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ON THE FEKETE-SZEGŐ THEOREM FOR CLOSE-TO-CONVEX FUNCTIONS

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Abstract. Let $K(\beta)$ be the class of normalised close-to-convex functions with order $\beta \geq 0$, defined in the unit disc D by

$$\left|\arg e^{i\lambda}\frac{zf'(z)}{g(z)}\right| \leq \frac{\pi\beta}{2},$$

for $|\lambda| < \pi/2$ and g starlike in D. For $f \in K(\beta)$ with $f(z) = z + a_2z^2 + a_3z^3 + \dots$ and $z \in D$, sharp bounds are given for $|a_3 - \mu a_2^2|$ for real μ.

Let S denote the class of analytic univalent functions f, defined for $z \in D = \{z : |z| < 1\}$ by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Fekete and Szegö [**6**], showed that for $f \in S$, given by (1),

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, & \text{if } \mu \leq 0 \\ 1 + 2e^{-2\mu/(1-\mu)}, & \text{if } 0 \leq \mu < 1 \\ 4\mu - 3, & \text{if } \mu \geq 1. \end{cases}$$

The inequalities are sharp in the sense that for each μ, there exists a function in S such that equality holds. Pfluger [**11**], [**12**] has recently considered the problem for complex μ.

Let S^* and K denote the classes of normalised starlike and close-to-convex functions respectively. Thus $f \in K$ if, and only if, there exists $g \in S^*$ and a real λ, with $|\lambda| < \pi/2$, such that for $z \in D$,

$$\operatorname{Re} e^{i\lambda}\frac{zf'(z)}{g(z)} > 0.$$

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Let K_0 be the subset of K when $\lambda = 0$. For $f \in K_0$, Keogh and Merkes [**8**] showed that

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu, & \text{if } \mu \leq 1/3, \\ 1/3 + 4/9\mu, & \text{if } 1/3 \leq \mu \leq 2/3, \\ 1, & \text{if } 2/3 \leq \mu \leq 1, \\ 4\mu - 3, & \text{if } \mu \geq 1. \end{cases}$$

Again, for each μ, there are functions in K_0 such that equality holds in all cases.

Eenigenburg and Silvia [**5**] were able to extend the result of Keogh and Merkes to the whole class K, whilst Koepf [**9**], apparently unaware of [**5**] and [**8**], gave a proof for $\mu \in [0, 1]$.

Denote by $K(\beta)$ the class of close-to-convex functions of order $\beta \geq 0$. Thus $f \in K(\beta)$, if, and only if, for $\beta \geq 0$, there exists $g \in S^*$ and a real λ, with $|\lambda| < \pi/2$, such that for $z \in D$,

$$\left| \arg e^{i\lambda} \frac{zf'(z)}{g(z)} \right| \leq \frac{\beta\pi}{2}. \tag{2}$$

Clearly for $0 \leq \beta \leq 1$, $K(\beta)$ is a subset of S, whilst for $\beta > 1$, $K(\beta)$ can contain functions with infinite valence [**7**]. We also note that $K(0) = C$, the class of normalised convex functions. For C, the Fekete-Szegö problem has been solved in [**8**]. Let $K_0(\beta)$ be the subset of $K(\beta)$ when $\lambda = 0$. Then in [**1**] it was shown that the result of Keogh and Merkes extends to:

Theorem A. *Let $f \in K_0(\beta)$ and be given by* (1)*. Then for $0 \leq \beta \leq 1$,*

$$|a_3 - \mu a_2^2| \leq \begin{cases} 1 - \mu + \dfrac{\beta(2-3\mu)(\beta+2)}{3}, & \textit{if } \mu \leq \dfrac{2\beta}{3(\beta+1)}, \\ 1 - \mu + \dfrac{2\beta}{3} + \dfrac{\beta(2-3\mu)^2}{3[2-\beta(2-3\mu)]}, & \textit{if } \dfrac{2\beta}{3(\beta+1)} \leq \mu \leq \dfrac{2}{3}, \\ \dfrac{2\beta+1}{3}, & \textit{if } \dfrac{2}{3} \leq \mu \leq \dfrac{2(\beta+2)}{3(\beta+1)}, \\ \mu - 1 + \dfrac{\beta(3\mu-2)(\beta+2)}{3}, & \textit{if } \mu \geq \dfrac{2(\beta+2)}{3(\beta+1)}, \end{cases}$$

whilst for $\beta > 1$, the first two inequalities hold. For each μ there are functions in $K_0(\beta)$ such that equality holds in all cases.

Koepf [**10**] considered the problem for the class $K(\beta)$ and gave the solution when $\mu = 2/3$. He also showed that the first inequality in Theorem A extends to $K(\beta)$ in the case $\beta \geq 1$, for all $|\lambda| < \pi/2$, and established the sharp inequalities

$$|a_3 - a_2^2| \leq \begin{cases} \dfrac{2\beta+1}{3}, & \text{if } 0 \leq \beta \leq 1 \\ \dfrac{\beta(\beta+2)}{3}, & \text{if } \beta \geq 1. \end{cases} \tag{3}$$

The purpose of this paper is to examine the question of extending Theorem A to $K(\beta)$.

Results

THEOREM 1. *Let* $f \in K(\beta)$ *and be given by* (1), *then for* $\beta \geq 0$,

$$|a_3 - \mu a_2^2| \leq \begin{cases} 1 - \mu + \dfrac{\beta(2-3\mu)(\beta+2)}{3}, & \text{if } \mu \leq \dfrac{2\beta}{3(\beta+1)}, \\ 1 - \mu + \dfrac{2\beta}{3} + \dfrac{\beta(2-3\mu)^2}{3[2-\beta(2-3\mu)]}, & \text{if } \dfrac{2\beta}{3(\beta+1)} \leq \mu \leq \dfrac{2}{3}, \end{cases}$$

provided $\cos^2\lambda \leq 1/2$ *or* $\lambda = 0$.

The inequalities are sharp in the sense that for each μ, *there exists a function in* $K(\beta)$, *such that equality holds.*

Proof. From (2) write

$$zf'(z) = g(z)\tilde{p}(z)^{\beta}, \tag{4}$$

for $g \in S^*$ given by $g(z) = z + b_2z^2 + b_3z^3 + \cdots$ and $\operatorname{Re} e^{i\lambda}\tilde{p}(z) > 0$ with $\tilde{p}(z) = 1 + \tilde{p}_1 z + \tilde{p}_2 z^2 + \cdots$. Thus for some p satisfying $\operatorname{Re} p(z) > 0$ and given by $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$, we have $\tilde{p}_n = p_n e^{-i\lambda}\cos\lambda$, so that $|\tilde{p}_n| = |p_n|\cos\lambda$ for $n \geq 1$.

Equating coefficients in (4) we have

$$\begin{aligned} 2a_2 &= b_2 + \beta\tilde{p}_1, \\ 3a_3 &= b_3 + \frac{\beta(\beta-1)}{2}\tilde{p}_1^2 + \beta\tilde{p}_2 + \beta\tilde{p}_1 b_2. \end{aligned}$$

and so

$$\begin{aligned} a_3 - \mu a_2^2 = \frac{1}{3}\left(b_3 - \frac{3}{4}\mu b_2^2\right) + \frac{\beta}{3}\left(\tilde{p}_2 + \left(\frac{\beta(2-3\mu)}{4} - \frac{1}{2}\right)\tilde{p}_1^2\right) \\ + \beta\left(\frac{1}{3} - \frac{\mu}{2}\right)\tilde{p}_1 b_2. \end{aligned} \tag{5}$$

Since $\dfrac{2\beta}{3(\beta+1)} \leq \mu \leq \dfrac{2}{3}$, it follows from (5) that

$$\begin{aligned} |a_3 - \mu a_2^2| \leq 1 - \mu + \frac{\beta\cos\lambda}{3}\left(2 - \frac{|p_1|^2}{2}(1 - |\sin\lambda|)\right) \\ + \frac{\beta^2}{12}(2-3\mu)|p_1|^2\cos^2\lambda + \frac{\beta(2-3\mu)}{3}|p_1|\cos\lambda, \end{aligned} \tag{6}$$

where we have used the inequalities $|b_3 - \nu b_2^2| \leq \max\{1, |4\nu - 3|\}$ for $g \in S^*$ with ν real [**8**], $|b_2| \leq 2$ and

$$\left|\tilde{p}_2 - \frac{\tilde{p}_1^2}{2}\right| \leq \cos\lambda\left(2 - \frac{|p_1|^2}{2}(1 - |\sin\lambda|)\right),$$

proved in [**9**].

Now write $u = |p_1|$ and $v = \cos\lambda$. Then (6) can be written as $|a_3 - \mu a_2^2| \leq \phi(u,v)$, where

$$\begin{aligned}\phi(u,v) =& 1-\mu+\frac{\beta v}{3}\left(2-\frac{u^2}{2}\left(1-\sqrt{1-v^2}\right)\right)\\ &+\frac{\beta^2u^2v^2}{12}(2-3\mu)+\frac{\beta uv}{3}(2-3\mu),\end{aligned}$$

where, since $|p_1| \leq 2$, it follows that $(u,v) \in [0,2]\times[0,1]$.

Fix $v = v_0$ and assume first that $\phi(u, v_0)$ has a turning point at u. Then $\phi'(u, v_0) = 0$ implies that

$$2u\left(1-\sqrt{1-v_0^2}\right) = \beta u v_0 X + 2X, \tag{7}$$

where $X = 2-3\mu$, so that $0 \leq X \leq 2/(1+\beta)$.

From (6) and (7) one obtains

$$\begin{aligned}\phi(u,v_0) &= 1-\mu+\frac{2\beta v_0}{3}+\frac{\beta u v_0}{6}X\\ &\leq 1-\mu+\frac{2\beta v_0}{3}+\frac{\beta v_0 X}{3}\\ &\leq 1-\mu+\frac{2\beta}{3}+\frac{\beta X^2}{3[2-\beta X]},\end{aligned}$$

if

$$v_0 \leq \frac{2(2-\beta X)+X^2}{(2+X)(2-\beta X)} = \Psi(\beta,X) \qquad \text{say.}$$

An elementary argument shows that $\Psi(\beta, X)$ has a minimum value of $2\sqrt{2}-2$ when $\beta \geq 0$. Next suppose that $u = 0$. Then $\phi(0,v) = 1-\mu+2\beta v/3 \leq 1-\mu+2\beta/3$. Finally let $u = 2$. Then if $v \leq 1/\sqrt{2}$,

$$\begin{aligned}\phi(2,v) &= 1-\mu+\frac{2\beta v}{3}\sqrt{1-v^2}+\frac{\beta^2v^2}{3}X+\frac{2\beta v}{3}X,\\ &\leq 1-\mu+\frac{\beta}{3}+\frac{\beta^2}{6}X+\frac{\sqrt{2}\beta}{3}X,\\ &\leq 1-\mu+\frac{2\beta}{3}+\frac{\beta X^2}{3[2-\beta X]},\end{aligned}$$

since $0 \leq X \leq 2/(1+\beta)$.

Thus in all cases, the second inequality in Theorem 1 is established, provided $v \leq 1/\sqrt{2}$.

Choosing $\lambda = 0$, $b_2 = p_2 = 2$, $b_3 = 3$ and $p_1 = \dfrac{2(2-3\mu)}{2-\beta(2-3\mu)}$ shows that the inequality is sharp on the interval $\dfrac{2\beta}{3(\beta+1)} \leq \mu \leq \dfrac{2}{3}$, since $|p_1| \leq 2$.

Next consider the case $\mu \le \dfrac{2\beta}{3(\beta+1)}$. Then

$$\begin{aligned}|a_3-\mu a_2^2| &\le \left|a_3-\frac{2\beta}{3(\beta+1)}a_2^2\right|+\left(\frac{2\beta}{3(\beta+1)}-\mu\right)|a_2|^2,\\ &\le 1+\frac{2\beta}{3}+\left(\frac{2\beta}{3(\beta+1)}-\mu\right)(\beta+1)^2\\ &=1-\mu+\frac{\beta(2-3\mu)(\beta+2)}{3},\end{aligned}$$

for $\beta \ge 0$, where we have used the result already proved in the case $\mu = 2\beta/3(\beta+1)$, and the fact that for $f \in K(\beta)$, the inequality $|a_2| \le \beta+1$ holds [**2**], [**3**], [**4**]. Equality is attained on choosing $\lambda = 0$, $p_1 = p_2 = b_2 = 2$ and $b_3 = 3$.

Remark 1. As mentioned above, Koepf [**10**] established the first inequality of Theorem 1 for all λ, such that $|\lambda| < \pi/2$, provided $\beta \ge 1$ and $\mu \ge 0$. We note that maximising the expression for $H_\beta(y)$ on page 424 gives another proof of the same inequality when $0 \le \beta \le 1$, provided $\cos^2\lambda \le 1/2$ or $\lambda = 0$.

We now consider the case $\mu \ge 2/3$. We prove:

THEOREM 2. *Let $f \in K(\beta)$ and be given by* (1). *Then*

$$|a_3-\mu a_2^2| \le \begin{cases} \dfrac{2\beta+1}{3}, & \text{if } \dfrac{2}{3} \le \mu \le \dfrac{2(\beta+2)}{3(\beta+1)},\\[2ex] \mu-1+\dfrac{\beta(3\mu-2)(\beta+2)}{3}, & \text{if } \mu \ge \dfrac{2(\beta+2)}{3(\beta+1)},\end{cases}$$

for $0 \le \beta \le 1$ if $\cos^2\lambda \le 1/2$ or if $\lambda = 0$. For $\beta \ge 1$, the inequalities hold if $\cos^2\lambda \le (3-\sqrt{5})/2$. The inequalities are sharp.

Proof. We first deal with the case when $\mu = \dfrac{2(\beta+2)}{3(\beta+1)}$. In [**8**] it was shown that for $g \in S^*$ given by $g(z) = z + b_2z^2 + b_3z^3 + \cdots$

$$\left|b_3-\frac{3\mu}{4}b_2^2\right| \le 1+\frac{|b_2^2|}{4}(3|\mu-1|-1). \tag{8}$$

Also, since $\mathrm{Re}\,p(z) > 0$, it follows that (see e.g. [**8**])

$$\left|p_2-\frac{p_1^2}{2}\right| \le 2-\frac{|p_1^2|}{2}. \tag{9}$$

Thus with $\mu = \dfrac{2(\beta+2)}{3(\beta+1)}$ we have from (8) that if $0 \le \beta \le 1$,

$$\left|b_3-\frac{3\mu}{4}b_2^2\right| \le 1-\frac{\beta|b_2^2|}{2(1+\beta)}, \tag{10}$$

and so from (5), (9) and (10) we obtain

$$\begin{aligned}
\left|a_3-\frac{2(\beta+2)}{3(\beta+1)}a_2^2\right| &\le \frac{1}{3}\left(1-\frac{\beta}{2(1+\beta)}|b_2^2|\right)+\frac{\beta\cos\lambda}{3}\left(2-\frac{|p_1^2|}{2}\right)\\
&\quad+\frac{\beta|p_1^2|\cos\lambda}{6}\sqrt{1-\left(\frac{1+2\beta}{(1+\beta)^2}\right)\cos^2\lambda}+\frac{\beta|p_1b_2|\cos\lambda}{3(1+\beta)}\\
&=\frac{2\beta\cos\lambda+1}{3}-\frac{\beta}{6(1+\beta)}\left(|b_2|-|p_1|\cos\lambda\right)^2-\frac{\beta|p_1^2|\cos\lambda}{6}\\
&\quad+\frac{\beta|p_1^2|\cos\lambda}{6}\sqrt{1-\left(\frac{1+2\beta}{(1+\beta)^2}\right)\cos^2\lambda}+\frac{\beta|p_1^2|\cos^2\lambda}{6(1+\beta)}\\
&\le\frac{2\beta\cos\lambda+1}{3}+\frac{\beta|p_1^2|\cos\lambda}{6}\left[-1+\sqrt{1-\left(\frac{1+2\beta}{(1+\beta)^2}\right)\cos^2\lambda}+\frac{\cos\lambda}{1+\beta}\right]\\
&\le\frac{2\beta+1}{3},
\end{aligned}$$

if $\cos^2\lambda \le (1+\beta)/2$, or if $\cos^2\lambda = 1$, where we have used the inequality $|p_1| \le 2$. Since $(1+\beta)/2$ increases for $0 \le \beta \le 1$, the above inequality is valid for $\cos^2\lambda \le 1/2$.

For $\beta \ge 1$ and $\mu = \dfrac{2(\beta+2)}{3(\beta+1)}$, it follows from (8) that

$$\left|b_3-\frac{3\mu}{4}b_2^2\right| \le 1-\frac{|b_2^2|}{2(1+\beta)},$$

and so again using (5) and (9) we obtain

$$\begin{aligned}
\left|a_3-\frac{2(\beta+2)}{3(\beta+1)}a_2^2\right| &\le \frac{1}{3}\left(1-\frac{|b_2^2|}{2(1+\beta)}\right)+\frac{\beta\cos\lambda}{3}\left(2-\frac{|p_1^2|}{2}\right)\\
&\quad+\frac{\beta|p_1^2|\cos\lambda}{6}\sqrt{1-\left(\frac{1+2\beta}{(1+\beta)^2}\right)\cos^2\lambda}+\frac{\beta|p_1b_2|\cos\lambda}{3(1+\beta)}\\
&=\frac{2\beta\cos\lambda+1}{3}-\frac{1}{6(1+\beta)}\left(|b_2|-\beta|p_1|\cos\lambda\right)^2-\frac{\beta|p_1^2|\cos\lambda}{6}\\
&\quad+\frac{\beta|p_1^2|\cos^2\lambda}{6}\sqrt{1-\left(\frac{1+2\beta}{(1+\beta)^2}\right)\cos^2\lambda}+\frac{\beta^2|p_1^2|\cos^2\lambda}{6(1+\beta)}\\
&\le\frac{2\beta\cos\lambda+1}{3}+\frac{\beta|p_1^2|\cos\lambda}{6}\left[-1+\sqrt{1-\left(\frac{1+2\beta}{(1+\beta)^2}\right)\cos^2\lambda}+\frac{\beta\cos\lambda}{1+\beta}\right]\\
&\le\frac{2\beta+1}{3},
\end{aligned}$$

if $\cos^2\lambda \le \left(1+3\beta-\sqrt{(5\beta+3)(\beta-1)}\right)/\left(2(1+\beta)\right)$, again since $|p_1| \le 2$. Since $\left(1+3\beta-\sqrt{5(\beta+3)(\beta-1)}\right)/\left(2(1+\beta)\right)$ decreases for $\beta \ge 1$, the inequality is valid for $\cos^2\lambda \le (3-\sqrt{5})/2$.

Next suppose that $\frac{2}{3} \le \mu \le \frac{2(\beta+2)}{3(\beta+1)}$. Then writing

$$a_3 - \mu a_2^2 = \frac{(\beta+1)(3\mu-2)}{2}\left(a_3 - \frac{2(\beta+2)}{3(\beta+1)}a_2^2\right) + \frac{3(\beta+1)}{2}\left(\frac{2(\beta+2)}{3(\beta+1)} - \mu\right)\left(a_3 - \frac{2}{3}a_2^2\right),$$

the result follows on using the Theorem already proved at the end points $\mu = 2/3$ and $\mu = \frac{2(\beta+2)}{3(\beta+1)}$.

Finally let $\mu \ge \frac{2(\beta+2)}{3(\beta+1)}$. Then, since

$$a_3 - \mu a_2^2 = \left(a_3 - \frac{2(\beta+2)}{3(\beta+1)}a_2^2\right) + \left(\frac{2(\beta+2)}{3(\beta+1)} - \mu\right)a_2^2,$$

the result follows again on using the case $\mu = \frac{2(\beta+2)}{3(\beta+1)}$ already established and the inequality $|a_2| \le 1+\beta$, proved in [**7**]. Equality is attained when $\lambda = 0$, $p_1 = p_2 = b_2 = 2$ and $b_3 = 3$.

Remark 2. An examination of the proof of Theorem 2 in the case $0 \le \beta \le 1$ when $\mu = \frac{2(\beta+2)}{3(\beta+1)}$ shows that

$$\left|a_3 - \frac{2(\beta+2)}{3(\beta+1)}a_2^2\right| \le \frac{1}{3} + \frac{2\beta}{3}\psi_1(\cos\lambda),$$

where

$$\psi_1(t) = t\left[\sqrt{1 - \left(\frac{1+2\beta}{(1+\beta)^2}\right)t^2} + \frac{t}{1+\beta}\right].$$

An elementary argument shows that ψ_1 attains its maximum at $t_0 \in (0,1)$ when

$$t_0^2 = \frac{2(1+\beta)^2 + (1+\beta)\sqrt{2(1+\beta)}}{4(1+2\beta)},$$

and that

$$\psi_1(t_0) = \frac{(1+\beta)[\sqrt{2(1+\beta)}+1]}{2(1+2\beta)}.$$

Thus if $0 \le \beta \le 1$ and $|\lambda| < \pi/2$,

$$\left|a_3 - \frac{2(\beta+2)}{3(\beta+1)}a_2^2\right| \le \frac{1}{3} + \frac{\beta(1+\beta)[\sqrt{2(1+\beta)}+1]}{3(1+2\beta)}. \tag{11}$$

Similarly, for $\beta \ge 1$, one obtains

$$\left|a_3 - \frac{2(\beta+2)}{3(\beta+1)}a_2^2\right| \le \frac{1}{3} + \frac{2\beta}{3}\psi_2(\cos\lambda),$$

where

$$\psi_2(t) = t\left[\sqrt{1-\left(\frac{1+2\beta}{(1+\beta)^2}\right)t^2}+\frac{\beta t}{1+\beta}\right].$$

It is easy to see that ψ_2 increases on $[0,1]$ and so for $|\lambda| < \pi/2$,

$$\left|a_3 - \frac{2(\beta+2)}{3(\beta+1)}a_2^2\right| \le \frac{1}{3} + \frac{4\beta^2}{3(1+\beta)}. \tag{12}$$

It is unlikely that either of (11) or (12) is sharp.

Finally, it is easy to see that, using (3), the following result obtains:

THEOREM 3. *Let $f \in K(\beta)$ and be given by* (1). *Then if* $0 \le \beta \le 1$,

$$|a_3 - \mu a_2^2| \le \begin{cases} \dfrac{2\beta+1}{3}, & \text{if } \dfrac{2}{3} \le \mu \le 1 \\ \dfrac{2\beta+1}{3} + (\mu-1)(1+\beta)^2, & \text{if } \mu \ge 1, \end{cases}$$

and if $\beta \ge 1$,

$$|a_3 - \mu a_2^2| \le \begin{cases} \dfrac{2\beta+1}{3} + \dfrac{(\beta^2-1)(3\mu-2)}{3}, & \text{if } \dfrac{2}{3} \le \mu \le 1 \\ \mu - 1 + \dfrac{\beta(3\mu-2)(\beta+2)}{3}, & \text{if } \mu \ge 1. \end{cases}$$

We note that if $0 \le \beta \le 1$, the inequality for $2/3 \le \mu \le 1$ is sharp when $\lambda = 0$, $b_2 = 0$, $b_3 = 1$, $p_1 = 0$ and $p_2 = 2$. When $\beta \ge 1$, the inequality for $\mu \ge 1$ is sharp for $\lambda = 0$, $p_1 = p_2 = b_2 = 2$ and $b_3 = 3$. The inequality for $0 \le \beta \le 1$ and $\mu \ge 1$ appears sharp only when $\mu = 1$, and the inequality for $\beta \ge 1$ when $2/3 \le \mu \le 1$ appears sharp only at the end points $\mu = 2/3$ and $\mu = 1$. However, in view of Theorem A, splitting the real line at $\mu = 1$ is probably not optimum, unless $\beta = 1$.

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