(pz^{p-1}) \prec h$ and $(D^{n+p}g)'/(pz^{p-1}) \prec h$. Hence there exist points z_1, z_2 in E such that
$$\frac{(D^{n+p}f)'}{pz^{p-1}} = h(z_1) \quad \text{and} \quad \frac{(D^{n+p}g)'}{pz^{p-1}} = h(z_2).$$
Let $F = tf + (1-t)g$, $0 < t < 1$. Then
$$\frac{(D^{n+p}F)'}{pz^{p-1}} = t\,\frac{(D^{n+p}f)'}{pz^{p-1}} + (1-t)\frac{(D^{n+p}g)'}{pz^{p-1}} = th(z_1) + (1-t)h(z_2) = h(z_3)$$
for some z_3 in E, because h is convex. In other words $F \in T_{n,p}(h)$.

A connection between the classes $T_{n-1,p}(h)$ **and** $T_{m-1,p}(h)$**.** We now prove the following

THEOREM 4. *Let* $f \in T_{n-1,p}(h)$ *and let*

(16) $$g(z) = (m+p+1)!z^{1-m} \times$$
$$\times \int_0^z \int_0^{x_{m+p-1}} \cdots \int_0^{x_2} \left[\frac{x_1^{n-1} f(x_1)}{(n+p-1)!}\right]^{(n+p-1)} dx_1 dx_2 \ldots dx_{m+p-1}.$$
Then $g \in T_{m-1,p}(h)$.

Proof. From (16) we have,
$$\frac{g(z)z^{m-1}}{(m+p-1)!} = \int_0^z \int_0^{x_{m+p-1}} \cdots \int_0^{x_2} \left[\frac{x_1^{n-1} f(x_1)}{(n+p-1)!}\right]^{(n+p-1)} dx_1 dx_2 \ldots dx_{m+p-1}.$$
Differentiating $(m+p-1)$ times we get
$$\frac{\left(g(z)z^{m-1}\right)^{(m+p-1)}}{(m+p-1)!} = \frac{\left(z^{n-1}f(z)\right)^{(n+p-1)}}{(n+p-1)!}.$$
Since
$$D^{n+p-1}f(z) = \frac{z^p\left(z^{n-1}f(z)\right)^{(n+p-1)}}{(n+p-1)!},$$
it follows that
$$\frac{\left(D^{m+p-1}g(z)\right)'}{pz^{p-1}} = \frac{\left(D^{n+p-1}f(z)\right)'}{pz^{p-1}}.$$
Therefore $f \in T_{n-1,p}(h)$ if and only if $g \in T_{m-1,p}(h)$.

Coefficient estimates. THEOREM 5. *If $f \in T_{n-1,p}(h)$, where h is univalently convex in E with $h(z) = 1 + \sum_{k=1}^{\infty} h_k z^k$, then*

$$|a_{p+k}| \leq \frac{|h_1|p}{(p+k)\binom{n+p+k-1}{k}}, \qquad k = 1, 2, 3, \dots . \tag{17}$$

Proof. Our hypothesis implies

$$\frac{(D^{n+p-1}f)'}{pz^{p-1}} = 1 + \sum_{k=1}^{\infty} \binom{n+p+k-1}{k} \frac{(p+k)}{p} a_{p+k} z^k \prec h(z). \tag{18}$$

Let $h(z) = 1 + h_1 z + h_2 z^2 + \cdots$, $z \in E$. From (18) we have, for $k = 1, 2, 3, \dots$

$$\frac{p+k}{p} \binom{n+p+k-1}{k} |a_{p+k}| \leq |h_1|$$

using a result due to Rogosinski [**5**]. Choosing $h(z) = [1 + (2\alpha - 1)z]/(1+z)$, $0 \leq \alpha < 1$, we get the sharp coefficient estimate

$$|a_{p+k}| \leq \frac{2p(1-\alpha)}{(p+k)\binom{n+p+k-1}{k}}$$

attained for

$$f(z) = z^p + 2p(1-\alpha) \sum_{k=1}^{\infty} \frac{(-1)^k z^{p+k}}{(p+k)\binom{n+p+k-1}{k}}.$$

THEOREM 6. *If $f \in T_{n-1,p}(h)$, where h is convex univalent in E, with $h(z) = 1 + \sum_{k=1}^{\infty} h_k z^k$, then for any complex number γ,*

$$|a_{p+2} - \gamma a_{p+1}^2| \leq \frac{2p|h_1| \max(1, |\mu|)}{(n+p+1)(n+p)(p+2)} \tag{19}$$

where

$$\mu = \left(\frac{\gamma(n+p+1)(p+2)ph_1^2}{2(n+p)(p+1)^2} - h_2 \right). \tag{20}$$

The result is sharp.

Proof. Our hypothesis on f enables us to write

$$(D^{n+p-1}f)'/(pz^{p-1}) = h(\omega(z)), \tag{21}$$

where ω is analytic and $|\omega(z)| \leq |z|$ in $|z| < 1$. Let $\omega = \sum_{j=1}^{\infty} c_j z^j$; then,

$$\left\{ 1 + \sum_{k=1}^{\infty} \binom{n+p+k-1}{k} a_{p+k} \frac{(p+k)}{p} z^k \right\} = \{1 + h_1(c_1 z + c_2 z^2 + \cdots) \\ + h_2(c_1 z + c_2 z^2 + \cdots)^2 + \cdots\}.$$

Equating the same powers of z we get

$$c_1 = \frac{(n+p)(p+1)}{h_1 p} a_{p+1} \tag{22}$$

$$c_2 = \frac{1}{h_1}\left[\frac{(n+p+1)(n+p)(p+2)}{2p} a_{p+2} - \frac{h_2}{h_1^2 p^2}(p+1)^2(n+p)^2 a_{p+1}^2\right]. \tag{23}$$

Define μ by (20). Then we have,

$$\begin{aligned} |c_2 - \mu c_1^2| &= \frac{1}{|h_1|}\bigg|\frac{(n+p+1)(n+p)(p+2)}{2p} a_{p+2} \\ &\qquad - \frac{(h_1\mu + h_2)}{h_1^2}\,\frac{(p+1)^2(n+p)^2}{p^2} a_{p+1}^2\bigg| \\ &= \frac{(n+p+1)(n+p)(p+2)}{|h_1|^{2p}} |a_{p+2} - \gamma a_{p+1}^2|. \end{aligned} \tag{24}$$

Using the coefficient inequality,

$$|c_2 - \mu c_1^2| \le \max(1, |\mu|) \tag{25}$$

due to Keog and Merkes [**4**], in (24) we obtain (19).

The equality is attained in (19) for the function $f(z)$ given by (21) when we choose $\omega(z) = z$ or $\omega(z) = z^2$.

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Ramanujan Institute for Advanced Study in Mathematics,
University of Madras,
Madras – 600 005, India

(Received 23 03 1990)

Department of Mathematics,
Queen Mary's College,
Madras – 600 004, India