PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 52 (66), 1992, 10–12

POWER MOMENTS OF THE ERROR TERM FOR THE APPROXIMATE FUNCTIONAL EQUATION OF THE RIEMANN ZETA-FUNCTION

Isao Kiuchi

Abstract. Let $\zeta(s)$ be the Riemann zeta-function, $d(n)$ the number of positive divisors of the integer n, and

$$R(s;t/2\pi)=\zeta^2(s)-\sum_{n\le t/2\pi}{}' \, d(n)n^{-s}-\chi^2(s)\sum_{n\le t/2\pi}{}' \, d(n)n^{s-1},$$

where

$$\chi(s)=2^s\pi^{s-1}\sin\left(\tfrac12\pi s\right)\Gamma(1-s).$$

We obtain the following power moment estimates:

$$\int_1^T\left|R\left(\tfrac12+it;t/2\pi\right)\right|^A dt\ll\begin{cases}T^{1-\frac14 A+\varepsilon}, & 0\le A\le 4,\\ 1, & A>4.\end{cases}$$

1. Statement of results. Let $d(n)$ denote the number of positive divisors of n, γ the Euler constant, and let

$$\Delta(x)=\sum_{n\le x}{}' d(n)-x(\log x+2\gamma-1)-\frac14 \tag{1}$$

where the symbol $\sum'$ indicates that the last term is to be halved if x is an integer. Kolesnik [**7**] proved the sharper estimate of (1):

$$\Delta(x)\ll x^{35/108+\varepsilon}. \tag{2}$$

Recently this was improved by Iwaniec and Mozzochi [**3**].

The asymptotic formula

$$\int_1^T\Delta^2(x)\,dx=\frac{1}{6\pi^2}\left\{\sum_{n=1}^{\infty}d^2(n)n^{-3/2}\right\}T^{3/2}+O(T\log^5T) \tag{3}$$

AMS Subject Classification (1990): Primary 11 M 06

was proved by Tong [**10**], who improved an earlier result of Cramér (see [**2**, Theorem 13.5]). The error term of (3) has been improved to $O(T\log^4 T)$ by Preissmann [**9**]. Now we suppose that A is a fixed positive number (not necessarily an integer). Ivić [**1**] has shown the power moment estimates for $\Delta(x)$:

$$\int_1^T |\Delta(t)|^A\,dt \ll \begin{cases} T^{1+(A/4)+\varepsilon}, & 0\le A\le 35/4, \qquad (4)\\ T^{19/54+35A/108+\varepsilon}, & A\ge 35/4, \qquad (5)\end{cases}$$

by using Kolesnik's result (2).

Let $s=\sigma+it$ $(0\le\sigma\le 1,\, t\ge 1)$ be a complex variable, and $\zeta(s)$ the Riemann zeta-function. We now define

$$R(s;t/2\pi)=\zeta^2(s)-\sum_{n\le t/2\pi}{}' \, d(n)n^{-s}-\chi^2(s)\sum_{n\le t/2\pi}{}' \, d(n)n^{s-1},$$

where $\chi(s)=2^s\pi^{s-1}\sin\left(\frac12\pi s\right)\Gamma(1-s)$. It has been shown by Motohashi [**8**] that

$$\chi(1-s)R(s;t/2\pi)=-\sqrt{2}(t/2\pi)^{-1/2}\Delta(t/2\pi)+O(t^{-1/4}). \qquad (6)$$

We note that Jutila [**4**] gives another proof of Motohashi's result (6). The asymptotic formula

$$\int_1^T \left|R\left(\tfrac12+it;t/2\pi\right)\right|^2 dt=\sqrt{2\pi}\left\{\sum_{n=1}^{\infty} d^2(n)h^2(n)n^{-1/2}\right\}T^{1/2}+O(T^{1/4}\log T)$$

was proved by Kiuchi and Matsumoto [**5**], and the error term has been improved to $O(\log^5 T)$ by Kiuchi [**6**], where

$$h(n)=(2/\pi)^{1/2}\int_0^\infty (y+\pi n)^{-1/2}\cos\left(y+\tfrac14\pi\right)dy.$$

The purpose of this paper is to prove the power moment estimates for $\left|R\left(\frac12+it;t/2\pi\right)\right|$. In view of the relation (6), to search analogues of (4) and (5) for $R(s;t/2\pi)$ is an interesting problem in itself and we can prove the following estimates:

Theorem. *For $T\ge 1$, we have*

$$\int_1^T \left|R\left(\tfrac12+it;t/2\pi\right)\right|^A dt \ll \begin{cases} T^{1-(A/4)+\varepsilon}, & 0\le A\le 4,\\ 1, & A>4.\end{cases}$$

2. Proof of Theorem. In case $\sigma=1/2$ from the inequality $(a+b)^A\ll a^A+b^A$ $(a>0,\, b>0)$, and (6), it follows that

$$\left|R\left(\tfrac12+it;t/2\pi\right)\right|^A \ll (t/2\pi)^{-A/2}|\Delta(t/2\pi)|^A+t^{-A/4} \qquad (7)$$

where A is a fixed positive number.

From (7) and Schwarz's inequality, it follows that

$$\int_{T/2}^{T} \left|R\left(\tfrac{1}{2}+it;t/2\pi\right)\right|^{A} dt \ll \left\{\int_{T/2}^{T} (t/2\pi)^{-A}\, dt\right\}^{1/2} \left\{\int_{T/2}^{T} |\Delta(t/2\pi)|^{2A}\, dt\right\}^{1/2} + T^{1-A/4}.$$

From (4) and (5), we have the following estimates:

$$\int_{T/2}^{T} \left|R\left(\tfrac{1}{2}+it;t/2\pi\right)\right|^{A} dt \ll \begin{cases} T^{1-(A/4)+\varepsilon}, & 0 \le A \le 35/8, \\ T^{(73-19A)/108+\varepsilon}, & A \ge 35/8. \end{cases}$$

Replacing T by $T/2$, $T/4$, and so on, and adding we have the theorem.

REFERENCES

[**1**] A. Ivić, *Large values of the error term in the divisor problem*, Invent. Math. **71** (1983), 513–520.

[**2**] A. Ivić, *The Riemann Zeta-Function*, John Wiley & Sons, New York, 1985.

[**3**] H. Iwaniec and C.J. Mozzochi, *On the divisor and circle problems*, J. Number Theory **29** (1988), 60–93.

[**4**] M. Jutila, *On the approximate functional equation for $\zeta^2(s)$ and other Dirichlet series*, Quart. J. Math. Oxford (2) **37** (1986), 193–209.

[**5**] I. Kiuchi and K. Matsumoto, *Mean value results for the approximate functional equation of the square of the Riemann zeta-function*, Acta Arith. **61** (1992), 337–345.

[**6**] I. Kiuchi, *An improvement on the mean value formula for the approximate functional equation of the square of the Riemann zeta-function*, J. Number Theory. (to appear).

[**7**] G. Kolesnik, *On the estimation of $\Delta(R)$ and $\zeta(1/2+it)$*, Pacific J. Math **98** (1982), 107–122.

[**8**] Y. Motohashi, *A note on the approximate functional equation for $\zeta^2(s)$*, Proc. Japan Acad. Ser. A **59** (1983), 393–396.

[**9**] E. Preissmann, *Sur la moyenne quadratique du terme de reste du problème du cercle*, C. R. Acad. Sci. Paris **306** (1988), 151–154.

[**10**] K.-C. Tong, *On divisor problems III*, Acta Math. Sinica **6** (1956), 515–545.

Department of Mathematics
Keio University
14-1, Hiyoshi, 3 chome, Kohoku-ku
Yokohama, 223 Japan

(Received 26 11 1991)